UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2015

List of Courses

Paper 1, Section I

3F Analysis I

Find the following limits:

(a) $\lim_{x \to 0} \frac{\sin x}{x}$

(b) $\lim_{x \to 0} (1+x)^{1/x}$

(c) $\lim_{x \to \infty} \frac{(1+x)^{\frac{x}{1+x}} \cos^4 x}{e^x}$

Carefully justify your answers.

[You may use standard results provided that they are clearly stated.]

Paper 1, Section I

4E Analysis I

Let $\sum_{n \geqslant 0} a_n z^n$ be a complex power series. State carefully what it means for the power series to have *radius of convergence* R, with $0 \leqslant R \leqslant \infty$.

Find the radius of convergence of $\sum_{n \geqslant 0} p(n) z^n$, where $p(n)$ is a fixed polynomial in n with coefficients in $\mathbb{C}$.

Paper 1, Section II

9F Analysis I

Let $(a_n), (b_n)$ be sequences of real numbers. Let $S_n = \sum_{j=1}^{n} a_j$ and set $S_0 = 0$. Show that for any $1 \leqslant m \leqslant n$ we have

$$\sum_{j=m}^{n} a_j b_j = S_n b_n - S_{m-1} b_m + \sum_{j=m}^{n-1} S_j (b_j - b_{j+1}).$$

Suppose that the series $\sum_{n \geqslant 1} a_n$ converges and that (b_n) is bounded and monotonic. Does $\sum_{n \geqslant 1} a_n b_n$ converge?

Assume again that $\sum_{n \geqslant 1} a_n$ converges. Does $\sum_{n \geqslant 1} n^{1/n} a_n$ converge?

Justify your answers.

[You may use the fact that a sequence of real numbers converges if and only if it is a Cauchy sequence.]

Paper 1, Section II

10D Analysis I

(a) For real numbers a, b such that $a < b$, let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Prove that f is bounded on $[a,b]$, and that f attains its supremum and infimum on $[a,b]$.

(b) For $x \in \mathbb{R}$, define

$$g(x) = \begin{cases} |x|^{\frac{1}{2}} \sin(1/\sin x), & x \neq n\pi \\ 0, & x = n\pi \end{cases} \qquad (n \in \mathbb{Z}).$$

Find the set of points $x \in \mathbb{R}$ at which $g(x)$ is continuous.

Does g attain its supremum on $[0, \pi]$?

Does g attain its supremum on $[\pi, 3\pi/2]$?

Justify your answers.

Paper 1, Section II

11D Analysis I

(i) State and prove the intermediate value theorem.

(ii) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. The chord joining the points $\big(\alpha, f(\alpha)\big)$ and $\big(\beta, f(\beta)\big)$ of the curve $y = f(x)$ is said to be *horizontal* if $f(\alpha) = f(\beta)$. Suppose that the chord joining the points $\big(0, f(0)\big)$ and $\big(1, f(1)\big)$ is horizontal. By considering the function g defined on $[0, \frac{1}{2}]$ by

$$g(x) = f(x + \tfrac{1}{2}) - f(x),$$

or otherwise, show that the curve $y = f(x)$ has a horizontal chord of length $\frac{1}{2}$ in $[0,1]$. Show, more generally, that it has a horizontal chord of length $\frac{1}{n}$ for each positive integer n.

Paper 1, Section II

12E Analysis I

Let $f\colon [0,1] \to \mathbb{R}$ be a bounded function, and let $\mathcal{D}_n$ denote the dissection $0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1$ of $[0,1]$. Prove that f is Riemann integrable if and only if the difference between the upper and lower sums of f with respect to the dissection $\mathcal{D}_n$ tends to zero as n tends to infinity.

Suppose that f is Riemann integrable and $g\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Prove that $g \circ f$ is Riemann integrable.

[You may use the mean value theorem provided that it is clearly stated.]

Paper 2, Section I

1B Differential Equations

Find the general solution of the equation

$$\frac{dy}{dx} - 2y = e^{\lambda x}, \qquad (*)$$

where λ is a constant not equal to 2.

By subtracting from the particular integral an appropriate multiple of the complementary function, obtain the limit as $\lambda \to 2$ of the general solution of $(*)$ and confirm that it yields the general solution for $\lambda = 2$.

Solve equation $(*)$ with $\lambda = 2$ and $y(1) = 2$.

Paper 2, Section I

2B Differential Equations

Find the general solution of the equation

$$2\frac{dy}{dt} = y - y^3.$$

Compute all possible limiting values of y as $t \to \infty$.

Find a non-zero value of $y(0)$ such that $y(t) = y(0)$ for all t.

Paper 2, Section II

5B Differential Equations

Write as a system of two first-order equations the second-order equation

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt}\left|\frac{d\theta}{dt}\right| + \sin\theta = 0, \qquad (*)$$

where c is a small, positive constant, and find its equilibrium points. What is the nature of these points?

Draw the trajectories in the (θ, ω) plane, where $\omega = d\theta/dt$, in the neighbourhood of two typical equilibrium points.

By considering the cases of $\omega > 0$ and $\omega < 0$ separately, find explicit expressions for ω^2 as a function of θ. Discuss how the second term in $(*)$ affects the nature of the equilibrium points.

Paper 2, Section II

6B Differential Equations

Consider the equation

$$2\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} - 7\frac{\partial^2 u}{\partial x\,\partial y} = 0 \qquad (*)$$

for the function $u(x,y)$, where x and y are real variables. By using the change of variables

$$\xi = x + \alpha y, \qquad \eta = \beta x + y,$$

where α and β are appropriately chosen integers, transform $(*)$ into the equation

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0.$$

Hence, solve equation $(*)$ supplemented with the boundary conditions

$$u(0,y) = 4y^2, \qquad u(-2y,y) = 0, \qquad \text{for all } y.$$

Paper 2, Section II

7B Differential Equations

Suppose that $u(x)$ satisfies the equation

$$\frac{d^2u}{dx^2} - f(x)u = 0,$$

where $f(x)$ is a given non-zero function. Show that under the change of coordinates $x = x(t)$,

$$\frac{d^2u}{dt^2} - \frac{\ddot{x}}{\dot{x}}\frac{du}{dt} - \dot{x}^2 f(x)u = 0,$$

where a dot denotes differentiation with respect to t. Furthermore, show that the function

$$U(t) = \dot{x}^{-\frac{1}{2}}u(x)$$

satisfies

$$\frac{d^2U}{dt^2} - \left[\dot{x}^2 f(x) + \dot{x}^{-\frac{1}{2}}\left(\frac{\ddot{x}}{\dot{x}}\frac{d}{dt}(\dot{x}^{\frac{1}{2}}) - \frac{d^2}{dt^2}(\dot{x}^{\frac{1}{2}})\right)\right]U = 0.$$

Choosing $\dot{x} = \big(f(x)\big)^{-\frac{1}{2}}$, deduce that

$$\frac{d^2U}{dt^2} - \big(1 + F(t)\big)U = 0,$$

for some appropriate function $F(t)$. Assuming that F may be neglected, deduce that $u(x)$ can be approximated by

$$u(x) \approx A(x)\big(c_+e^{G(x)} + c_-e^{-G(x)}\big),$$

where c_+, c_- are constants and A, G are functions that you should determine in terms of $f(x)$.

Paper 2, Section II

8B Differential Equations

Suppose that $\mathbf{x}(t) \in \mathbb{R}^3$ obeys the differential equation

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x}, \qquad (*)$$

where M is a constant 3×3 real matrix.

(i) Suppose that M has *distinct* eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Explain why $\mathbf{x}$ may be expressed in the form $\sum_{i=1}^{3} a_i(t)\mathbf{e}_i$ and deduce by substitution that the general solution of $(*)$ is

$$\mathbf{x} = \sum_{i=1}^{3} A_i e^{\lambda_i t}\mathbf{e}_i,$$

where A_1, A_2, A_3 are constants.

(ii) What is the general solution of $(*)$ if $\lambda_2 = \lambda_3 \neq \lambda_1$, but there are still three linearly independent eigenvectors?

(iii) Suppose again that $\lambda_2 = \lambda_3 \neq \lambda_1$, but now there are only two linearly independent eigenvectors: $\mathbf{e}_1$ corresponding to λ_1 and $\mathbf{e}_2$ corresponding to λ_2. Suppose that a vector $\mathbf{v}$ satisfying the equation $(M - \lambda_2 I)\mathbf{v} = \mathbf{e}_2$ exists, where I denotes the identity matrix. Show that $\mathbf{v}$ is linearly independent of $\mathbf{e}_1$ and $\mathbf{e}_2$, and hence or otherwise find the general solution of $(*)$.

Paper 4, Section I

3C Dynamics and Relativity

Find the moment of inertia of a uniform sphere of mass M and radius a about an axis through its centre.

The kinetic energy T of any rigid body with total mass M, centre of mass $\mathbf{R}$, moment of inertia I about an axis of rotation through $\mathbf{R}$, and angular velocity ω about that same axis, is given by $T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}I\omega^2$. What physical interpretation can be given to the two parts of this expression?

A spherical marble of uniform density and mass M rolls without slipping at speed V along a flat surface. Explaining any relationship that you use between its speed and angular velocity, show that the kinetic energy of the marble is $\frac{7}{10}MV^2$.

Paper 4, Section I

4C Dynamics and Relativity

Write down the 4-momentum of a particle with energy E and 3-momentum $\mathbf{p}$. State the relationship between the energy E and wavelength λ of a photon.

An electron of mass m is at rest at the origin of the laboratory frame: write down its 4-momentum. The electron is scattered by a photon of wavelength λ_1 travelling along the x-axis: write down the initial 4-momentum of the photon. Afterwards, the photon has wavelength λ_2 and has been deflected through an angle θ. Show that

$$\lambda_2 - \lambda_1 = \frac{2h}{mc}\sin^2(\tfrac{1}{2}\theta)$$

where c is the speed of light and h is Planck's constant.

Paper 4, Section II

9C Dynamics and Relativity

A particle is projected vertically upwards at speed V from the surface of the Earth, which may be treated as a perfect sphere. The variation of gravity with height should not be ignored, but the rotation of the Earth should be. Show that the height $z(t)$ of the particle obeys

$$\ddot{z} = -\frac{gR^2}{(R+z)^2},$$

where R is the radius of the Earth and g is the acceleration due to gravity measured at the Earth's surface.

Using dimensional analysis, show that the maximum height H of the particle and the time T taken to reach that height are given by

$$H = RF(\lambda) \quad \text{and} \quad T = \frac{V}{g}G(\lambda),$$

where F and G are functions of $\lambda = V^2/gR$.

Write down the equation of conservation of energy and deduce that

$$T = \int_0^H \sqrt{\frac{R+z}{V^2R - (2gR - V^2)z}}\, dz.$$

Hence or otherwise show that

$$F(\lambda) = \frac{\lambda}{2-\lambda} \quad \text{and} \quad G(\lambda) = \int_0^1 \sqrt{\frac{2-\lambda+\lambda x}{(2-\lambda)^3(1-x)}}\, dx.$$

Paper 4, Section II

10C Dynamics and Relativity

A particle of mass m and charge q has position vector $\mathbf{r}(t)$ and moves in a constant, uniform magnetic field $\mathbf{B}$ so that its equation of motion is

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}.$$

Let $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$ be the particle's angular momentum. Show that

$$\mathbf{L} \cdot \mathbf{B} + \tfrac{1}{2}q|\mathbf{r} \times \mathbf{B}|^2$$

is a constant of the motion. Explain why the kinetic energy T is also constant, and show that it may be written in the form

$$T = \tfrac{1}{2}m\mathbf{u} \cdot \big((\mathbf{u} \cdot \mathbf{v})\mathbf{v} - r^2\ddot{\mathbf{u}}\big),$$

where $\mathbf{v} = \dot{\mathbf{r}}$, $r = |\mathbf{r}|$ and $\mathbf{u} = \mathbf{r}/r$.

[*Hint: Consider* $\mathbf{u} \cdot \dot{\mathbf{u}}$.]

Paper 4, Section II

11C Dynamics and Relativity

Consider a particle with position vector $\mathbf{r}(t)$ moving in a plane described by polar coordinates (r, θ). Obtain expressions for the radial (r) and transverse (θ) components of the velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$.

A charged particle of unit mass moves in the electric field of another charge that is fixed at the origin. The electrostatic force on the particle is $-p/r^2$ in the radial direction, where p is a positive constant. The motion takes place in an unusual medium that resists radial motion but not tangential motion, so there is an additional radial force $-k\dot{r}/r^2$ where k is a positive constant. Show that the particle's motion lies in a plane. Using polar coordinates in that plane, show also that its angular momentum $h = r^2\dot{\theta}$ is constant.

Obtain the equation of motion

$$\frac{d^2u}{d\theta^2} + \frac{k}{h}\frac{du}{d\theta} + u = \frac{p}{h^2},$$

where $u = r^{-1}$, and find its general solution assuming that $k/|h| < 2$. Show that so long as the motion remains bounded it eventually becomes circular with radius h^2/p.

Obtain the expression

$$E = \tfrac{1}{2}h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) - pu$$

for the particle's total energy, that is, its kinetic energy plus its electrostatic potential energy. Hence, or otherwise, show that the energy is a decreasing function of time.

Paper 4, Section II

12C Dynamics and Relativity

Write down the Lorentz transform relating the components of a 4-vector between two inertial frames.

A particle moves along the x-axis of an inertial frame. Its position at time t is $x(t)$, its velocity is $u = dx/dt$, and its 4-position is $X = (ct, x)$, where c is the speed of light. The particle's 4-velocity is given by $U = dX/d\tau$ and its 4-acceleration is $A = dU/d\tau$, where *proper time* τ is defined by $c^2 d\tau^2 = c^2 dt^2 - dx^2$. Show that

$$U = \gamma\,(c, u) \quad \text{and} \quad A = \gamma^4 \dot{u}\,(u/c, 1)$$

where $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$ and $\dot{u} = du/dt$.

The *proper 3-acceleration* a of the particle is defined to be the spatial component of its 4-acceleration measured in the particle's instantaneous rest frame. By transforming A to the rest frame, or otherwise, show that

$$a = \gamma^3 \dot{u} = \frac{d}{dt}(\gamma u).$$

Given that the particle moves with constant proper 3-acceleration starting from rest at the origin, show that

$$x(t) = \frac{c^2}{a}\left(\sqrt{1 + \frac{a^2 t^2}{c^2}} - 1\right),$$

and that, if $at \ll c$, then $x \approx \frac{1}{2} a t^2$.

Paper 3, Section I

1D Groups

Say that a group is *dihedral* if it has two generators x and y, such that x has order n (greater than or equal to 2 and possibly infinite), y has order 2, and $yxy^{-1} = x^{-1}$. In particular the groups C_2 and $C_2 \times C_2$ are regarded as dihedral groups. Prove that:

(i) any dihedral group can be generated by two elements of order 2;

(ii) any group generated by two elements of order 2 is dihedral; and

(iii) any non-trivial quotient group of a dihedral group is dihedral.

Paper 3, Section I

2D Groups

How many cyclic subgroups (including the trivial subgroup) does S_5 contain? Exhibit two isomorphic subgroups of S_5 which are not conjugate.

Paper 3, Section II

5D Groups

What does it mean for a group G to *act on* a set X? For $x \in X$, what is meant by the *orbit* $\mathrm{Orb}(x)$ to which x belongs, and by the *stabiliser* G_x of x? Show that G_x is a subgroup of G. Prove that, if G is finite, then $|G| = |G_x| \cdot |\mathrm{Orb}(x)|$.

(a) Prove that the symmetric group S_n acts on the set $P^{(n)}$ of all polynomials in n variables $x_1, \ldots, x_n$, if we define $\sigma \cdot f$ to be the polynomial given by

$$(\sigma \cdot f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

for $f \in P^{(n)}$ and $\sigma \in S_n$. Find the orbit of $f = x_1x_2 + x_3x_4 \in P^{(4)}$ under S_4. Find also the order of the stabiliser of f.

(b) Let r, n be fixed positive integers such that $r \leqslant n$. Let B_r be the set of all subsets of size r of the set $\{1, 2, \ldots, n\}$. Show that S_n acts on B_r by defining $\sigma \cdot U$ to be the set $\{\sigma(u) : u \in U\}$, for any $U \in B_r$ and $\sigma \in S_n$. Prove that S_n is transitive in its action on B_r. Find also the size of the stabiliser of $U \in B_r$.

Paper 3, Section II

6D Groups

Let G, H be groups and let $\varphi\colon G \to H$ be a function. What does it mean to say that φ is a *homomorphism* with *kernel* K? Show that if $K = \{e, \xi\}$ has order 2 then $x^{-1}\xi x = \xi$ for each $x \in G$. [If you use any general results about kernels of homomorphisms, then you should prove them.]

Which of the following four statements are true, and which are false? Justify your answers.

(a) There is a homomorphism from the orthogonal group $\mathrm{O}(3)$ to a group of order 2 with kernel the special orthogonal group $\mathrm{SO}(3)$.

(b) There is a homomorphism from the symmetry group S_3 of an equilateral triangle to a group of order 2 with kernel of order 3.

(c) There is a homomorphism from $\mathrm{O}(3)$ to $\mathrm{SO}(3)$ with kernel of order 2.

(d) There is a homomorphism from S_3 to a group of order 3 with kernel of order 2.

Paper 3, Section II

7D Groups

(a) State and prove Lagrange's theorem.

(b) Let G be a group and let H, K be fixed subgroups of G. For each $g \in G$, any set of the form $HgK = \{hgk : h \in H, k \in K\}$ is called an (H, K) *double coset*, or simply a double coset if H and K are understood. Prove that every element of G lies in some (H, K) double coset, and that any two (H, K) double cosets either coincide or are disjoint.

Let G be a finite group. Which of the following three statements are true, and which are false? Justify your answers.

(i) The size of a double coset divides the order of G.

(ii) Different double cosets for the same pair of subgroups have the same size.

(iii) The number of double cosets divides the order of G.

Paper 3, Section II

8D Groups

(a) Let G be a non-trivial group and let $Z(G) = \{h \in G : gh = hg \text{ for all } g \in G\}$. Show that $Z(G)$ is a normal subgroup of G. If the order of G is a power of a prime, show that $Z(G)$ is non-trivial.

(b) The *Heisenberg group* H is the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$. Show that H is a subgroup of the group of non-singular real matrices under matrix multiplication.

Find $Z(H)$ and show that $H/Z(H)$ is isomorphic to $\mathbb{R}^2$ under vector addition.

(c) For p prime, the *modular Heisenberg group* H_p is defined as in (b), except that x, y and z now lie in the field of p elements. Write down $|H_p|$. Find both $Z(H_p)$ and $H_p/Z(H_p)$ in terms of generators and relations.

Paper 4, Section I

1E Numbers and Sets

(a) Find all integers x and y such that

$$6x + 2y \equiv 3 \pmod{53} \qquad \text{and} \qquad 17x + 4y \equiv 7 \pmod{53}.$$

(b) Show that if an integer $n > 4$ is composite then $(n-1)! \equiv 0 \pmod{n}$.

Paper 4, Section I

2E Numbers and Sets

State the Chinese remainder theorem and Fermat's theorem. Prove that

$$p^4 \equiv 1 \pmod{240}$$

for any prime $p > 5$.

Paper 4, Section II

5E Numbers and Sets

(i) Let $\sim$ be an equivalence relation on a set X. What is an *equivalence class* of $\sim$? What is a *partition* of X? Prove that the equivalence classes of $\sim$ form a partition of X.

(ii) Let $\sim$ be the relation on the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ defined by

$$m \sim n \iff \exists\, a, b \in \mathbb{N} \text{ such that } m \text{ divides } n^a \text{ and } n \text{ divides } m^b.$$

Show that $\sim$ is an equivalence relation, and show that it has infinitely many equivalence classes, all but one of which are infinite.

Paper 4, Section II

6E Numbers and Sets

Let p be a prime. A *base p expansion* of an integer k is an expression

$$k = k_0 + p \cdot k_1 + p^2 \cdot k_2 + \cdots + p^\ell \cdot k_\ell$$

for some natural number ℓ, with $0 \leqslant k_i < p$ for $i = 0, 1, \ldots, \ell$.

(i) Show that the sequence of coefficients $k_0, k_1, k_2, \ldots, k_\ell$ appearing in a base p expansion of k is unique, up to extending the sequence by zeroes.

(ii) Show that

$$\binom{p}{j} \equiv 0 \pmod{p}, \quad 0 < j < p,$$

and hence, by considering the polynomial $(1+x)^p$ or otherwise, deduce that

$$\binom{p^i}{j} \equiv 0 \pmod{p}, \quad 0 < j < p^i.$$

(iii) If $n_0 + p \cdot n_1 + p^2 \cdot n_2 + \cdots + p^\ell \cdot n_\ell$ is a base p expansion of n, then, by considering the polynomial $(1+x)^n$ or otherwise, show that

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_\ell}{k_\ell} \pmod{p}.$$

Paper 4, Section II

7E Numbers and Sets

State the inclusion–exclusion principle.

Let $n \in \mathbb{N}$. A permutation σ of the set $\{1, 2, 3, \ldots, n\}$ is said to *contain a transposition* if there exist i, j with $1 \leqslant i < j \leqslant n$ such that $\sigma(i) = j$ and $\sigma(j) = i$. Derive a formula for the number, $f(n)$, of permutations which do not contain a transposition, and show that

$$\lim_{n\to\infty} \frac{f(n)}{n!} = e^{-\frac{1}{2}}.$$

Paper 4, Section II

8E Numbers and Sets

What does it mean for a set to be *countable*? Prove that

(a) if B is countable and $f\colon A \to B$ is injective, then A is countable;

(b) if A is countable and $f\colon A \to B$ is surjective, then B is countable.

Prove that $\mathbb{N} \times \mathbb{N}$ is countable, and deduce that

(i) if X and Y are countable, then so is $X \times Y$;

(ii) $\mathbb{Q}$ is countable.

Let $\mathcal{C}$ be a collection of circles in the plane such that for each point a on the x-axis, there is a circle in $\mathcal{C}$ passing through the point a which has the x-axis tangent to the circle at a. Show that $\mathcal{C}$ contains a pair of circles that intersect.

Paper 2, Section I
3F Probability

Let U be a uniform random variable on $(0,1)$, and let $\lambda > 0$.

(a) Find the distribution of the random variable $-(\log U)/\lambda$.

(b) Define a new random variable X as follows: suppose a fair coin is tossed, and if it lands heads we set $X = U^2$ whereas if it lands tails we set $X = 1 - U^2$. Find the probability density function of X.

Paper 2, Section I
4F Probability

Let A, B be events in the sample space Ω such that $0 < P(A) < 1$ and $0 < P(B) < 1$. The event B is said to *attract* A if the conditional probability $P(A|B)$ is greater than $P(A)$, otherwise it is said that A *repels* B. Show that if B attracts A, then A attracts B. Does $B^c = \Omega \setminus B$ repel A?

Paper 2, Section II
9F Probability

Lionel and Cristiana have a and b million pounds, respectively, where $a, b \in \mathbb{N}$. They play a series of independent football games in each of which the winner receives one million pounds from the loser (a draw cannot occur). They stop when one player has lost his or her entire fortune. Lionel wins each game with probability $0 < p < 1$ and Cristiana wins with probability $q = 1 - p$, where $p \neq q$. Find the expected number of games before they stop playing.

Paper 2, Section II

10F Probability

Consider the function

$$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2},\quad x\in\mathbb{R}.$$

Show that ϕ defines a probability density function. If a random variable X has probability density function ϕ, find the moment generating function of X, and find all moments $E[X^k]$, $k\in\mathbb{N}$.

Now define

$$r(x)=\frac{P(X>x)}{\phi(x)}.$$

Show that for every $x>0$,

$$\frac{1}{x}-\frac{1}{x^3}<r(x)<\frac{1}{x}.$$

Paper 2, Section II

11F Probability

State and prove Markov's inequality and Chebyshev's inequality, and deduce the weak law of large numbers.

If X is a random variable with mean zero and finite variance σ^2, prove that for any $a>0$,

$$P(X\geqslant a)\leqslant\frac{\sigma^2}{\sigma^2+a^2}.$$

[*Hint: Show first that* $P(X\geqslant a)\leqslant P\left((X+b)^2\geqslant(a+b)^2\right)$ *for every* $b>0$.]

Paper 2, Section II

12F Probability

When coin A is tossed it comes up heads with probability $\frac{1}{4}$, whereas coin B comes up heads with probability $\frac{3}{4}$. Suppose one of these coins is randomly chosen and is tossed twice. If both tosses come up heads, what is the probability that coin B was tossed? Justify your answer.

In each draw of a lottery, an integer is picked independently at random from the first n integers $1,2,\ldots,n$, with replacement. What is the probability that in a sample of r successive draws the numbers are drawn in a non-decreasing sequence? Justify your answer.

Paper 3, Section I

3A Vector Calculus

(i) For $r = |\mathbf{x}|$ with $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, show that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad (i = 1, 2, 3).$$

(ii) Consider the vector fields $\mathbf{F}(\mathbf{x}) = r^2\mathbf{x}$, $\mathbf{G}(\mathbf{x}) = (\mathbf{a}\cdot\mathbf{x})\mathbf{x}$ and $\mathbf{H}(\mathbf{x}) = \mathbf{a}\times\hat{\mathbf{x}}$, where $\mathbf{a}$ is a constant vector in $\mathbb{R}^3$ and $\hat{\mathbf{x}}$ is the unit vector in the direction of $\mathbf{x}$. Using suffix notation, or otherwise, find the divergence and the curl of each of $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$.

Paper 3, Section I

4A Vector Calculus

The smooth curve $\mathcal{C}$ in $\mathbb{R}^3$ is given in parametrised form by the function $\mathbf{x}(u)$. Let s denote arc length measured along the curve.

(a) Express the tangent $\mathbf{t}$ in terms of the derivative $\mathbf{x}' = d\mathbf{x}/du$, and show that $du/ds = |\mathbf{x}'|^{-1}$.

(b) Find an expression for $d\mathbf{t}/ds$ in terms of derivatives of $\mathbf{x}$ with respect to u, and show that the curvature κ is given by

$$\kappa = \frac{|\mathbf{x}' \times \mathbf{x}''|}{|\mathbf{x}'|^3}.$$

[*Hint: You may find the identity* $(\mathbf{x}'\cdot\mathbf{x}'')\mathbf{x}' - (\mathbf{x}'\cdot\mathbf{x}')\mathbf{x}'' = \mathbf{x}' \times (\mathbf{x}' \times \mathbf{x}'')$ *helpful.*]

(c) For the curve

$$\mathbf{x}(u) = \begin{pmatrix} u\cos u \\ u\sin u \\ 0 \end{pmatrix},$$

with $u \geqslant 0$, find the curvature as a function of u.

Paper 3, Section II

9A Vector Calculus

The vector field $\mathbf{F}(\mathbf{x})$ is given in terms of cylindrical polar coordinates (ρ, ϕ, z) by

$$\mathbf{F}(\mathbf{x}) = f(\rho)\mathbf{e}_\rho,$$

where f is a differentiable function of ρ, and $\mathbf{e}_\rho = \cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y$ is the unit basis vector with respect to the coordinate ρ. Compute the partial derivatives $\partial F_1/\partial x$, $\partial F_2/\partial y$, $\partial F_3/\partial z$ and hence find the divergence $\nabla \cdot \mathbf{F}$ in terms of ρ and ϕ.

The domain V is bounded by the surface $z = (x^2+y^2)^{-1}$, by the cylinder $x^2+y^2 = 1$, and by the planes $z = \frac{1}{4}$ and $z = 1$. Sketch V and compute its volume.

Find the most general function $f(\rho)$ such that $\nabla \cdot \mathbf{F} = 0$, and verify the divergence theorem for the corresponding vector field $\mathbf{F}(\mathbf{x})$ in V.

Paper 3, Section II

10A Vector Calculus

State Stokes' theorem.

Let S be the surface in $\mathbb{R}^3$ given by $z^2 = x^2 + y^2 + 1 - \lambda$, where $0 \leqslant z \leqslant 1$ and λ is a positive constant. Sketch the surface S for representative values of λ and find the surface element $\mathbf{dS}$ with respect to the Cartesian coordinates x and y.

Compute $\nabla \times \mathbf{F}$ for the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$$

and verify Stokes' theorem for $\mathbf{F}$ on the surface S for every value of λ.

Now compute $\nabla \times \mathbf{G}$ for the vector field

$$\mathbf{G}(\mathbf{x}) = \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

and find the line integral $\int_{\partial S} \mathbf{G} \cdot \mathbf{dx}$ for the boundary ∂S of the surface S. Is it possible to obtain this result using Stokes' theorem? Justify your answer.

Paper 3, Section II

11A Vector Calculus

(i) Starting with the divergence theorem, derive Green's first theorem

$$\int_V (\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi)\, dV = \int_{\partial V} \psi \frac{\partial \phi}{\partial n}\, dS.$$

(ii) The function $\phi(\mathbf{x})$ satisfies Laplace's equation $\nabla^2 \phi = 0$ in the volume V with given boundary conditions $\phi(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in \partial V$. Show that $\phi(\mathbf{x})$ is the only such function. Deduce that if $\phi(\mathbf{x})$ is constant on ∂V then it is constant in the whole volume V.

(iii) Suppose that $\phi(\mathbf{x})$ satisfies Laplace's equation in the volume V. Let V_r be the sphere of radius r centred at the origin and contained in V. The function $f(r)$ is defined by

$$f(r) = \frac{1}{4\pi r^2} \int_{\partial V_r} \phi(\mathbf{x})\, dS.$$

By considering the derivative df/dr, and by introducing the Jacobian in spherical polar coordinates and using the divergence theorem, or otherwise, show that $f(r)$ is constant and that $f(r) = \phi(\mathbf{0})$.

(iv) Let M denote the maximum of ϕ on ∂V_r and m the minimum of ϕ on ∂V_r. By using the result from (iii), or otherwise, show that $m \leqslant \phi(\mathbf{0}) \leqslant M$.

Paper 3, Section II

12A Vector Calculus

(a) Let t_{ij} be a rank 2 tensor whose components are invariant under rotations through an angle π about each of the three coordinate axes. Show that t_{ij} is diagonal.

(b) An array of numbers a_{ij} is given in one orthonormal basis as $\delta_{ij} + \epsilon_{1ij}$ and in another rotated basis as δ_{ij}. By using the invariance of the determinant of any rank 2 tensor, or otherwise, prove that a_{ij} is not a tensor.

(c) Let a_{ij} be an array of numbers and b_{ij} a tensor. Determine whether the following statements are true or false. Justify your answers.

(i) If $a_{ij} b_{ij}$ is a scalar for any rank 2 tensor b_{ij}, then a_{ij} is a rank 2 tensor.

(ii) If $a_{ij} b_{ij}$ is a scalar for any symmetric rank 2 tensor b_{ij}, then a_{ij} is a rank 2 tensor.

(iii) If a_{ij} is antisymmetric and $a_{ij} b_{ij}$ is a scalar for any symmetric rank 2 tensor b_{ij}, then a_{ij} is an antisymmetric rank 2 tensor.

(iv) If a_{ij} is antisymmetric and $a_{ij} b_{ij}$ is a scalar for any antisymmetric rank 2 tensor b_{ij}, then a_{ij} is an antisymmetric rank 2 tensor.

Paper 1, Section I

1B Vectors and Matrices

(a) Describe geometrically the curve

$$|\alpha z + \beta \bar{z}| = \sqrt{\alpha\beta}\,(z + \bar{z}) + (\alpha - \beta)^2,$$

where $z \in \mathbb{C}$ and α, β are positive, distinct, real constants.

(b) Let θ be a real number not equal to an integer multiple of 2π. Show that

$$\sum_{m=1}^{N} \sin(m\theta) = \frac{\sin\theta + \sin(N\theta) - \sin(N\theta + \theta)}{2(1 - \cos\theta)},$$

and derive a similar expression for $\displaystyle\sum_{m=1}^{N} \cos(m\theta)$.

Paper 1, Section I

2C Vectors and Matrices

Precisely one of the four matrices specified below is not orthogonal. Which is it? Give a brief justification.

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -\sqrt{3} & \sqrt{2} \\ 1 & \sqrt{3} & \sqrt{2} \\ -2 & 0 & \sqrt{2} \end{pmatrix} \quad \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix} \quad \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -2 & 1 \\ -\sqrt{6} & 0 & \sqrt{6} \\ 1 & 1 & 1 \end{pmatrix} \quad \frac{1}{9}\begin{pmatrix} 7 & -4 & -4 \\ -4 & 1 & -8 \\ -4 & -8 & 1 \end{pmatrix}$$

Given that the four matrices represent transformations of $\mathbb{R}^3$ corresponding (in no particular order) to a rotation, a reflection, a combination of a rotation and a reflection, and none of these, identify each matrix. Explain your reasoning.

[*Hint: For* **two** *of the matrices, A and B say, you may find it helpful to calculate* $\det(A - I)$ *and* $\det(B - I)$*, where I is the identity matrix.*]

Paper 1, Section II

5B Vectors and Matrices

(i) State and prove the Cauchy–Schwarz inequality for vectors in $\mathbb{R}^n$. Deduce the inequalities

$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}|+|\mathbf{b}| \quad \text{and} \quad |\mathbf{a}+\mathbf{b}+\mathbf{c}| \leqslant |\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|$$

for $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}^n$.

(ii) Show that every point on the intersection of the planes

$$\mathbf{x}\cdot\mathbf{a} = A, \quad \mathbf{x}\cdot\mathbf{b} = B,$$

where $\mathbf{a} \neq \mathbf{b}$, satisfies

$$|\mathbf{x}|^2 \geqslant \frac{(A-B)^2}{|\mathbf{a}-\mathbf{b}|^2}.$$

What happens if $\mathbf{a}=\mathbf{b}$?

(iii) Using your results from part (i), or otherwise, show that for any $\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2 \in \mathbb{R}^n$,

$$|\mathbf{x}_1-\mathbf{y}_1| - |\mathbf{x}_1-\mathbf{y}_2| \leqslant |\mathbf{x}_2-\mathbf{y}_1| + |\mathbf{x}_2-\mathbf{y}_2|.$$

Paper 1, Section II

6C Vectors and Matrices

(i) Consider the map from $\mathbb{R}^4$ to $\mathbb{R}^3$ represented by the matrix

$$\begin{pmatrix} \alpha & 1 & 1 & -1 \\ 2 & -\alpha & 0 & -2 \\ -\alpha & 2 & 1 & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$. Find the image and kernel of the map for each value of α.

(ii) Show that any linear map $f\colon \mathbb{R}^n \to \mathbb{R}$ may be written in the form $f(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x}$ for some fixed vector $\mathbf{a} \in \mathbb{R}^n$. Show further that $\mathbf{a}$ is uniquely determined by f.

It is given that $n=4$ and that the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

lie in the kernel of f. Determine the set of possible values of $\mathbf{a}$.

Paper 1, Section II
7A Vectors and Matrices

(i) Find the eigenvalues and eigenvectors of the following matrices and show that both are diagonalisable:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 4 & -3 \\ -4 & 10 & -4 \\ -3 & 4 & 1 \end{pmatrix}.$$

(ii) Show that, if two real $n \times n$ matrices can both be diagonalised using the same basis transformation, then they commute.

(iii) Suppose now that two real $n \times n$ matrices C and D commute and that D has n distinct eigenvalues. Show that for any eigenvector $\mathbf{x}$ of D the vector $C\mathbf{x}$ is a scalar multiple of $\mathbf{x}$. Deduce that there exists a common basis transformation that diagonalises both matrices.

(iv) Show that A and B satisfy the conditions in (iii) and find a matrix S such that both of the matrices $S^{-1}AS$ and $S^{-1}BS$ are diagonal.

Paper 1, Section II
8A Vectors and Matrices

(a) A matrix is called *normal* if $A^\dagger A = AA^\dagger$. Let A be a normal $n \times n$ complex matrix.

(i) Show that for any vector $\mathbf{x} \in \mathbb{C}^n$,

$$|A\mathbf{x}| = |A^\dagger \mathbf{x}|.$$

(ii) Show that $A - \lambda I$ is also normal for any $\lambda \in \mathbb{C}$, where I denotes the identity matrix.

(iii) Show that if $\mathbf{x}$ is an eigenvector of A with respect to the eigenvalue $\lambda \in \mathbb{C}$, then $\mathbf{x}$ is also an eigenvector of $A^\dagger$, and determine the corresponding eigenvalue.

(iv) Show that if $\mathbf{x}_\lambda$ and $\mathbf{x}_\mu$ are eigenvectors of A with respect to distinct eigenvalues λ and μ respectively, then $\mathbf{x}_\lambda$ and $\mathbf{x}_\mu$ are orthogonal.

(v) Show that if A has a basis of eigenvectors, then A can be diagonalised using an orthonormal basis. Justify your answer.

[You may use standard results provided that they are clearly stated.]

(b) Show that any matrix A satisfying $A^\dagger = A$ is normal, and deduce using results from (a) that its eigenvalues are real.

(c) Show that any matrix A satisfying $A^\dagger = -A$ is normal, and deduce using results from (a) that its eigenvalues are purely imaginary.

(d) Show that any matrix A satisfying $A^\dagger = A^{-1}$ is normal, and deduce using results from (a) that its eigenvalues have unit modulus.