UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 28 May, 2015 9:00 am to 12:00 noon

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Vectors and Matrices

(a) Describe geometrically the curve

$$|\alpha z + \beta \bar{z}| = \sqrt{\alpha\beta}\,(z + \bar{z}) + (\alpha - \beta)^2,$$

where $z \in \mathbb{C}$ and α, β are positive, distinct, real constants.

(b) Let θ be a real number not equal to an integer multiple of 2π. Show that

$$\sum_{m=1}^{N} \sin(m\theta) = \frac{\sin\theta + \sin(N\theta) - \sin(N\theta + \theta)}{2(1 - \cos\theta)},$$

and derive a similar expression for $\displaystyle\sum_{m=1}^{N} \cos(m\theta)$.

2C Vectors and Matrices

Precisely one of the four matrices specified below is not orthogonal. Which is it? Give a brief justification.

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -\sqrt{3} & \sqrt{2} \\ 1 & \sqrt{3} & \sqrt{2} \\ -2 & 0 & \sqrt{2} \end{pmatrix} \quad \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix} \quad \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -2 & 1 \\ -\sqrt{6} & 0 & \sqrt{6} \\ 1 & 1 & 1 \end{pmatrix} \quad \frac{1}{9}\begin{pmatrix} 7 & -4 & -4 \\ -4 & 1 & -8 \\ -4 & -8 & 1 \end{pmatrix}$$

Given that the four matrices represent transformations of $\mathbb{R}^3$ corresponding (in no particular order) to a rotation, a reflection, a combination of a rotation and a reflection, and none of these, identify each matrix. Explain your reasoning.

[*Hint: For* ***two*** *of the matrices, A and B say, you may find it helpful to calculate* $\det(A - I)$ *and* $\det(B - I)$*, where I is the identity matrix.*]

3F Analysis I

Find the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

(b) $\displaystyle\lim_{x\to 0} (1 + x)^{1/x}$

(c) $\displaystyle\lim_{x\to\infty} \frac{(1 + x)^{\frac{x}{1+x}} \cos^4 x}{e^x}$

Carefully justify your answers.

[You may use standard results provided that they are clearly stated.]

4E Analysis I

Let $\sum_{n\geqslant 0} a_n z^n$ be a complex power series. State carefully what it means for the power series to have *radius of convergence* R, with $0 \leqslant R \leqslant \infty$.

Find the radius of convergence of $\sum_{n\geqslant 0} p(n) z^n$, where $p(n)$ is a fixed polynomial in n with coefficients in $\mathbb{C}$.

SECTION II

5B Vectors and Matrices

(i) State and prove the Cauchy–Schwarz inequality for vectors in $\mathbb{R}^n$. Deduce the inequalities

$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}|+|\mathbf{b}| \quad \text{and} \quad |\mathbf{a}+\mathbf{b}+\mathbf{c}| \leqslant |\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|$$

for $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}^n$.

(ii) Show that every point on the intersection of the planes

$$\mathbf{x}\cdot\mathbf{a} = A, \quad \mathbf{x}\cdot\mathbf{b} = B,$$

where $\mathbf{a} \neq \mathbf{b}$, satisfies

$$|\mathbf{x}|^2 \geqslant \frac{(A-B)^2}{|\mathbf{a}-\mathbf{b}|^2}.$$

What happens if $\mathbf{a} = \mathbf{b}$?

(iii) Using your results from part (i), or otherwise, show that for any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n$,

$$|\mathbf{x}_1-\mathbf{y}_1| - |\mathbf{x}_1-\mathbf{y}_2| \leqslant |\mathbf{x}_2-\mathbf{y}_1| + |\mathbf{x}_2-\mathbf{y}_2|.$$

6C Vectors and Matrices

(i) Consider the map from $\mathbb{R}^4$ to $\mathbb{R}^3$ represented by the matrix

$$\begin{pmatrix} \alpha & 1 & 1 & -1 \\ 2 & -\alpha & 0 & -2 \\ -\alpha & 2 & 1 & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$. Find the image and kernel of the map for each value of α.

(ii) Show that any linear map $f\colon \mathbb{R}^n \to \mathbb{R}$ may be written in the form $f(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x}$ for some fixed vector $\mathbf{a} \in \mathbb{R}^n$. Show further that $\mathbf{a}$ is uniquely determined by f.

It is given that $n = 4$ and that the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

lie in the kernel of f. Determine the set of possible values of $\mathbf{a}$.

7A Vectors and Matrices

(i) Find the eigenvalues and eigenvectors of the following matrices and show that both are diagonalisable:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 4 & -3 \\ -4 & 10 & -4 \\ -3 & 4 & 1 \end{pmatrix}.$$

(ii) Show that, if two real $n \times n$ matrices can both be diagonalised using the same basis transformation, then they commute.

(iii) Suppose now that two real $n \times n$ matrices C and D commute and that D has n distinct eigenvalues. Show that for any eigenvector $\mathbf{x}$ of D the vector $C\mathbf{x}$ is a scalar multiple of $\mathbf{x}$. Deduce that there exists a common basis transformation that diagonalises both matrices.

(iv) Show that A and B satisfy the conditions in (iii) and find a matrix S such that both of the matrices $S^{-1}AS$ and $S^{-1}BS$ are diagonal.

8A Vectors and Matrices

(a) A matrix is called *normal* if $A^\dagger A = AA^\dagger$. Let A be a normal $n \times n$ complex matrix.

(i) Show that for any vector $\mathbf{x} \in \mathbb{C}^n$,

$$|A\mathbf{x}| = |A^\dagger \mathbf{x}|.$$

(ii) Show that $A - \lambda I$ is also normal for any $\lambda \in \mathbb{C}$, where I denotes the identity matrix.

(iii) Show that if $\mathbf{x}$ is an eigenvector of A with respect to the eigenvalue $\lambda \in \mathbb{C}$, then $\mathbf{x}$ is also an eigenvector of $A^\dagger$, and determine the corresponding eigenvalue.

(iv) Show that if $\mathbf{x}_\lambda$ and $\mathbf{x}_\mu$ are eigenvectors of A with respect to distinct eigenvalues λ and μ respectively, then $\mathbf{x}_\lambda$ and $\mathbf{x}_\mu$ are orthogonal.

(v) Show that if A has a basis of eigenvectors, then A can be diagonalised using an orthonormal basis. Justify your answer.

[You may use standard results provided that they are clearly stated.]

(b) Show that any matrix A satisfying $A^\dagger = A$ is normal, and deduce using results from (a) that its eigenvalues are real.

(c) Show that any matrix A satisfying $A^\dagger = -A$ is normal, and deduce using results from (a) that its eigenvalues are purely imaginary.

(d) Show that any matrix A satisfying $A^\dagger = A^{-1}$ is normal, and deduce using results from (a) that its eigenvalues have unit modulus.

9F Analysis I

Let $(a_n), (b_n)$ be sequences of real numbers. Let $S_n = \sum_{j=1}^{n} a_j$ and set $S_0 = 0$. Show that for any $1 \leqslant m \leqslant n$ we have

$$\sum_{j=m}^{n} a_j b_j = S_n b_n - S_{m-1} b_m + \sum_{j=m}^{n-1} S_j (b_j - b_{j+1}).$$

Suppose that the series $\sum_{n \geqslant 1} a_n$ converges and that (b_n) is bounded and monotonic. Does $\sum_{n \geqslant 1} a_n b_n$ converge?

Assume again that $\sum_{n \geqslant 1} a_n$ converges. Does $\sum_{n \geqslant 1} n^{1/n} a_n$ converge?

Justify your answers.

[You may use the fact that a sequence of real numbers converges if and only if it is a Cauchy sequence.]

10D Analysis I

(a) For real numbers a, b such that $a < b$, let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Prove that f is bounded on $[a,b]$, and that f attains its supremum and infimum on $[a,b]$.

(b) For $x \in \mathbb{R}$, define

$$g(x) = \begin{cases} |x|^{\frac{1}{2}} \sin(1/\sin x), & x \neq n\pi \\ 0, & x = n\pi \end{cases} \qquad (n \in \mathbb{Z}).$$

Find the set of points $x \in \mathbb{R}$ at which $g(x)$ is continuous.

Does g attain its supremum on $[0, \pi]$?

Does g attain its supremum on $[\pi, 3\pi/2]$?

Justify your answers.

11D Analysis I

(i) State and prove the intermediate value theorem.

(ii) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. The chord joining the points $\big(\alpha, f(\alpha)\big)$ and $\big(\beta, f(\beta)\big)$ of the curve $y = f(x)$ is said to be *horizontal* if $f(\alpha) = f(\beta)$. Suppose that the chord joining the points $\big(0, f(0)\big)$ and $\big(1, f(1)\big)$ is horizontal. By considering the function g defined on $[0, \frac{1}{2}]$ by

$$g(x) = f(x + \tfrac{1}{2}) - f(x),$$

or otherwise, show that the curve $y = f(x)$ has a horizontal chord of length $\frac{1}{2}$ in $[0,1]$. Show, more generally, that it has a horizontal chord of length $\frac{1}{n}$ for each positive integer n.

12E Analysis I

Let $f\colon [0,1] \to \mathbb{R}$ be a bounded function, and let $\mathcal{D}_n$ denote the dissection $0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1$ of $[0,1]$. Prove that f is Riemann integrable if and only if the difference between the upper and lower sums of f with respect to the dissection $\mathcal{D}_n$ tends to zero as n tends to infinity.

Suppose that f is Riemann integrable and $g\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Prove that $g \circ f$ is Riemann integrable.

[You may use the mean value theorem provided that it is clearly stated.]

END OF PAPER