UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday, 29 May, 2015 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Differential Equations

Find the general solution of the equation

$$\frac{dy}{dx} - 2y = e^{\lambda x}, \qquad (*)$$

where λ is a constant not equal to 2.

By subtracting from the particular integral an appropriate multiple of the complementary function, obtain the limit as $\lambda \to 2$ of the general solution of $(*)$ and confirm that it yields the general solution for $\lambda = 2$.

Solve equation $(*)$ with $\lambda = 2$ and $y(1) = 2$.

2B Differential Equations

Find the general solution of the equation

$$2\frac{dy}{dt} = y - y^3.$$

Compute all possible limiting values of y as $t \to \infty$.

Find a non-zero value of $y(0)$ such that $y(t) = y(0)$ for all t.

3F Probability

Let U be a uniform random variable on $(0, 1)$, and let $\lambda > 0$.

(a) Find the distribution of the random variable $-(\log U)/\lambda$.

(b) Define a new random variable X as follows: suppose a fair coin is tossed, and if it lands heads we set $X = U^2$ whereas if it lands tails we set $X = 1 - U^2$. Find the probability density function of X.

4F Probability

Let A, B be events in the sample space Ω such that $0 < P(A) < 1$ and $0 < P(B) < 1$. The event B is said to *attract* A if the conditional probability $P(A|B)$ is greater than $P(A)$, otherwise it is said that A *repels* B. Show that if B attracts A, then A attracts B. Does $B^c = \Omega \setminus B$ repel A?

SECTION II

5B Differential Equations

Write as a system of two first-order equations the second-order equation

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt}\left|\frac{d\theta}{dt}\right| + \sin\theta = 0, \qquad (*)$$

where c is a small, positive constant, and find its equilibrium points. What is the nature of these points?

Draw the trajectories in the (θ, ω) plane, where $\omega = d\theta/dt$, in the neighbourhood of two typical equilibrium points.

By considering the cases of $\omega > 0$ and $\omega < 0$ separately, find explicit expressions for ω^2 as a function of θ. Discuss how the second term in $(*)$ affects the nature of the equilibrium points.

6B Differential Equations

Consider the equation

$$2\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} - 7\frac{\partial^2 u}{\partial x\,\partial y} = 0 \qquad (*)$$

for the function $u(x, y)$, where x and y are real variables. By using the change of variables

$$\xi = x + \alpha y, \qquad \eta = \beta x + y,$$

where α and β are appropriately chosen integers, transform $(*)$ into the equation

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0.$$

Hence, solve equation $(*)$ supplemented with the boundary conditions

$$u(0, y) = 4y^2, \qquad u(-2y, y) = 0, \qquad \text{for all } y.$$

7B Differential Equations

Suppose that $u(x)$ satisfies the equation

$$\frac{d^2u}{dx^2} - f(x)u = 0,$$

where $f(x)$ is a given non-zero function. Show that under the change of coordinates $x = x(t)$,

$$\frac{d^2u}{dt^2} - \frac{\ddot{x}}{\dot{x}}\frac{du}{dt} - \dot{x}^2 f(x)u = 0,$$

where a dot denotes differentiation with respect to t. Furthermore, show that the function

$$U(t) = \dot{x}^{-\frac{1}{2}}u(x)$$

satisfies

$$\frac{d^2U}{dt^2} - \left[\dot{x}^2 f(x) + \dot{x}^{-\frac{1}{2}}\left(\frac{\ddot{x}}{\dot{x}}\frac{d}{dt}(\dot{x}^{\frac{1}{2}}) - \frac{d^2}{dt^2}(\dot{x}^{\frac{1}{2}})\right)\right]U = 0.$$

Choosing $\dot{x} = \big(f(x)\big)^{-\frac{1}{2}}$, deduce that

$$\frac{d^2U}{dt^2} - \big(1 + F(t)\big)U = 0,$$

for some appropriate function $F(t)$. Assuming that F may be neglected, deduce that $u(x)$ can be approximated by

$$u(x) \approx A(x)\big(c_+ e^{G(x)} + c_- e^{-G(x)}\big),$$

where c_+, c_- are constants and A, G are functions that you should determine in terms of $f(x)$.

8B Differential Equations

Suppose that $\mathbf{x}(t) \in \mathbb{R}^3$ obeys the differential equation

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x}, \qquad (*)$$

where M is a constant 3×3 real matrix.

(i) Suppose that M has *distinct* eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Explain why $\mathbf{x}$ may be expressed in the form $\sum_{i=1}^{3} a_i(t)\mathbf{e}_i$ and deduce by substitution that the general solution of $(*)$ is

$$\mathbf{x} = \sum_{i=1}^{3} A_i e^{\lambda_i t}\mathbf{e}_i,$$

where A_1, A_2, A_3 are constants.

(ii) What is the general solution of $(*)$ if $\lambda_2 = \lambda_3 \neq \lambda_1$, but there are still three linearly independent eigenvectors?

(iii) Suppose again that $\lambda_2 = \lambda_3 \neq \lambda_1$, but now there are only two linearly independent eigenvectors: $\mathbf{e}_1$ corresponding to λ_1 and $\mathbf{e}_2$ corresponding to λ_2. Suppose that a vector $\mathbf{v}$ satisfying the equation $(M - \lambda_2 I)\mathbf{v} = \mathbf{e}_2$ exists, where I denotes the identity matrix. Show that $\mathbf{v}$ is linearly independent of $\mathbf{e}_1$ and $\mathbf{e}_2$, and hence or otherwise find the general solution of $(*)$.

9F Probability

Lionel and Cristiana have a and b million pounds, respectively, where $a, b \in \mathbb{N}$. They play a series of independent football games in each of which the winner receives one million pounds from the loser (a draw cannot occur). They stop when one player has lost his or her entire fortune. Lionel wins each game with probability $0 < p < 1$ and Cristiana wins with probability $q = 1 - p$, where $p \neq q$. Find the expected number of games before they stop playing.

10F Probability

Consider the function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}.$$

Show that ϕ defines a probability density function. If a random variable X has probability density function ϕ, find the moment generating function of X, and find all moments $E[X^k]$, $k \in \mathbb{N}$.

Now define

$$r(x) = \frac{P(X > x)}{\phi(x)}.$$

Show that for every $x > 0$,

$$\frac{1}{x} - \frac{1}{x^3} < r(x) < \frac{1}{x}.$$

11F Probability

State and prove Markov's inequality and Chebyshev's inequality, and deduce the weak law of large numbers.

If X is a random variable with mean zero and finite variance σ^2, prove that for any $a > 0$,

$$P(X \geqslant a) \leqslant \frac{\sigma^2}{\sigma^2 + a^2}.$$

[*Hint: Show first that* $P(X \geqslant a) \leqslant P\left((X+b)^2 \geqslant (a+b)^2\right)$ *for every* $b > 0$.]

12F Probability

When coin A is tossed it comes up heads with probability $\frac{1}{4}$, whereas coin B comes up heads with probability $\frac{3}{4}$. Suppose one of these coins is randomly chosen and is tossed twice. If both tosses come up heads, what is the probability that coin B was tossed? Justify your answer.

In each draw of a lottery, an integer is picked independently at random from the first n integers $1, 2, \ldots, n$, with replacement. What is the probability that in a sample of r successive draws the numbers are drawn in a non-decreasing sequence? Justify your answer.

END OF PAPER