UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Wednesday, 3 June, 2015 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

(a) Find all integers x and y such that

$$6x + 2y \equiv 3 \pmod{53} \qquad \text{and} \qquad 17x + 4y \equiv 7 \pmod{53}.$$

(b) Show that if an integer $n > 4$ is composite then $(n-1)! \equiv 0 \pmod{n}$.

2E Numbers and Sets

State the Chinese remainder theorem and Fermat's theorem. Prove that

$$p^4 \equiv 1 \pmod{240}$$

for any prime $p > 5$.

3C Dynamics and Relativity

Find the moment of inertia of a uniform sphere of mass M and radius a about an axis through its centre.

The kinetic energy T of any rigid body with total mass M, centre of mass $\mathbf{R}$, moment of inertia I about an axis of rotation through $\mathbf{R}$, and angular velocity ω about that same axis, is given by $T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}I\omega^2$. What physical interpretation can be given to the two parts of this expression?

A spherical marble of uniform density and mass M rolls without slipping at speed V along a flat surface. Explaining any relationship that you use between its speed and angular velocity, show that the kinetic energy of the marble is $\frac{7}{10}MV^2$.

4C Dynamics and Relativity

Write down the 4-momentum of a particle with energy E and 3-momentum $\mathbf{p}$. State the relationship between the energy E and wavelength λ of a photon.

An electron of mass m is at rest at the origin of the laboratory frame: write down its 4-momentum. The electron is scattered by a photon of wavelength λ_1 travelling along the x-axis: write down the initial 4-momentum of the photon. Afterwards, the photon has wavelength λ_2 and has been deflected through an angle θ. Show that

$$\lambda_2 - \lambda_1 = \frac{2h}{mc}\sin^2(\tfrac{1}{2}\theta)$$

where c is the speed of light and h is Planck's constant.

SECTION II

5E Numbers and Sets

(i) Let $\sim$ be an equivalence relation on a set X. What is an *equivalence class* of $\sim$? What is a *partition* of X? Prove that the equivalence classes of $\sim$ form a partition of X.

(ii) Let $\sim$ be the relation on the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ defined by

$$m \sim n \iff \exists\, a, b \in \mathbb{N} \text{ such that } m \text{ divides } n^a \text{ and } n \text{ divides } m^b.$$

Show that $\sim$ is an equivalence relation, and show that it has infinitely many equivalence classes, all but one of which are infinite.

6E Numbers and Sets

Let p be a prime. A *base p expansion* of an integer k is an expression

$$k = k_0 + p \cdot k_1 + p^2 \cdot k_2 + \cdots + p^\ell \cdot k_\ell$$

for some natural number ℓ, with $0 \leqslant k_i < p$ for $i = 0, 1, \ldots, \ell$.

(i) Show that the sequence of coefficients $k_0, k_1, k_2, \ldots, k_\ell$ appearing in a base p expansion of k is unique, up to extending the sequence by zeroes.

(ii) Show that

$$\binom{p}{j} \equiv 0 \pmod{p}, \quad 0 < j < p,$$

and hence, by considering the polynomial $(1+x)^p$ or otherwise, deduce that

$$\binom{p^i}{j} \equiv 0 \pmod{p}, \quad 0 < j < p^i.$$

(iii) If $n_0 + p \cdot n_1 + p^2 \cdot n_2 + \cdots + p^\ell \cdot n_\ell$ is a base p expansion of n, then, by considering the polynomial $(1+x)^n$ or otherwise, show that

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1} \cdots \binom{n_\ell}{k_\ell} \pmod{p}.$$

7E Numbers and Sets

State the inclusion–exclusion principle.

Let $n \in \mathbb{N}$. A permutation σ of the set $\{1, 2, 3, \ldots, n\}$ is said to *contain a transposition* if there exist i, j with $1 \leqslant i < j \leqslant n$ such that $\sigma(i) = j$ and $\sigma(j) = i$. Derive a formula for the number, $f(n)$, of permutations which do not contain a transposition, and show that

$$\lim_{n\to\infty} \frac{f(n)}{n!} = e^{-\frac{1}{2}}.$$

8E Numbers and Sets

What does it mean for a set to be *countable*? Prove that

(a) if B is countable and $f\colon A \to B$ is injective, then A is countable;

(b) if A is countable and $f\colon A \to B$ is surjective, then B is countable.

Prove that $\mathbb{N} \times \mathbb{N}$ is countable, and deduce that

(i) if X and Y are countable, then so is $X \times Y$;

(ii) $\mathbb{Q}$ is countable.

Let $\mathcal{C}$ be a collection of circles in the plane such that for each point a on the x-axis, there is a circle in $\mathcal{C}$ passing through the point a which has the x-axis tangent to the circle at a. Show that $\mathcal{C}$ contains a pair of circles that intersect.

9C Dynamics and Relativity

A particle is projected vertically upwards at speed V from the surface of the Earth, which may be treated as a perfect sphere. The variation of gravity with height should not be ignored, but the rotation of the Earth should be. Show that the height $z(t)$ of the particle obeys

$$\ddot{z} = -\frac{gR^2}{(R+z)^2},$$

where R is the radius of the Earth and g is the acceleration due to gravity measured at the Earth's surface.

Using dimensional analysis, show that the maximum height H of the particle and the time T taken to reach that height are given by

$$H = RF(\lambda) \quad \text{and} \quad T = \frac{V}{g}G(\lambda),$$

where F and G are functions of $\lambda = V^2/gR$.

Write down the equation of conservation of energy and deduce that

$$T = \int_0^H \sqrt{\frac{R+z}{V^2R - (2gR - V^2)z}}\, dz.$$

Hence or otherwise show that

$$F(\lambda) = \frac{\lambda}{2-\lambda} \quad \text{and} \quad G(\lambda) = \int_0^1 \sqrt{\frac{2-\lambda+\lambda x}{(2-\lambda)^3(1-x)}}\, dx.$$

10C Dynamics and Relativity

A particle of mass m and charge q has position vector $\mathbf{r}(t)$ and moves in a constant, uniform magnetic field $\mathbf{B}$ so that its equation of motion is

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}.$$

Let $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$ be the particle's angular momentum. Show that

$$\mathbf{L} \cdot \mathbf{B} + \tfrac{1}{2}q|\mathbf{r} \times \mathbf{B}|^2$$

is a constant of the motion. Explain why the kinetic energy T is also constant, and show that it may be written in the form

$$T = \tfrac{1}{2}m\mathbf{u} \cdot \big((\mathbf{u} \cdot \mathbf{v})\mathbf{v} - r^2\ddot{\mathbf{u}}\big),$$

where $\mathbf{v} = \dot{\mathbf{r}}$, $r = |\mathbf{r}|$ and $\mathbf{u} = \mathbf{r}/r$.

[*Hint: Consider* $\mathbf{u} \cdot \dot{\mathbf{u}}$.]

11C Dynamics and Relativity

Consider a particle with position vector $\mathbf{r}(t)$ moving in a plane described by polar coordinates (r, θ). Obtain expressions for the radial (r) and transverse (θ) components of the velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$.

A charged particle of unit mass moves in the electric field of another charge that is fixed at the origin. The electrostatic force on the particle is $-p/r^2$ in the radial direction, where p is a positive constant. The motion takes place in an unusual medium that resists radial motion but not tangential motion, so there is an additional radial force $-k\dot{r}/r^2$ where k is a positive constant. Show that the particle's motion lies in a plane. Using polar coordinates in that plane, show also that its angular momentum $h = r^2\dot{\theta}$ is constant.

Obtain the equation of motion

$$\frac{d^2u}{d\theta^2} + \frac{k}{h}\frac{du}{d\theta} + u = \frac{p}{h^2},$$

where $u = r^{-1}$, and find its general solution assuming that $k/|h| < 2$. Show that so long as the motion remains bounded it eventually becomes circular with radius h^2/p.

Obtain the expression

$$E = \tfrac{1}{2}h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) - pu$$

for the particle's total energy, that is, its kinetic energy plus its electrostatic potential energy. Hence, or otherwise, show that the energy is a decreasing function of time.

12C Dynamics and Relativity

Write down the Lorentz transform relating the components of a 4-vector between two inertial frames.

A particle moves along the x-axis of an inertial frame. Its position at time t is $x(t)$, its velocity is $u = dx/dt$, and its 4-position is $X = (ct, x)$, where c is the speed of light. The particle's 4-velocity is given by $U = dX/d\tau$ and its 4-acceleration is $A = dU/d\tau$, where *proper time* τ is defined by $c^2 d\tau^2 = c^2 dt^2 - dx^2$. Show that

$$U = \gamma\,(c, u) \quad \text{and} \quad A = \gamma^4 \dot{u}\,(u/c, 1)$$

where $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$ and $\dot{u} = du/dt$.

The *proper 3-acceleration* a of the particle is defined to be the spatial component of its 4-acceleration measured in the particle's instantaneous rest frame. By transforming A to the rest frame, or otherwise, show that

$$a = \gamma^3 \dot{u} = \frac{d}{dt}(\gamma u).$$

Given that the particle moves with constant proper 3-acceleration starting from rest at the origin, show that

$$x(t) = \frac{c^2}{a}\left(\sqrt{1 + \frac{a^2 t^2}{c^2}} - 1\right),$$

and that, if $at \ll c$, then $x \approx \frac{1}{2} a t^2$.

END OF PAPER