UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Wednesday, 4 June, 2014 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *. . .,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Number Theory

Show that the continued fraction for $\sqrt{51}$ is $[7; \overline{7, 14}]$.

Hence, or otherwise, find positive integers x and y that satisfy the equation $x^2 - 51y^2 = 1$.

Are there integers x and y such that $x^2 - 51y^2 = -1$?

2G Topics in Analysis

State Runge's theorem about uniform approximation of holomorphic functions by polynomials.

Let $\mathbb{R}_+ \subset \mathbb{C}$ be the subset of non-negative real numbers and let

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}.$$

Prove that there is a sequence of complex polynomials which converges to the function $1/z$ uniformly on each compact subset of $\Delta \setminus \mathbb{R}_+$.

3F Geometry and Groups

Let $\mathbb{H}^2$ denote the hyperbolic plane, and $T \subset \mathbb{H}^2$ be a non-degenerate triangle, i.e. the bounded region enclosed by three finite-length geodesic arcs. Prove that the three angle bisectors of T meet at a point.

Must the three vertices of T lie on a hyperbolic circle? Justify your answer.

4I Coding and Cryptography

Let A be a random variable that takes values in the finite alphabet $\mathcal{A}$. Prove that there is a decodable binary code $c : \mathcal{A} \to \{0,1\}^*$ that satisfies

$$H(A) \leqslant \mathbb{E}(l(A)) \leqslant H(A) + 1 ,$$

where $l(a)$ is the length of the code word $c(a)$ and $H(A)$ is the entropy of A.

Is it always possible to find such a code with $\mathbb{E}(l(A)) = H(A)$? Justify your answer.

5K Statistical Modelling

In an experiment to study factors affecting the production of the plastic polyvinyl chloride (PVC), three experimenters each used eight devices to produce the PVC and measured the sizes of the particles produced. For each of the 24 combinations of device and experimenter, two size measurements were obtained.

The experimenters and devices used for each of the 48 measurements are stored in `R` as factors in the objects `experimenter` and `device` respectively, with the measurements themselves stored in the vector `psize`. The following analysis was performed in `R`.

```
> fit0 <- lm(psize ~ experimenter + device)
> fit <- lm(psize ~ experimenter + device + experimenter:device)
> anova(fit0, fit)
Analysis of Variance Table

Model 1: psize ~ experimenter + device
Model 2: psize ~ experimenter + device + experimenter:device
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     38 49.815
2     24 35.480 14    14.335 0.6926 0.7599
```

Let X and X_0 denote the design matrices obtained by `model.matrix(fit)` and `model.matrix(fit0)` respectively, and let Y denote the response `psize`. Let P and P_0 denote orthogonal projections onto the column spaces of X and X_0 respectively.

For each of the following quantities, write down their numerical values if they appear in the analysis of variance table above; otherwise write 'unknown'.

1. $\|(I-P)Y\|^2$
2. $\|X(X^TX)^{-1}X^TY\|^2$
3. $\|(I-P_0)Y\|^2 - \|(I-P)Y\|^2$
4. $\dfrac{\|(P-P_0)Y\|^2/14}{\|(I-P)Y\|^2/24}$
5. $\sum_{i=1}^{48} Y_i/48$

Out of the two models that have been fitted, which appears to be the more appropriate for the data according to the analysis performed, and why?

6B Mathematical Biology

An epidemic model is given by

$$\frac{dS}{dt} = -rIS\,,$$
$$\frac{dI}{dt} = +rIS - aI\,,$$

where $S(t)$ are the susceptibles, $I(t)$ are the infecteds, and a and r are positive parameters. The basic reproduction ratio is defined as $R_0 = rN/a$, where N is the total population size. Find a condition on R_0 for an epidemic to be possible if, initially, $S \approx N$ and I is small but non-zero.

Now suppose a proportion p of the population was vaccinated (with a completely effective vaccine) so that initially $S \approx (1-p)N$. On a sketch of the (R_0, p) plane, mark the regions where an epidemic is still possible, where the vaccination will prevent an epidemic, and where no vaccination was necessary.

For the case when an epidemic is possible, show that σ, the proportion of the initially susceptible population that has not been infected by the end of an epidemic, satisfies

$$\sigma - \frac{1}{(1-p)R_0}\log\sigma \approx 1.$$

7D Dynamical Systems

Define the *Poincaré index* of a closed curve $\mathcal{C}$ for a vector field $\mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$.

Explain carefully why the index of $\mathcal{C}$ is fully determined by the fixed points of the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ that lie within $\mathcal{C}$.

What is the Poincaré index for a closed curve $\mathcal{C}$ if it (a) encloses only a saddle point, (b) encloses only a focus and (c) encloses only a node?

What is the Poincaré index for a closed curve $\mathcal{C}$ that is a periodic trajectory of the dynamical system?

A dynamical system in $\mathbb{R}^2$ has 2 saddle points, 1 focus and 1 node. What is the maximum number of different periodic orbits? [For the purposes of this question, two orbits are said to be different if they enclose different sets of fixed points.]

8B Further Complex Methods

State the conditions for a point $z = z_0$ to be a *regular singular point* of a linear second-order homogeneous ordinary differential equation in the complex plane.

Find all singular points of the Airy equation

$$w''(z) - zw(z) = 0\,,$$

and determine whether they are regular or irregular.

9A Classical Dynamics

(a) The action for a one-dimensional dynamical system with a generalized coordinate q and Lagrangian L is given by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt\,.$$

State the principle of least action. Write the expression for the Hamiltonian in terms of the generalized velocity $\dot{q}$, the generalized momentum p and the Lagrangian L. Use it to derive Hamilton's equations from the principle of least action.

(b) The motion of a particle of charge q and mass m in an electromagnetic field with scalar potential $\phi(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$ is characterized by the Lagrangian

$$L = \frac{m\dot{\mathbf{r}}^2}{2} - q(\phi - \dot{\mathbf{r}} \cdot \mathbf{A})\,.$$

(i) Write down the Hamiltonian of the particle.

(ii) Consider a particle which moves in three dimensions in a magnetic field with $\mathbf{A} = (0, Bx, 0)$, where B is a constant. There is no electric field. Obtain Hamilton's equations for the particle.

10E Cosmology

Consider a finite sphere of zero-pressure material of uniform density $\rho(t)$ which expands with radius $r(t) = a(t)r_0$, where r_0 is an arbitary constant, due to the evolution of the expansion scale factor $a(t)$. The sphere has constant total mass M and its radius satisfies

$$\ddot{r} = -\frac{d\Phi}{dr}\,,$$

where

$$\Phi(r) = -\frac{GM}{r} - \frac{1}{6}\Lambda r^2 c^2,$$

with Λ constant. Show that the scale factor obeys the equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{Kc^2}{a^2} + \frac{1}{3}\Lambda c^2,$$

where K is a constant. Explain why the sign, but not the magnitude, of K is important. Find exact solutions of this equation for $a(t)$ when

(i) $K = \Lambda = 0$ and $\rho(t) \neq 0$,

(ii) $\rho = K = 0$ and $\Lambda > 0$,

(iii) $\rho = \Lambda = 0$ and $K \neq 0$.

Which two of the solutions (i)–(iii) are relevant for describing the evolution of the universe after the radiation-dominated era?

SECTION II

11F Number Theory

State and prove Lagrange's theorem about polynomial congruences modulo a prime.

Define the *Euler totient function* ϕ.

Let p be a prime and let d be a positive divisor of $p-1$. Show that there are exactly $\phi(d)$ elements of $(\mathbb{Z}/p\mathbb{Z})^{\times}$ with order d.

Deduce that $(\mathbb{Z}/p\mathbb{Z})^{\times}$ is cyclic.

Let g be a primitive root modulo p^2. Show that g must be a primitive root modulo p.

Let g be a primitive root modulo p. Must it be a primitive root modulo p^2? Give a proof or a counterexample.

12G Topics in Analysis

Define what is meant by a *nowhere dense* set in a metric space. State a version of the Baire Category Theorem. Show that any complete non-empty metric space without isolated points is uncountable.

Let A be the set of real numbers whose decimal expansion does *not* use the digit 6. (A terminating decimal representation is used when it exists.) Show that there exists a real number which cannot be written as $a+q$ with $a \in A$ and $q \in \mathbb{Q}$.

13B Mathematical Biology

A discrete-time model for breathing is given by

$$\begin{aligned} V_{n+1} &= \alpha C_{n-k}\,, && (1)\\ C_{n+1}-C_n &= \gamma - \beta V_{n+1}\,, && (2) \end{aligned}$$

where V_n is the volume of each breath in time step n and C_n is the concentration of carbon dioxide in the blood at the end of time step n. The parameters α, β and γ are all positive. Briefly explain the biological meaning of each of the above equations.

Find the steady state. For $k=0$ and $k=1$ determine the stability of the steady state.

For general (integer) $k>1$, by seeking parameter values when the modulus of a perturbation to the steady state is constant, find the range of parameters where the solution is stable. What is the periodicity of the constant-modulus solution at the edge of this range? Comment on how the size of the range depends on k.

This can be developed into a more realistic model by changing the term $-\beta V_{n+1}$ to $-\beta C_n V_{n+1}$ in (2). Briefly explain the biological meaning of this change. Show that for both $k=0$ and $k=1$ the new steady state is stable if $0<a<1$, where $a=\sqrt{\alpha\beta\gamma}$.

14D Dynamical Systems

Let $f : I \to I$ be a continuous one-dimensional map of an interval $I \subset \mathbb{R}$. Explain what is meant by saying that f has a *horseshoe*.

A map g on the interval $[a, b]$ is a *tent map* if

(i) $g(a) = a$ and $g(b) = a$;

(ii) for some c with $a < c < b$, g is linear and increasing on the interval $[a, c]$, linear and decreasing on the interval $[c, b]$, and continuous at c.

Consider the tent map defined on the interval $[0, 1]$ by

$$f(x) = \begin{cases} \mu x & 0 \leqslant x \leqslant \frac{1}{2} \\ \mu(1-x) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

with $1 < \mu \leqslant 2$. Find the corresponding expressions for $f^2(x) = f(f(x))$.

Find the non-zero fixed point x_0 and the points $x_{-1} < \frac{1}{2} < x_{-2}$ that satisfy

$$f^2(x_{-2}) = f(x_{-1}) = x_0 = f(x_0)\,.$$

Sketch graphs of f and f^2 showing the points corresponding to x_{-2}, x_{-1} and x_0. Indicate the values of f and f^2 at their maxima and minima and also the gradients of each piece of their graphs.

Identify a subinterval of $[0, 1]$ on which f^2 is a tent map. Hence demonstrate that f^2 has a horseshoe if $\mu \geqslant 2^{1/2}$.

Explain briefly why f^4 has a horseshoe when $\mu \geqslant 2^{1/4}$.

Why are there periodic points of f arbitrarily close to x_0 for $\mu \geqslant 2^{1/2}$, but no such points for $2^{1/4} \leqslant \mu < 2^{1/2}$? Explain carefully any results or terms that you use.

15E Cosmology

The luminosity distance to an astronomical light source is given by $d_L = \chi/a(t)$, where $a(t)$ is the expansion scale factor and χ is the comoving distance in the universe defined by $dt = a(t)d\chi$. A zero-curvature Friedmann universe containing pressure-free matter and a cosmological constant with density parameters Ω_m and $\Omega_\Lambda \equiv 1 - \Omega_m$, respectively, obeys the Friedmann equation

$$H^2 = H_0^2 \left(\frac{\Omega_{m0}}{a^3} + \Omega_{\Lambda 0} \right),$$

where $H = (da/dt)/a$ is the Hubble expansion rate of the universe and the subscript $_0$ denotes present-day values, with $a_0 \equiv 1$.

If z is the redshift, show that

$$d_L(z) = \frac{1+z}{H_0} \int_0^z \frac{dz'}{\left[(1-\Omega_{\Lambda 0})(1+z')^3 + \Omega_{\Lambda 0}\right]^{1/2}}.$$

Find $d_L(z)$ when $\Omega_{\Lambda 0} = 0$ and when $\Omega_{m0} = 0$. Roughly sketch the form of $d_L(z)$ for these two cases. What is the effect of a cosmological constant Λ on the luminosity distance at a fixed value of z? Briefly describe how the relation between luminosity distance and redshift has been used to establish the acceleration of the expansion of the universe.

16I Logic and Set Theory

Explain what is meant by a *structure* for a first-order signature Σ, and describe briefly how first-order terms and formulae in the language over Σ are interpreted in a structure. Suppose that A and B are Σ-structures, and that ϕ is a conjunction of atomic formulae over Σ: show that an n-tuple $((a_1, b_1), \ldots, (a_n, b_n))$ belongs to the interpretation $[\![\phi]\!]_{A\times B}$ of ϕ in $A \times B$ if and only if $(a_1, \ldots, a_n) \in [\![\phi]\!]_A$ and $(b_1, \ldots, b_n) \in [\![\phi]\!]_B$.

A first-order theory $\mathbb{T}$ is called *regular* if its axioms all have the form

$$(\forall \vec{x})(\phi \Rightarrow (\exists \vec{y})\psi),$$

where $\vec{x}$ and $\vec{y}$ are (possibly empty) strings of variables and ϕ and ψ are conjunctions of atomic formulae (possibly the empty conjunction $\top$). Show that if A and B are models of a regular theory $\mathbb{T}$, then so is $A \times B$.

Now suppose that $\mathbb{T}$ is a regular theory, and that a sentence of the form

$$(\forall \vec{x})(\phi \Rightarrow (\psi_1 \vee \psi_2 \vee \cdots \vee \psi_n))$$

is derivable from the axioms of $\mathbb{T}$, where ϕ and the ψ_i are conjunctions of atomic formulae. Show that the sentence $(\forall \vec{x})(\phi \Rightarrow \psi_i)$ is derivable for some i. [*Hint: Suppose not, and use the Completeness Theorem to obtain a suitable family of* $\mathbb{T}$*-models* $A_1, \ldots, A_n$.]

17I Graph Theory

Prove that $\chi(G) \leqslant \Delta(G) + 1$ for every graph G. Prove further that, if $\kappa(G) \geqslant 3$, then $\chi(G) \leqslant \Delta(G)$ unless G is complete.

Let $k \geqslant 2$. A graph G is said to be k-*critical* if $\chi(G) = k + 1$, but $\chi(G - v) = k$ for every vertex v of G. Show that, if G is k-critical, then $\kappa(G) \geqslant 2$.

Let $k \geqslant 2$, and let H be the graph K_{k+1} with an edge removed. Show that H has the following property: it has two vertices which receive the same colour in every k-colouring of H. By considering two copies of H, construct a k-colourable graph G of order $2k + 1$ with the following property: it has three vertices which receive the same colour in every k-colouring of G.

Construct, for all integers $k \geqslant 2$ and $\ell \geqslant 2$, a k-critical graph G of order $\ell k + 1$ with $\kappa(G) = 2$.

18H Galois Theory

Let L/K be an algebraic extension of fields, and $x \in L$. What does it mean to say that x is separable over K? What does it mean to say that L/K is separable?

Let $K = \mathbb{F}_p(t)$ be the field of rational functions over $\mathbb{F}_p$.

(i) Show that if x is inseparable over K then $K(x)$ contains a pth root of t.

(ii) Show that if L/K is finite there exists $n \geqslant 0$ and $y \in L$ such that $y^{p^n} = t$ and $L/K(y)$ is separable.

Show that $Y^2 + tY + t$ is an irreducible separable polynomial over the field of rational functions $K = \mathbb{F}_2(t)$. Find the degree of the splitting field of $X^4 + tX^2 + t$ over K.

19H Representation Theory

(i) State Frobenius' theorem for transitive permutation groups acting on a finite set. Define *Frobenius group* and show that any finite Frobenius group (with an appropriate action) satisfies the hypotheses of Frobenius' theorem.

(ii) Consider the group

$$F_{p,q} := \langle a, b : a^p = b^q = 1,\ b^{-1}ab = a^u \rangle,$$

where p is prime, q divides $p-1$ (q not necessarily prime), and u has multiplicative order q modulo p (such elements u exist since q divides $p-1$). Let S be the subgroup of $\mathbb{Z}_p^\times$ consisting of the powers of u, so that $|S| = q$. Write $r = (p-1)/q$, and let $v_1, \ldots, v_r$ be coset representatives for S in $\mathbb{Z}_p^\times$.

(a) Show that $F_{p,q}$ has $q+r$ conjugacy classes and that a complete list of the classes comprises $\{1\}$, $\{a^{v_j s} : s \in S\}$ $(1 \leqslant j \leqslant r)$ and $\{a^m b^n : 0 \leqslant m \leqslant p-1\}$ $(1 \leqslant n \leqslant q-1)$.

(b) By observing that the derived subgroup $F'_{p,q} = \langle a \rangle$, find q 1-dimensional characters of $F_{p,q}$. [Appropriate results may be quoted without proof.]

(c) Let $\varepsilon = e^{2\pi i/p}$. For $v \in \mathbb{Z}_p^\times$ denote by ψ_v the character of $\langle a \rangle$ defined by $\psi_v(a^x) = \varepsilon^{vx}$ $(0 \leqslant x \leqslant p-1)$. By inducing these characters to $F_{p,q}$, or otherwise, find r distinct irreducible characters of degree q.

20F Algebraic Topology

Let K be a simplicial complex in $\mathbb{R}^N$, which we may also consider as lying in $\mathbb{R}^{N+1}$ using the first N coordinates. Write $c = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{N+1}$. Show that if $\langle v_0, v_1, \ldots, v_n \rangle$ is a simplex of K then $\langle v_0, v_1, \ldots, v_n, c \rangle$ is a simplex in $\mathbb{R}^{N+1}$.

Let $L \leqslant K$ be a subcomplex and let $\overline{K}$ be the collection

$$K \cup \{\langle v_0, v_1, \ldots, v_n, c \rangle \mid \langle v_0, v_1, \ldots, v_n \rangle \in L\} \cup \{\langle c \rangle\}$$

of simplices in $\mathbb{R}^{N+1}$. Show that $\overline{K}$ is a simplicial complex.

If $|K|$ is a Möbius band, and $|L|$ is its boundary, show that

$$H_i(\overline{K}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/2 & \text{if } i = 1 \\ 0 & \text{if } i \geqslant 2. \end{cases}$$

21G Linear Analysis

(i) State carefully the theorems of Stone–Weierstrass and Arzelá–Ascoli (work with real scalars only).

(ii) Let $\mathcal{F}$ denote the family of functions on $[0,1]$ of the form

$$f(x)=\sum_{n=1}^{\infty} a_n \sin(nx),$$

where the a_n are real and $|a_n| \leqslant 1/n^3$ for all $n \in \mathbb{N}$. Prove that any sequence in $\mathcal{F}$ has a subsequence that converges uniformly on $[0,1]$.

(iii) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $f(0)=0$ and $f'(0)$ exists. Show that for each $\varepsilon > 0$ there exists a real polynomial p having only odd powers, i.e. of the form

$$p(x) = a_1 x + a_3 x^3 + \cdots + a_{2m-1} x^{2m-1} \ ,$$

such that $\sup_{x\in[0,1]} |f(x) - p(x)| < \varepsilon$. Show that the same holds without the assumption that f is differentiable at 0.

22H Riemann Surfaces

State the Uniformization Theorem.

Show that any domain of $\mathbb{C}$ whose complement has more than one point is uniformized by the unit disc Δ. [You may use the fact that for $\mathbb{C}_\infty$ the group of automorphisms consists of Möbius transformations, and for $\mathbb{C}$ it consists of maps of the form $z \mapsto az+b$ with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$.]

Let X be the torus $\mathbb{C}/\Lambda$, where Λ is a lattice. Given $p \in X$, show that $X \setminus \{p\}$ is uniformized by the unit disc Δ.

Is it true that a holomorphic map from $\mathbb{C}$ to a compact Riemann surface of genus two must be constant? Justify your answer.

23H Algebraic Geometry

Let $f \in k[x]$ be a polynomial with distinct roots, $\deg f = d > 2$, $\operatorname{char} k = 0$, and let $C \subseteq \mathbf{P}^2$ be the projective closure of the affine curve

$$y^{d-1} = f(x).$$

Show that C is smooth, with a single point at ∞.

Pick an appropriate $\omega \in \Omega^1_{k(C)/k}$ and compute the valuation $v_q(\omega)$ for all $q \in C$.

Hence determine $\deg \mathcal{K}_C$.

24G Differential Geometry

Let $\alpha : I \to S$ be a parametrized curve on a smooth embedded surface $S \subset \mathbf{R}^3$. Define what is meant by a *vector field* V along α and the concept of such a vector field being *parallel*. If V and W are both parallel vector fields along α, show that the inner product $\langle V(t), W(t)\rangle$ is constant.

Given a local parametrization $\phi : U \to S$, define the *Christoffel symbols* Γ^i_{jk} on U. Given a vector $v_0 \in T_{\alpha(0)}S$, prove that there exists a unique parallel vector field $V(t)$ along α with $V(0) = v_0$ (recall that $V(t)$ is called the *parallel transport* of v_0 along α).

Suppose now that the image of α also lies on another smooth embedded surface $S' \subset \mathbf{R}^3$ and that $T_{\alpha(t)}S = T_{\alpha(t)}S'$ for all $t \in I$. Show that parallel transport of a vector v_0 is the same whether calculated on S or S'. Suppose S is the unit sphere in $\mathbf{R}^3$ with centre at the origin and let $\alpha : [0, 2\pi] \to S$ be the curve on S given by

$$\alpha(t) = (\sin\phi\cos t,\ \sin\phi\sin t,\ \cos\phi)$$

for some fixed angle ϕ. Suppose $v_0 \in T_PS$ is the unit tangent vector to α at $P = \alpha(0) = \alpha(2\pi)$ and let v_1 be its image in T_PS under parallel transport along α. Show that the angle between v_0 and v_1 is $2\pi\cos\phi$.
[*Hint: You may find it useful to consider the circular cone S' which touches the sphere S along the curve α.*]

25K Probability and Measure

(i) Let $(E, \mathcal{E}, \mu)$ be a measure space. What does it mean to say that a function $\theta : E \to E$ is a *measure-preserving transformation*?

What does it mean to say that θ is *ergodic*?

State Birkhoff's almost everywhere ergodic theorem.

(ii) Consider the set $E = (0, 1]^2$ equipped with its Borel σ-algebra and Lebesgue measure. Fix an irrational number $a \in (0, 1]$ and define $\theta : E \to E$ by

$$\theta(x_1, x_2) = (x_1 + a, x_2 + a),$$

where addition in each coordinate is understood to be modulo 1. Show that θ is a measure-preserving transformation. Is θ ergodic? Justify your answer.

Let f be an integrable function on E and let $\bar{f}$ be the invariant function associated with f by Birkhoff's theorem. Write down a formula for $\bar{f}$ in terms of f. [You are not expected to justify this answer.]

26J Applied Probability

(i) Define a *Poisson process* $(N_t,\ t \geqslant 0)$ with intensity λ. Specify without justification the distribution of N_t. Let $T_1, T_2, \ldots$ denote the jump times of $(N_t, t \geqslant 0)$. Derive the joint distribution of $(T_1, \ldots, T_n)$ given $\{N_t = n\}$.

(ii) Let $(N_t, t \geqslant 0)$ be a Poisson process with intensity $\lambda > 0$ and let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, independent of $(N_t, t \geqslant 0)$, all having the same distribution as a random variable X. Show that if $g(s, x)$ is a real-valued function of real variables s, x, and T_j are the jump times of $(N_t, t \geqslant 0)$ then

$$\mathbb{E}\left[\exp\left\{\theta \sum_{j=1}^{N_t} g(T_j, X_j)\right\}\right] = \exp\left\{\lambda \int_0^t (\mathbb{E}(e^{\theta g(s,X)}) - 1) ds\right\},$$

for all $\theta \in \mathbb{R}$. [*Hint: Condition on* $\{N_t = n\}$ *and* $T_1, \ldots, T_n$*, using* (i).]

(iii) A university library is open from 9am to 5pm. Students arrive at times of a Poisson process with intensity λ. Each student spends a random amount of time in the library, independently of the other students. These times are identically distributed for all students and have the same distribution as a random variable X. Show that the number of students in the library at 5pm is a Poisson random variable with a mean that you should specify.

27J Principles of Statistics

State and prove Wilks' theorem about testing the simple hypothesis $H_0 : \theta = \theta_0$, against the alternative $H_1 : \theta \in \Theta \setminus \{\theta_0\}$, in a one-dimensional regular parametric model $\{f(\cdot, \theta) : \theta \in \Theta\}, \Theta \subseteq \mathbb{R}$. [You may use without proof the results from lectures on the consistency and asymptotic distribution of maximum likelihood estimators, as well as on uniform laws of large numbers. Necessary regularity conditions can be assumed without statement.]

Find the maximum likelihood estimator $\hat{\theta}_n$ based on i.i.d. observations $X_1, \ldots, X_n$ in a $N(0, \theta)$-model, $\theta \in \Theta = (0, \infty)$. Deduce the limit distribution as $n \to \infty$ of the sequence of statistics

$$-n\left(\log(\overline{X^2}) - (\overline{X^2} - 1)\right),$$

where $\overline{X^2} = (1/n)\sum_{i=1}^n X_i^2$ and $X_1, \ldots, X_n$ are i.i.d. $N(0, 1)$.

28J Optimization and Control

A particle follows a discrete-time trajectory on $\mathbb{R}$ given by

$$x_{t+1} = Ax_t + \xi_t u_t + \epsilon_t$$

for $t = 1, 2, \ldots, T$, where $T \geqslant 2$ is a fixed integer, A is a real constant, x_t is the position of the particle and u_t is the control action at time t, and $(\xi_t, \epsilon_t)_{t=1}^T$ is a sequence of independent random vectors with $\mathbb{E}\,\xi_t = \mathbb{E}\,\epsilon_t = 0$, $\operatorname{var}(\xi_t) = V_\xi > 0$, $\operatorname{var}(\epsilon_t) = V_\epsilon > 0$ and $\operatorname{cov}(\xi_t, \epsilon_t) = 0$.

Find the closed-loop control, i.e. the control action u_t defined as a function of $(x_1, \ldots, x_t; u_1, \ldots, u_{t-1})$, that minimizes

$$\sum_{t=1}^{T} x_t^2 + c \sum_{t=1}^{T-1} u_t\,,$$

where $c > 0$ is given. [Note that this function is quadratic in x, but linear in u.]

Does the closed-loop control depend on V_ϵ or on V_ξ? Deduce the form of the optimal open-loop control.

29K Stochastic Financial Models

Derive the Black–Scholes formula $C(S_0, K, r, T, \sigma)$ for the time-0 price of a European call option with expiry T and strike K written on an asset with volatility σ and time-0 price S_0, and where r is the riskless rate of interest. Explain what is meant by the *delta hedge* for this option, and determine it explicitly.

In terms of the Black–Scholes call option price formula C, find the time-0 price of a forward-starting option, which pays $(S_T - \lambda S_t)^+$ at time T, where $0 < t < T$ and $\lambda > 0$ are given. Find the price of an option which pays $\max\{S_T, \lambda S_t\}$ at time T. How would this option be hedged?

30D Partial Differential Equations

(a) Consider variable-coefficient operators of the form

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u + cu$$

whose coefficients are defined on a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let a_{jk} satisfy the condition of uniform ellipticity, namely

$$m\|\xi\|^2 \leqslant \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \leqslant M\|\xi\|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n$$

for suitably chosen positive numbers m, M.

State and prove the weak maximum principle for solutions of $Pu = 0$. [Any results from linear algebra and calculus needed in your proof should be stated clearly, but need not be proved.]

(b) Consider the nonlinear elliptic equation

$$-\Delta u + e^u = f \tag{1}$$

for $u : \mathbb{R}^n \to \mathbb{R}$ satisfying the additional condition

$$\lim_{|x|\to\infty} u(x) = 0\,. \tag{2}$$

Assume that $f \in \mathcal{S}(\mathbb{R}^n)$. Prove that any two C^2 solutions of (1) which also satisfy (2) are equal.

Now let $u \in C^2(\mathbb{R}^n)$ be a solution of (1) and (2). Prove that if $f(x) < 1$ for all x then $u(x) < 0$ for all x. Prove that if $\max_x f(x) = L \geqslant 1$ then $u(x) \leqslant \ln L$ for all x.

31C Asymptotic Methods

(a) Find the Stokes ray for the function $f(z)$ as $z \to 0$ with $0 < \arg z < \pi$, where

$$f(z) = \sinh(z^{-1}).$$

(b) Describe how the leading-order asymptotic behaviour as $x \to \infty$ of

$$I(x) = \int_a^b f(t) e^{ixg(t)} dt$$

may be found by the method of stationary phase, where f and g are real functions and the integral is taken along the real line. You should consider the cases for which:

(i) $g'(t)$ is non-zero in $[a, b)$ and has a simple zero at $t = b$.

(ii) $g'(t)$ is non-zero apart from having one simple zero at $t = t_0$, where $a < t_0 < b$.

(iii) $g'(t)$ has more than one simple zero in (a, b) with $g'(a) \neq 0$ and $g'(b) \neq 0$.

Use the method of stationary phase to find the leading-order asymptotic form as $x \to \infty$ of

$$J(x) = \int_0^1 \cos\left(x(t^4 - t^2)\right) dt.$$

[*You may assume that* $\displaystyle\int_{-\infty}^{\infty} e^{iu^2} du = \sqrt{\pi} e^{i\pi/4}$.]

32D Integrable Systems

What does it mean to say that a finite-dimensional Hamiltonian system is *integrable*? State the Arnold–Liouville theorem.

A six-dimensional dynamical system with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ is governed by the differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j(y_i - y_j)}{(x_i - x_j)^2 + (y_i - y_j)^2}, \quad \frac{\mathrm{d}y_i}{\mathrm{d}t} = \frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j(x_i - x_j)}{(x_i - x_j)^2 + (y_i - y_j)^2}$$

for $i = 1, 2, 3$, where $\{\Gamma_i\}_{i=1}^3$ are positive constants. Show that these equations can be written in the form

$$\Gamma_i\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial F}{\partial y_i}, \quad \Gamma_i\frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{\partial F}{\partial x_i}, \quad i = 1, 2, 3$$

for an appropriate function F. By introducing the coordinates

$$\mathbf{q} = (x_1, x_2, x_3)\,, \quad \mathbf{p} = (\Gamma_1 y_1, \Gamma_2 y_2, \Gamma_3 y_3)\,,$$

show that the system can be written in Hamiltonian form

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial H}{\partial \mathbf{p}}\,, \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{q}}$$

for some Hamiltonian $H = H(\mathbf{q}, \mathbf{p})$ which you should determine.

Show that the three functions

$$A = \sum_{i=1}^{3}\Gamma_i x_i, \quad B = \sum_{i=1}^{3}\Gamma_i y_i, \quad C = \sum_{i=1}^{3}\Gamma_i\left(x_i^2 + y_i^2\right)$$

are first integrals of the Hamiltonian system.

Making use of the fundamental Poisson brackets $\{q_i, q_j\} = \{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$, show that

$$\{A, C\} = 2B, \quad \{B, C\} = -2A.$$

Hence show that the Hamiltonian system is integrable.

33A Principles of Quantum Mechanics

Let $\mathbf{J} = (J_1, J_2, J_3)$ and $|j\, m\rangle$ denote the standard angular-momentum operators and states so that, in units where $\hbar = 1$,

$$\mathbf{J}^2|j\, m\rangle = j(j+1)|j\, m\rangle\,, \quad J_3|j\, m\rangle = m|j\, m\rangle\,.$$

Show that $U(\theta) = \exp(-i\theta J_2)$ is unitary. Define

$$J_i(\theta) \;=\; U(\theta)\, J_i\, U^{-1}(\theta) \quad \text{for } i = 1, 2, 3$$

and

$$|j\, m\rangle_\theta \;=\; U(\theta)|j\, m\rangle\,.$$

Find expressions for $J_1(\theta), J_2(\theta)$ and $J_3(\theta)$ as linear combinations of J_1, J_2 and J_3. Briefly explain why $U(\theta)$ represents a rotation of $\mathbf{J}$ through angle θ about the 2-axis.

Show that

$$J_3(\theta)|j\, m\rangle_\theta \;=\; m|j\, m\rangle_\theta\,. \qquad (*)$$

Express $|1\, 0\rangle_\theta$ as a linear combination of the states $|1\, m\rangle$, $m = -1, 0, 1$. By expressing J_1 in terms of $J_\pm$, use $(*)$ to determine the coefficients in this expansion.

A particle of spin 1 is in the state $|1\, 0\rangle$ at time $t = 0$. It is subject to the Hamiltonian

$$H = -\mu \mathbf{B} \cdot \mathbf{J}\,,$$

where $\mathbf{B} = (0, \mathbf{B}, 0)$. At time t the value of J_3 is measured and found to be $J_3 = 0$. At time $2t$ the value of J_3 is measured again and found to be $J_3 = 1$. Show that the joint probability for these two values to be measured is

$$\frac{1}{8}\sin^2(2\mu B t)\,.$$

[*The following result may be quoted:* $J_\pm\,|j\, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\;|j\;\; m \pm 1\rangle$.]

34A Applications of Quantum Mechanics

In the *nearly-free electron model* a particle of mass m moves in one dimension in a periodic potential of the form $V(x) = \lambda U(x)$, where $\lambda \ll 1$ is a dimensionless coupling and $U(x)$ has a Fourier series

$$U(x) = \sum_{l=-\infty}^{+\infty} U_l \exp\left(\frac{2\pi i}{a} lx\right),$$

with coefficients obeying $U_{-l} = U_l^*$ for all l.

Ignoring any degeneracies in the spectrum, the exact energy $E(k)$ of a Bloch state with wavenumber k can be expanded in powers of λ as

$$E(k) = E_0(k) + \lambda\langle k|U|k\rangle + \lambda^2 \sum_{k'\neq k} \frac{\langle k|U|k'\rangle\langle k'|U|k\rangle}{E_0(k) - E_0(k')} + O(\lambda^3), \tag{1}$$

where $|k\rangle$ is a normalised eigenstate of the free Hamiltonian $\hat{H}_0 = \hat{p}^2/2m$ with momentum $p = \hbar k$ and energy $E_0(k) = \hbar^2 k^2/2m$.

Working on a finite interval of length $L = Na$, where N is a positive integer, we impose periodic boundary conditions on the wavefunction:

$$\psi(x + Na) = \psi(x).$$

What are the allowed values of the wavenumbers k and k' which appear in (1)? For these values evaluate the matrix element $\langle k|U|k'\rangle$.

For what values of k and k' does (1) cease to be a good approximation? Explain your answer. Quoting any results you need from degenerate perturbation theory, calculate to $O(\lambda)$ the location and width of the gaps between allowed energy bands for the periodic potential $V(x)$, in terms of the Fourier coefficients U_l.

Hence work out the allowed energy bands for the following potentials:

$$\text{(i)} \qquad V(x) = 2\lambda \cos\left(\frac{2\pi x}{a}\right),$$

$$\text{(ii)} \qquad V(x) = \lambda a \sum_{n=-\infty}^{+\infty} \delta(x - na).$$

35E Statistical Physics

In the grand canonical ensemble, at temperature T and chemical potential μ, what is the probability of finding a system in a state with energy E and particle number N?

A particle with spin degeneracy g_s and mass m moves in $d \geqslant 2$ spatial dimensions with dispersion relation $E = \hbar^2 k^2/2m$. Compute the density of states $g(E)$. [You may denote the area of a unit $(d-1)$-dimensional sphere as S_{d-1}.]

Treating the particles as non-interacting fermions, determine the energy E of a gas in terms of the pressure P and volume V.

Derive an expression for the Fermi energy in terms of the number density of particles. Compute the degeneracy pressure at zero temperature in terms of the number of particles and the Fermi energy.

Show that at high temperatures the gas obeys the ideal gas law (up to small corrections which you need not compute).

36C Electrodynamics

The 4-vector potential $A^\mu(t, \mathbf{x})$ (in the Lorenz gauge $\partial_\mu A^\mu = 0$) due to a localised source with conserved 4-vector current J^μ is

$$A^\mu(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{J^\mu(t_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3\mathbf{x}' \, ,$$

where $t_{\text{ret}} = t - |\mathbf{x} - \mathbf{x}'|/c$. For a source that varies slowly in time, show that the spatial components of A^μ at a distance $r = |\mathbf{x}|$ that is large compared to the spatial extent of the source are

$$\mathbf{A}(t, \mathbf{x}) \approx \frac{\mu_0}{4\pi r} \left. \frac{d\mathbf{P}}{dt} \right|_{t-r/c} ,$$

where $\mathbf{P}$ is the electric dipole moment of the source, which you should define. Explain what is meant by the *far-field* region, and calculate the leading-order part of the magnetic field there.

A point charge q moves non-relativistically in a circle of radius a in the (x, y) plane with angular frequency ω (such that $a\omega \ll c$). Show that the magnetic field in the far-field at the point $\mathbf{x}$ with spherical polar coordinates r, θ and ϕ has components along the θ and ϕ directions given by

$$\begin{aligned} B_\theta &\approx -\frac{\mu_0 \omega^2 q a}{4\pi r c} \sin[\omega(t - r/c) - \phi] \, , \\ B_\phi &\approx \frac{\mu_0 \omega^2 q a}{4\pi r c} \cos[\omega(t - r/c) - \phi] \cos\theta \, . \end{aligned}$$

Calculate the total power radiated by the charge.

37E General Relativity

The vector field V^a is the normalised ($V_a V^a = -c^2$) tangent to a congruence of timelike geodesics, and $B_{ab} = \nabla_b V_a$.

Show that:

(i) $V^a B_{ab} = V^b B_{ab} = 0$;

(ii) $V^c \nabla_c B_{ab} = -B^c{}_b B_{ac} - R^d{}_{acb} V^c V_d$.

[You may use the Ricci identity $\nabla_c \nabla_b X_a = \nabla_b \nabla_c X_a - R^d{}_{acb} X_d$.]

Now assume that B_{ab} is symmetric and let $\theta = B_a{}^a$. By writing $B_{ab} = \widetilde{B}_{ab} + \frac{1}{4}\theta g_{ab}$, or otherwise, show that

$$\frac{d\theta}{d\tau} \leqslant -\tfrac{1}{4}\theta^2 - R_{00} ,$$

where $R_{00} = R_{ab}V^a V^b$ and $\dfrac{d\theta}{d\tau} \equiv V^a \nabla_a \theta$. [You may use without proof the result that $\widetilde{B}_{ab}\widetilde{B}^{ab} \geqslant 0$.]

Assume, in addition, that the stress-energy tensor T_{ab} takes the perfect-fluid form $(\rho + p/c^2)V_a V_b + p g_{ab}$ and that $\rho c^2 + 3p > 0$. Show that

$$\frac{d\theta^{-1}}{d\tau} > \frac{1}{4} ,$$

and deduce that, if $\theta(0) < 0$, then $|\theta(\tau)|$ will become unbounded for some value of τ less than $4/|\theta(0)|$.

38B Fluid Dynamics II

A rigid sphere of radius a falls under gravity through an incompressible fluid of density ρ and viscosity μ towards a rigid horizontal plane. The minimum gap $h_0(t)$ between the sphere and the plane satisfies $h_0 \ll a$. Find an approximation for the gap thickness $h(r,t)$ between the sphere and the plane in the region $r \ll a$, where r is the distance from the axis of symmetry.

For a prescribed value of $\dot{h}_0 = dh_0/dt$, use lubrication theory to find the radial velocity and the fluid pressure in the region $r \ll a$. Explain why the approximations of lubrication theory require $h_0 \ll a$ and $\rho h_0 \dot{h}_0 \ll \mu$.

Calculate the total vertical force due to the motion that is exerted by the fluid on the sphere. Deduce that if the sphere is settling under its own weight (corrected for buoyancy) then $h_0(t)$ decreases exponentially. What is the exponential decay rate for a solid sphere of density ρ_s in a fluid of density ρ_f?

39C Waves

The equations describing small-amplitude motions in a stably stratified, incompressible, inviscid fluid are

$$\frac{\partial \tilde{\rho}}{\partial t} + w\frac{\mathrm{d}\rho_0}{\mathrm{d}z} = 0\,, \qquad \rho_0\frac{\partial \mathbf{u}}{\partial t} = \tilde{\rho}\mathbf{g} - \boldsymbol{\nabla}\tilde{p}\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0\,,$$

where $\rho_0(z)$ is the background stratification, $\tilde{\rho}(\mathbf{x},t)$ and $\tilde{p}(\mathbf{x},t)$ are the perturbations about an undisturbed hydrostatic state, $\mathbf{u}(\mathbf{x},t) = (u,v,w)$ is the velocity, and $\mathbf{g} = (0,0,-g)$.

Show that

$$\left[\frac{\partial^2}{\partial t^2}\nabla^2 + N^2\Big(\nabla^2 - \frac{\partial^2}{\partial z^2}\Big)\right] w = 0\,,$$

stating any approximation made, and define the Brunt–Väisälä frequency N.

Deduce the dispersion relation for plane harmonic waves with wavevector $\mathbf{k} = (k,0,m)$. Calculate the group velocity and verify that it is perpendicular to $\mathbf{k}$.

Such a stably stratified fluid with a uniform value of N occupies the region $z > h(x,t)$ above a moving lower boundary $z = h(x,t)$. Find the velocity field $w(x,z,t)$ generated by the boundary motion for the case $h = \epsilon\sin[k(x-Ut)]$, where $0 < \epsilon k \ll 1$ and $U > 0$ is a constant.

For the case $k^2 < N^2/U^2$, sketch the orientation of the wave crests, the direction of propagation of the crests, and the direction of the group velocity.

40D Numerical Analysis

Consider the linear system

$$Ax = b\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $b, x \in \mathbb{R}^n$.

(i) Define the *Jacobi iteration method with relaxation parameter* ω for solving (1).

(ii) Assume that A is a symmetric positive-definite matrix whose diagonal part D is such that the matrix $2D - A$ is also positive definite. Prove that the relaxed Jacobi iteration method always converges if the relaxation parameter ω is equal to 1.

(iii) Let A be the tridiagonal matrix with diagonal elements $a_{ii} = \alpha$ and off-diagonal elements $a_{i+1,i} = a_{i,i+1} = \beta$, where $0 < \beta < \frac{1}{2}\alpha$. For which values of ω (expressed in terms of α and β) does the relaxed Jacobi iteration method converge? What choice of ω gives the optimal convergence speed?

[You may quote without proof any relevant results about the convergence of iterative methods and about the eigenvalues of matrices.]

END OF PAPER