UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Monday, 2 June, 2014 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A****,* ***B****,* ***C****, ...,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet <u>must</u> bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Number Theory

Show that

$$\sum_{p \leqslant x} \frac{1}{p} \geqslant \log \log x - \frac{1}{2}.$$

Deduce that there are infinitely many primes.

2G Topics in Analysis

State Chebyshev's equal ripple criterion.

Let

$$h(t) = \prod_{\ell=1}^{n} \left(t - \cos \frac{(2\ell - 1)\pi}{2n} \right).$$

Show that if $q(t) = \sum_{j=0}^{n} a_j t^j$ where $a_0, \ldots, a_n$ are real constants with $|a_n| \geqslant 1$, then

$$\sup_{t \in [-1,1]} |h(t)| \leqslant \sup_{t \in [-1,1]} |q(t)|.$$

3F Geometry and Groups

Let g, h be non-identity Möbius transformations. Prove that g and h commute if and only if one of the following holds:

1. $\mathrm{Fix}(g) = \mathrm{Fix}(h)$;
2. g, h are involutions each of which exchanges the other's fixed points.

Give an example to show that the second case can occur.

4I Coding and Cryptography

Let $c : \mathcal{A} \to \{0,1\}^*$ be a decodable binary code defined on a finite alphabet $\mathcal{A}$. Let $l(a)$ be the length of the code word $c(a)$. Prove that

$$\sum_{a \in \mathcal{A}} 2^{-l(a)} \leqslant 1 .$$

Show that, for the decodable code $c : \mathcal{A} \to \{0,1\}^*$ described above, there is a prefix-free code $p : \mathcal{A} \to \{0,1\}^*$ with each code word $p(a)$ having length $l(a)$. [You may use, without proof, any standard results from the course.]

5K Statistical Modelling

Define the concept of an *exponential dispersion family.* Show that the family of scaled binomial distributions $\frac{1}{n}\text{Bin}(n,p)$, with $p \in (0,1)$ and $n \in \mathbb{N}$, is of exponential dispersion family form.

Deduce the mean of the scaled binomial distribution from the exponential dispersion family form.

What is the canonical link function in this case?

6B Mathematical Biology

Consider an experiment where two or three individuals are added to a population with probability λ_2 and λ_3 respectively per unit time. The death rate in the population is a constant β per individual per unit time.

Write down the master equation for the probability $p_n(t)$ that there are n individuals in the population at time t. From this, derive an equation for $\frac{\partial\phi}{\partial t}$, where ϕ is the generating function

$$\phi(s,t) = \sum_{n=0}^{\infty} s^n p_n(t).$$

Find the solution for ϕ in steady state, and show that the mean and variance of the population size are given by

$$\langle n\rangle = 3\frac{\lambda_3}{\beta} + 2\frac{\lambda_2}{\beta}, \qquad \text{var}(n) = 6\frac{\lambda_3}{\beta} + 3\frac{\lambda_2}{\beta}.$$

Hence show that, for a free choice of λ_2 and λ_3 subject to a given target mean, the experimenter can minimise the variance by only adding two individuals at a time.

7D Dynamical Systems

Consider the system

$$\begin{aligned} \dot{x} &= -x + y + y^2, \\ \dot{y} &= \mu - xy\,. \end{aligned}$$

Show that when $\mu = 0$ the fixed point at the origin has a stationary bifurcation.

Find the centre subspace of the extended system linearised about $(x,y,\mu) = (0,0,0)$.

Find an approximation to the centre manifold giving y as a function of x and μ, including terms up to quadratic order.

Hence deduce an expression for $\dot{x}$ on the centre manifold, and identify the type of bifurcation at $\mu = 0$.

8B Further Complex Methods

Suppose $z = 0$ is a regular singular point of a linear second-order homogeneous ordinary differential equation in the complex plane. Define the *monodromy matrix* M around $z = 0$.

Demonstrate that if

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then the differential equation admits a solution of the form $a(z) + b(z) \log z$, where $a(z)$ and $b(z)$ are single-valued functions.

9A Classical Dynamics

The components of the angular velocity $\boldsymbol{\omega}$ of a rigid body and of the position vector $\mathbf{r}$ are given in a body frame.

(a) The kinetic energy of the rigid body is defined as

$$T = \frac{1}{2} \int d^3\mathbf{r}\, \rho(\mathbf{r}) \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \,,$$

Given that the centre of mass is at rest, show that T can be written in the form

$$T = \frac{1}{2} I_{ab} \omega_a \omega_b \,,$$

where the explicit form of the tensor I_{ab} should be determined.

(b) Explain what is meant by the *principal moments of inertia*.

(c) Consider a rigid body with principal moments of inertia $I_1\,, I_2$ and I_3, which are all unequal. Derive Euler's equations of torque-free motion

$$\begin{aligned} I_1 \dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \,, \\ I_2 \dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \,, \\ I_3 \dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \,. \end{aligned}$$

(d) The body rotates about the principal axis with moment of inertia I_1. Derive the condition for stable rotation.

10E Cosmology

A self-gravitating fluid with density ρ, pressure $P(\rho)$ and velocity $\mathbf{v}$ in a gravitational potential Φ obeys the equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) &= 0\,, \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} + \frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{\nabla}\Phi &= \mathbf{0}\,, \\ \nabla^2 \Phi &= 4\pi G \rho\,. \end{aligned}$$

Assume that there exists a static constant solution of these equations with $\mathbf{v} = \mathbf{0}$, $\rho = \rho_0$ and $\Phi = \Phi_0$, for which $\boldsymbol{\nabla}\Phi_0$ can be neglected. This solution is perturbed. Show that, to first order in the perturbed quantities, the density perturbations satisfy

$$\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \nabla^2 \rho_1 + 4\pi G \rho_0 \rho_1\,,$$

where $\rho = \rho_0 + \rho_1(\mathbf{x}, t)$ and $c_s^2 = dP/d\rho$. Show that there are solutions to this equation of the form

$$\rho_1(\mathbf{x}, t) = A \exp[-i\mathbf{k} \cdot \mathbf{x} + i\omega t]\,,$$

where A, ω and $\mathbf{k}$ are constants and

$$\omega^2 = c_s^2\, \mathbf{k} \cdot \mathbf{k} - 4\pi G \rho_0\,.$$

Interpret these solutions physically in the limits of small and large $|\mathbf{k}|$, explaining what happens to density perturbations on large and small scales, and determine the critical wavenumber that divides the two distinct behaviours of the perturbation.

SECTION II

11G Topics in Analysis

Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous map never taking the value 0 and satisfying $\gamma(0) = \gamma(1)$. Define the *degree* (or *winding number*) $w(\gamma; 0)$ of γ about 0. Prove the following:

(i) $w(1/\gamma;\, 0) = w(\gamma^-; 0)$, where $\gamma^-(t) = \gamma(1-t)$.

(ii) If $\sigma : [0,1] \to \mathbb{C}$ is continuous, $\sigma(0) = \sigma(1)$ and $|\sigma(t)| < |\gamma(t)|$ for each $0 \leqslant t \leqslant 1$, then $w(\gamma + \sigma; 0) = w(\gamma; 0)$.

(iii) If $\gamma_m : [0,1] \to \mathbb{C}$, $m = 1, 2, \ldots$, are continuous maps with $\gamma_m(0) = \gamma_m(1)$, which converge to γ uniformly on $[0,1]$ as $m \to \infty$, then $w(\gamma_m; 0) = w(\gamma; 0)$ for sufficiently large m.

Let $\alpha : [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map such that $\alpha(0) = \alpha(1)$ and $|\alpha(t) - e^{2\pi i t}| \leqslant 1$ for each $t \in [0,1]$. Deduce from the results of (ii) and (iii) that $w(\alpha; 0) = 1$.

[You may not use homotopy invariance of the winding number without proof.]

12I Coding and Cryptography

What is the *information capacity* of a memoryless, time-independent channel? Compute the information capacity of a binary symmetric channel with probability p of error. Show the steps in your computation.

Binary digits are transmitted through a noisy channel, which is memoryless and time-independent. With probability α $(0 < \alpha < 1)$ the digit is corrupted and noise is received, otherwise the digit is transmitted unchanged. So, if we denote the input by 0 and 1 and the output as 0, $*$ and 1 with $*$ denoting the noise, the transition matrix is

$$\begin{pmatrix} 1-\alpha & 0 \\ \alpha & \alpha \\ 0 & 1-\alpha \end{pmatrix}.$$

Compute the information capacity of this channel.

Explain how to code a message for transmission through the channel described above, and how to decode it, so that the probability of error for each bit is arbitrarily small.

13B Mathematical Biology

An activator–inhibitor system is described by the equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{au}{v} - u^2 + d_1 \frac{\partial^2 u}{\partial x^2}\,, \\ \frac{\partial v}{\partial t} &= v^2 - \frac{v}{u^2} + d_2 \frac{\partial^2 v}{\partial x^2}\,, \end{aligned}$$

where $a, d_1, d_2 > 0$.

Find the range of a for which the spatially homogeneous system has a stable equilibrium solution with $u > 0$ and $v > 0$. Determine when the equilibrium is a stable focus, and sketch the phase diagram for this case (restricting attention to $u > 0$ and $v > 0$).

For the case when the homogeneous system is stable, consider spatial perturbations proportional to $\cos(kx)$ of the solution found above. Briefly explain why the system will be stable to spatial perturbations with very small or very large k. Find conditions for the system to be unstable to a spatial perturbation (for some range of k which need not be given). Sketch the region satisfying these conditions in the $(a, d_1/d_2)$ plane.

Find k_c, the critical wavenumber at the onset of instability, in terms of a and d_1.

14B Further Complex Methods

Use the Euler product formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\dots(z+n)}$$

to show that:

(i) $\Gamma(z+1) = z\Gamma(z)$;

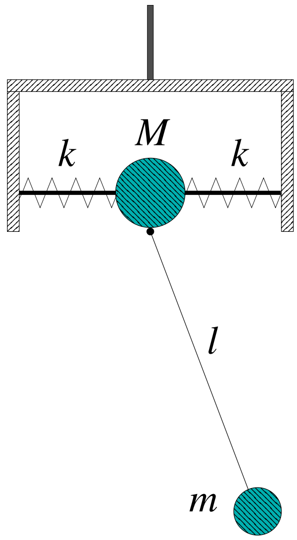
(ii) $\dfrac{1}{\Gamma(z)} = z e^{\gamma z} \displaystyle\prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}$, where $\gamma = \displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n\right)$.

Deduce that

$$\frac{d}{dz} \log\left(\Gamma(z)\right) = -\gamma - \frac{1}{z} + z \sum_{k=1}^{\infty} \frac{1}{k(z+k)}\,.$$

15A Classical Dynamics

A planar pendulum consists of a mass m at the end of a light rod of length l. The pivot of the pendulum is attached to a bead of mass M, which slides along a horizontal rod without friction. The bead is connected to the ends of the horizontal rod by two identical springs of force constant k. The pivot constrains the pendulum to swing in the vertical plane through the horizontal rod. The horizontal rod is mounted on a bracket, so the system could rotate about the vertical axis which goes through its centre as shown in the figure.



(a) Initially, the system is not allowed to rotate about the vertical axis.

(i) Identify suitable generalized coordinates and write down the Lagrangian of the system.

(ii) Write down expression(s) for any conserved quantities. Justify your answer.

(iii) Derive the equations of motion.

(iv) For $M = m/2$ and $gm/kl = 3$, find the frequencies of small oscillations around the stable equilibrium and the corresponding normal modes. Describe the respective motions of the system.

(b) Assume now that the system is free to rotate about the vertical axis without friction. Write down the Lagrangian of the system. Identify and calculate the additional conserved quantity.

16I Logic and Set Theory

Write down the recursive definitions of ordinal addition, multiplication and exponentiation. Show that, for any nonzero ordinal α, there exist unique ordinals β, γ and n such that $\alpha = \omega^{\beta}.n + \gamma$, $\gamma < \omega^{\beta}$ and $0 < n < \omega$.

Hence or otherwise show that α (that is, the set of ordinals less than α) is closed under addition if and only if $\alpha = \omega^{\beta}$ for some β. Show also that an infinite ordinal α is closed under multiplication if and only if $\alpha = \omega^{(\omega^{\gamma})}$ for some γ.

[You may assume the standard laws of ordinal arithmetic, and the fact that $\alpha \leqslant \omega^{\alpha}$ for all α.]

17I Graph Theory

Let k and n be integers with $1 \leqslant k < n$. Show that every connected graph of order n, in which $d(u) + d(v) \geqslant k$ for every pair u, v of non-adjacent vertices, contains a path of length k.

Let k and n be integers with $1 \leqslant k \leqslant n$. Show that a graph of order n that contains no path of length k has at most $(k-1)n/2$ edges, and that this value is achieved only if k divides n and G is the union of n/k disjoint copies of K_k. [*Hint: Proceed by induction on n and consider a vertex of minimum degree.*]

18H Galois Theory

Describe the Galois correspondence for a finite Galois extension L/K.

Let L be the splitting field of $X^4 - 2$ over $\mathbb{Q}$. Compute the Galois group G of $L/\mathbb{Q}$. For each subgroup of G, determine the corresponding subfield of L.

Let L/K be a finite Galois extension whose Galois group is isomorphic to S_n. Show that L is the splitting field of a separable polynomial of degree n.

19H Representation Theory

In this question work over $\mathbb{C}$. Let H be a subgroup of G. State Mackey's restriction formula, defining all the terms you use. Deduce Mackey's irreducibility criterion.

Let $G = \langle g, r : g^m = r^2 = 1,\ rgr^{-1} = g^{-1} \rangle$ (the dihedral group of order $2m$) and let $H = \langle g \rangle$ (the cyclic subgroup of G of order m). Write down the m inequivalent irreducible characters χ_k $(1 \leqslant k \leqslant m)$ of H. Determine the values of k for which the induced character $\mathrm{Ind}_H^G \chi_k$ is irreducible.

20F Number Fields

(i) Show that each prime ideal in a number field K divides a unique rational prime p. Define the *ramification index* and *residue class degree* of such an ideal. State and prove a formula relating these numbers, for all prime ideals dividing a given rational prime p, to the degree of K over $\mathbb{Q}$.

(ii) Show that if ζ_n is a primitive nth root of unity then $\prod_{j=1}^{n-1}(1-\zeta_n^j)=n$. Deduce that if $n=pq$, where p and q are distinct primes, then $1-\zeta_n$ is a unit in $\mathbb{Z}[\zeta_n]$.

(iii) Show that if $K=\mathbb{Q}(\zeta_p)$ where p is prime, then any prime ideal of K dividing p has ramification index at least $p-1$. Deduce that $[K:\mathbb{Q}]=p-1$.

21F Algebraic Topology

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with integer entries. Considering S^1 as the quotient space $\mathbb{R}/\mathbb{Z}$, show that the function

$$\begin{aligned} \varphi_A : S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ ([x],[y]) &\longmapsto ([ax+by],[cx+dy]) \end{aligned}$$

is well-defined and continuous. If in addition $\det(A)=\pm 1$, show that φ_A is a homeomorphism.

State the Seifert–van Kampen theorem. Let X_A be the space obtained by gluing together two copies of $S^1 \times D^2$ along their boundaries using the homeomorphism φ_A. Show that the fundamental group of X_A is cyclic and determine its order.

22G Linear Analysis

(a) Let X and Y be Banach spaces, and let $T\colon X \to Y$ be a surjective linear map. Assume that there is a constant $c > 0$ such that $\|Tx\| \geqslant c\|x\|$ for all $x \in X$. Show that T is continuous. [You may use any standard result from general Banach space theory provided you clearly state it.] Give an example to show that the assumption that X and Y are complete is necessary.

(b) Let C be a closed subset of a Banach space X such that

(i) $x_1 + x_2 \in C$ for each $x_1, x_2 \in C$;

(ii) $\lambda x \in C$ for each $x \in C$ and $\lambda > 0$;

(iii) for each $x \in X$, there exist $x_1, x_2 \in C$ such that $x = x_1 - x_2$.

Prove that, for some $M > 0$, the unit ball of X is contained in the closure of the set

$$\{x_1 - x_2 : x_i \in C, \ \ \|x_i\| \leqslant M \ (i = 1, 2)\} \ .$$

[You may use without proof any version of the Baire Category Theorem.] Deduce that, for some $K > 0$, every $x \in X$ can be written as $x = x_1 - x_2$ with $x_i \in C$ and $\|x_i\| \leqslant K\|x\|$ $(i = 1, 2)$.

23H Riemann Surfaces

State and prove the Valency Theorem and define the *degree* of a non-constant holomorphic map between compact Riemann surfaces.

Let X be a compact Riemann surface of genus g and $\pi : X \to \mathbb{C}_\infty$ a holomorphic map of degree two. Find the cardinality of the set R of ramification points of π. Find also the cardinality of the set of branch points of π. [You may use standard results from lectures provided they are clearly stated.]

Define $\sigma : X \to X$ as follows: if $p \in R$, then $\sigma(p) = p$; otherwise, $\sigma(p) = q$ where q is the unique point such that $\pi(q) = \pi(p)$ and $p \neq q$. Show that σ is a conformal equivalence with $\pi\sigma = \pi$ and $\sigma\sigma = \text{id}$.

24H Algebraic Geometry

(i) Let k be an algebraically closed field, $n \geqslant 1$, and S a subset of k^n.

Let $I(S) = \{f \in k[x_1, \ldots, x_n] \mid f(p) = 0 \text{ when } p \in S\}$. Show that $I(S)$ is an ideal, and that $k[x_1, \ldots, x_n]/I(S)$ does not have any non-zero nilpotent elements.

Let $X \subseteq \mathbf{A}^n$, $Y \subseteq \mathbf{A}^m$ be affine varieties, and $\Phi : k[Y] \to k[X]$ be a k-algebra homomorphism. Show that Φ determines a map of sets from X to Y.

(ii) Let X be an irreducible affine variety. Define the *dimension* of X, $\dim X$ (in terms of the tangent spaces of X) and the *transcendence dimension* of X, $\text{tr.dim}\, X$.

State the Noether normalization theorem. Using this, or otherwise, prove that the transcendence dimension of X equals the dimension of X.

25G Differential Geometry

Define the terms *Gaussian curvature* K and *mean curvature* H for a smooth embedded oriented surface $S \subset \mathbf{R}^3$. [You may assume the fact that the derivative of the Gauss map is self-adjoint.] If $K = H^2$ at all points of S, show that both H and K are locally constant. [*Hint: Use the symmetry of second partial derivatives of the field of unit normal vectors.*]

If $K = H^2 = 0$ at all points of S, show that the unit normal vector $\mathbf{N}$ to S is locally constant and that S is locally contained in a plane. If $K = H^2$ is a strictly positive constant on S and $\phi : U \to S$ is a local parametrization (where U is connected) on S with unit normal vector $\mathbf{N}(u, v)$ for $(u, v) \in U$, show that $\phi(u, v) + \mathbf{N}(u, v)/H$ is constant on U. Deduce that S is locally contained in a sphere of radius $1/|H|$.

If S is connected with $K = H^2$ at all points of S, deduce that S is contained in either a plane or a sphere.

26K Probability and Measure

State and prove the monotone convergence theorem.

Let $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$ be finite measure spaces. Define the *product σ-algebra* $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ on $E_1 \times E_2$.

Define the *product measure* $\mu = \mu_1 \otimes \mu_2$ on $\mathcal{E}$, and show carefully that μ is countably additive.

[You may use without proof any standard facts concerning measurability provided these are clearly stated.]

27J Applied Probability

(i) Explain what the *Moran model* and the *infinite alleles model* are. State Ewens' sampling formula for the distribution of the allelic frequency spectrum $(a_1, \ldots, a_n)$ in terms of θ where $\theta = Nu$ with u denoting the mutation rate per individual and N the population size.

Let K_n be the number of allelic types in a sample of size n. Give, without justification, an expression for $\mathbb{E}(K_n)$ in terms of θ.

(ii) Let K_n and θ be as above. Show that for $1 \leqslant k \leqslant n$ we have that

$$P(K_n = k) = C \frac{\theta^k}{\theta(\theta+1)\cdots(\theta+n-1)}$$

for some constant C that does not depend on θ.

Show that, given $\{K_n = k\}$, the distribution of the allelic frequency spectrum $(a_1, \ldots, a_n)$ does not depend on θ.

Show that the value of θ which maximises $\mathbb{P}(K_n = k)$ is the one for which $k = \mathbb{E}(K_n)$.

28J Principles of Statistics

In a general decision problem, define the concepts of a *Bayes rule* and of *admissibility*. Show that a unique Bayes rule is admissible.

Consider i.i.d. observations $X_1, \ldots, X_n$ from a Poisson(θ), $\theta \in \Theta = (0, \infty)$, model. Can the maximum likelihood estimator $\hat{\theta}_{MLE}$ of θ be a Bayes rule for estimating θ in quadratic risk for any prior distribution on θ that has a continuous probability density on $(0, \infty)$? Justify your answer.

Now model the X_i as i.i.d. copies of $X|\theta \sim$ Poisson(θ), where θ is drawn from a prior that is a Gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$ (given below). Show that the posterior distribution of $\theta | X_1, \ldots, X_n$ is a Gamma distribution and find its parameters. Find the Bayes rule $\hat{\theta}_{BAYES}$ for estimating θ in quadratic risk for this prior. [The Gamma probability density function with parameters $\alpha > 0, \lambda > 0$ is given by

$$f(\theta) = \frac{\lambda^\alpha \theta^{\alpha-1} e^{-\lambda\theta}}{\Gamma(\alpha)}, \qquad \theta > 0,$$

where $\Gamma(\alpha)$ is the usual Gamma function.]

Finally assume that the X_i have actually been generated from a fixed Poisson(θ_0) distribution, where $\theta_0 > 0$. Show that $\sqrt{n}(\hat{\theta}_{BAYES} - \hat{\theta}_{MLE})$ converges to zero in probability and deduce the asymptotic distribution of $\sqrt{n}(\hat{\theta}_{BAYES} - \theta_0)$ under the joint law $P_{\theta_0}^{\mathbb{N}}$ of the random variables $X_1, X_2, \ldots$. [You may use standard results from lectures without proof provided they are clearly stated.]

29J Optimization and Control

Describe the elements of a discrete-time stochastic dynamic programming equation for the problem of maximizing the expected sum of non-negative rewards over an infinite horizon. Give an example to show that there may not exist an optimal policy. Prove that if a policy has a value function that satisfies the dynamic programming equation then the policy is optimal.

A squirrel collects nuts for the coming winter. There are plenty of nuts lying around, but each time the squirrel leaves its lair it risks being caught by a predator. Assume that the outcomes of the squirrel's journeys are independent, that it is caught with probability p, and that it returns safely with a random weight of nuts, exponentially distributed with parameter λ. By solving the dynamic programming equation for the value function $F(x)$, find a policy maximizing the expected weight of nuts collected for the winter. Here the state variable x takes values in $\mathbb{R}_+$ (the weight of nuts so far collected) or -1 (a no-return state when the squirrel is caught).

30K Stochastic Financial Models

An agent has expected-utility preferences over his possible wealth at time 1; that is, the wealth Z is preferred to wealth Z' if and only if $E\ U(Z) \geqslant E\ U(Z')$, where the function $U : \mathbb{R} \to \mathbb{R}$ is strictly concave and twice continuously differentiable. The agent can trade in a market, with the time-1 value of his portfolio lying in an affine space $\mathcal{A}$ of random variables. Assuming cash can be held between time 0 and time 1, define the agent's time-0 *utility indifference price* $\pi(Y)$ for a contingent claim with time-1 value Y. Assuming any regularity conditions you may require, prove that the map $Y \mapsto \pi(Y)$ is concave.

Comment briefly on the limit $\lim_{\lambda \to 0} \pi(\lambda Y)/\lambda$.

Consider a market with two claims with time-1 values X and Y. Their joint distribution is

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} V_{XX} & V_{XY} \\ V_{YX} & V_{YY} \end{pmatrix} \right).$$

At time 0, arbitrary quantities of the claim X can be bought at price p, but Y is not marketed. Derive an explicit expression for $\pi(Y)$ in the case where

$$U(x) = -\exp(-\gamma x),$$

where $\gamma > 0$ is a given constant.

31D Partial Differential Equations

In this question, functions are all real-valued, and

$$H^s_{per} = \{u = \sum_{m\in\mathbb{Z}} \hat{u}(m)e^{imx} \in L^2 : \|u\|^2_{H^s} = \sum_{m\in\mathbb{Z}} (1+m^2)^s |\hat{u}(m)|^2 < \infty\}$$

are the Sobolev spaces of functions 2π-periodic in x, for $s = 0, 1, 2, \ldots$.

State Parseval's theorem. For $s = 0, 1$ prove that the norm $\|u\|_{H^s}$ is equivalent to the norm $\| \, \|_s$ defined by

$$\|u\|^2_s = \sum_{r=0}^{s} \int_{-\pi}^{+\pi} (\partial^r_x u)^2 \, dx \, .$$

Consider the Cauchy problem

$$u_t - u_{xx} = f \, , \qquad u(x,0) = u_0(x) \, , \qquad t \geqslant 0 \, , \tag{1}$$

where $f = f(x,t)$ is a smooth function which is 2π-periodic in x, and the initial value u_0 is also smooth and 2π-periodic. Prove that if u is a smooth solution which is 2π-periodic in x, then it satisfies

$$\int_0^T (\, u_t^2 + u_{xx}^2 \,) \, dt \leqslant C \left(\|u_0\|^2_{H^1} + \int_0^T \int_{-\pi}^{\pi} |f(x,t)|^2 \, dx \, dt \right)$$

for some number $C > 0$ which does not depend on u or f.

State the Lax–Milgram lemma. Prove, using the Lax–Milgram lemma, that if

$$f(x,t) = e^{\lambda t} g(x)$$

with $g \in H^0_{per}$ and $\lambda > 0$, then there exists a weak solution to (1) of the form $u(x,t) = e^{\lambda t}\phi(x)$ with $\phi \in H^1_{per}$. Does the same hold for all $\lambda \in \mathbb{R}$? Briefly explain your answer.

32D Integrable Systems

Let $u = u(x)$ be a smooth function that decays rapidly as $|x| \to \infty$ and let $L = -\partial_x^2 + u(x)$ denote the associated Schrödinger operator. Explain very briefly each of the terms appearing in the scattering data

$$S = \left\{ \{\chi_n, c_n\}_{n=1}^N, R(k) \right\},$$

associated with the operator L. What does it mean to say $u(x)$ is *reflectionless*?

Given S, define the function

$$F(x) = \sum_{n=1}^N c_n^2 e^{-\chi_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}kx} R(k)\, \mathrm{d}k\,.$$

If $K = K(x, y)$ is the unique solution to the GLM equation

$$K(x,y) + F(x+y) + \int_x^\infty K(x,z) F(z+y)\, \mathrm{d}z = 0\,,$$

what is the relationship between $u(x)$ and $K(x,x)$?

Now suppose that $u = u(x,t)$ is time dependent and that it solves the KdV equation $u_t + u_{xxx} - 6uu_x = 0$. Show that $L = -\partial_x^2 + u(x,t)$ obeys the Lax equation

$$L_t = [L, A], \quad \text{where } A = 4\partial_x^3 - 3(u\partial_x + \partial_x u)\,.$$

Show that the discrete eigenvalues of L are time independent.

In what follows you may assume the time-dependent scattering data take the form

$$S(t) = \left\{ \left\{ \chi_n, c_n e^{4\chi_n^3 t} \right\}_{n=1}^N, R(k,0) e^{8\mathrm{i}k^3 t} \right\}.$$

Show that if $u(x,0)$ is reflectionless, then the solution to the KdV equation takes the form

$$u(x,t) = -2\frac{\partial^2}{\partial x^2} \log\left[\det A(x,t)\right]\,,$$

where A is an $N \times N$ matrix which you should determine.

Assume further that $R(k,0) = k^2 f(k)$, where f is smooth and decays rapidly at infinity. Show that, for any fixed x,

$$\int_{-\infty}^{\infty} e^{\mathrm{i}kx} R(k,0)\, e^{8\mathrm{i}k^3 t}\, \mathrm{d}k = O(t^{-1}) \quad \text{as } t \to \infty\,.$$

Comment briefly on the significance of this result.

[*You may assume* $\dfrac{1}{\det A} \dfrac{\mathrm{d}}{\mathrm{d}x}(\det A) = \mathrm{tr}\left(A^{-1} \dfrac{\mathrm{d}A}{\mathrm{d}x} \right)$ *for a non-singular matrix* $A(x)$.]

33A Principles of Quantum Mechanics

(i) Let a and $a^\dagger$ be the annihilation and creation operators, respectively, for a simple harmonic oscillator whose Hamiltonian is

$$H_0 \;=\; \omega\left(a^\dagger a + \tfrac{1}{2}\right),$$

with $[a, a^\dagger] = 1$. Explain how the set of eigenstates $\{\,|n\rangle \,:\, n = 0, 1, 2, \ldots\}$ of H_0 is obtained and deduce the corresponding eigenvalues. Show that

$$a|0\rangle = 0\,,$$

$$a|n\rangle = \sqrt{n}|n-1\rangle\,, \qquad n \geqslant 1\,,$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,, \qquad n \geqslant 0\,.$$

(ii) Consider a system whose unperturbed Hamiltonian is

$$H_0 = \left(a^\dagger a + \tfrac{1}{2}\right) \;+\; 2\left(b^\dagger b + \tfrac{1}{2}\right),$$

where $[a, a^\dagger] = 1$, $[b, b^\dagger] = 1$ and all other commutators are zero. Find the degeneracies of the eigenvalues of H_0 with energies $E_0 = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$ and $\frac{11}{2}$.

The system is perturbed so that it is now described by the Hamiltonian

$$H = H_0 + \lambda H',$$

where $H' = (a^\dagger)^2 b + a^2 b^\dagger$. Using degenerate perturbation theory, calculate to $O(\lambda)$ the energies of the eigenstates associated with the level $E_0 = \frac{9}{2}$.

Write down the eigenstates, to $O(\lambda)$, associated with these perturbed energies. By explicit evaluation show that they are in fact exact eigenstates of H with these energies as eigenvalues.

34A Applications of Quantum Mechanics

(a) A classical particle of mass m scatters on a central potential $V(r)$ with energy E, impact parameter b, and scattering angle θ. Define the corresponding *differential cross-section.*

For particle trajectories in the Coulomb potential,

$$V_C(r) = \frac{e^2}{4\pi\epsilon_0 r}\,,$$

the impact parameter is given by

$$b = \frac{e^2}{8\pi\epsilon_0 E}\cot\left(\frac{\theta}{2}\right).$$

Find the differential cross-section as a function of E and θ.

(b) A quantum particle of mass m and energy $E = \hbar^2 k^2/2m$ scatters in a localised potential $V(\mathbf{r})$. With reference to the asymptotic form of the wavefunction at large $|\mathbf{r}|$, define the *scattering amplitude* $f(\mathbf{k},\mathbf{k}')$ as a function of the incident and outgoing wavevectors $\mathbf{k}$ and $\mathbf{k}'$ (where $|\mathbf{k}| = |\mathbf{k}'| = k$). Define the differential cross-section for this process and express it in terms of $f(\mathbf{k},\mathbf{k}')$.

Now consider a potential of the form $V(\mathbf{r}) = \lambda U(\mathbf{r})$, where $\lambda \ll 1$ is a dimensionless coupling and U does not depend on λ. You may assume that the Schrödinger equation for the wavefunction $\psi(\mathbf{k};\mathbf{r})$ of a scattering state with incident wavevector $\mathbf{k}$ may be written as the integral equation

$$\psi(\mathbf{k};\mathbf{r}) = \exp\left(i\mathbf{k}\cdot\mathbf{r}\right) \;+\; \frac{2m\lambda}{\hbar^2}\int d^3r'\,\mathcal{G}_0^{(+)}(k;\mathbf{r}-\mathbf{r}')\;U(\mathbf{r}')\,\psi(\mathbf{k};\mathbf{r}')\,,$$

where

$$\mathcal{G}_0^{(+)}(k;\mathbf{r}) = -\frac{1}{4\pi}\frac{\exp\left(ik|\mathbf{r}|\right)}{|\mathbf{r}|}\,.$$

Show that the corresponding scattering amplitude is given by

$$f(\mathbf{k},\mathbf{k}') = -\frac{m\lambda}{2\pi\hbar^2}\int d^3r'\,\exp\left(-i\mathbf{k}'\cdot\mathbf{r}'\right)\,U(\mathbf{r}')\,\psi(\mathbf{k};\mathbf{r}')\,.$$

By expanding the wavefunction in powers of λ and keeping only the leading term, calculate the leading-order contribution to the differential cross-section, and evaluate it for the case of the Yukawa potential

$$V(\mathbf{r}) = \lambda\frac{\exp(-\mu r)}{r}\,.$$

By taking a suitable limit, obtain the differential cross-section for quantum scattering in the Coulomb potential $V_C(r)$ defined in Part (a) above, correct to leading order in an expansion in powers of the constant $\tilde{\alpha} = e^2/4\pi\epsilon_0$. Express your answer as a function of the particle energy E and scattering angle θ, and compare it to the corresponding classical cross-section calculated in Part (a).

35E Statistical Physics

Briefly describe the *microcanonical*, *canonical* and *grand canonical ensembles*. Why do they agree in the thermodynamic limit?

A harmonic oscillator in one spatial dimension has Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 .$$

Here p and x are the momentum and position of the oscillator, m is its mass and ω its frequency. The harmonic oscillator is placed in contact with a heat bath at temperature T. What is the relevant ensemble?

Treating the harmonic oscillator classically, compute the mean energy $\langle E \rangle$, the energy fluctuation ΔE^2 and the heat capacity C.

Treating the harmonic oscillator quantum mechanically, compute the mean energy $\langle E \rangle$, the energy fluctuation ΔE^2 and the heat capacity C.

In what limit of temperature do the classical and quantum results agree? Explain why they differ away from this limit. Describe an experiment for which this difference has implications.

36E General Relativity

Show how the geodesic equations and hence the Christoffel symbols $\Gamma^a{}_{bc}$ can be obtained from a Lagrangian.

In units with $c = 1$, the FLRW spacetime line element is

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)\,.$$

Show that $\Gamma^1{}_{01} = \dot{a}/a$.

You are given that, for the above metric, $G_0{}^0 = -3\dot{a}^2/a^2$ and $G_1{}^1 = -2\ddot{a}/a - \dot{a}^2/a^2$, where $G_a{}^b$ is the Einstein tensor, which is diagonal. Verify by direct calculation that $\nabla_b G_a{}^b = 0$.

Solve the vacuum Einstein equations in the presence of a cosmological constant to determine the form of $a(t)$.

37B Fluid Dynamics II

Air is blown over the surface of a large, deep reservoir of water in such a way as to exert a tangential stress in the x-direction of magnitude Kx^2 for $x > 0$, with $K > 0$. The water is otherwise at rest and occupies the region $y > 0$. The surface $y = 0$ remains flat.

Find order-of-magnitude estimates for the boundary-layer thickness $\delta(x)$ and tangential surface velocity $U(x)$ in terms of the relevant physical parameters.

Using the boundary-layer equations, find the ordinary differential equation governing the dimensionless function f defined in the streamfunction

$$\psi(x, y) = U(x)\delta(x)f(\eta), \quad \text{where } \eta = y/\delta(x).$$

What are the boundary conditions on f?

Does $f \to 0$ as $\eta \to \infty$? Why, or why not?

The total horizontal momentum flux $P(X)$ across the vertical line $x = X$ is proportional to X^a for $X > 0$. Find the exponent a. By considering the steadiness of the momentum balance in the region $0 < x < X$, explain why the value of a is consistent with the form of the stress exerted on the boundary.

38C Waves

The function $\phi(x, t)$ satisfies the equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^4 \phi}{\partial x^2 \partial t^2} \,.$$

Derive the dispersion relation, and sketch graphs of frequency, phase velocity and group velocity as functions of the wavenumber. In the case of a localised initial disturbance, will it be the shortest or the longest waves that are to be found at the front of a dispersing wave packet? Do the wave crests move faster or slower than the wave packet?

Give the solution to the initial-value problem for which at $t = 0$

$$\phi = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk \quad \text{and} \quad \frac{\partial \phi}{\partial t} = 0\,,$$

and $\phi(x, 0)$ is real. Use the method of stationary phase to obtain an approximation for $\phi(Vt, t)$ for fixed $0 < V < 1$ and large t. If, in addition, $\phi(x, 0) = \phi(-x, 0)$, deduce an approximation for the sequence of times at which $\phi(Vt, t) = 0$.

You are given that $\phi(t, t)$ decreases like $t^{-1/4}$ for large t. Give a brief physical explanation why this rate of decay is slower than for $0 < V < 1$. What can be said about $\phi(Vt, t)$ for large t if $V > 1$? [Detailed calculation is not required in these cases.]

[*You may assume that* $\displaystyle\int_{-\infty}^{\infty} e^{-au^2} du = \sqrt{\frac{\pi}{a}}$ *for* $\mathrm{Re}(a) \geqslant 0,\ a \neq 0$.]

39D Numerical Analysis

Consider the one-dimensional advection equation

$$u_t = u_x\,, \quad -\infty < x < \infty\,, \quad t \geqslant 0\,,$$

subject to an initial condition $u(x,0) = \varphi(x)$. Consider discretization of this equation with finite differences on an equidistant space-time $\{(mh, nk),\ m \in \mathbb{Z},\ n \in \mathbb{Z}^+\}$ with step size $h > 0$ in space and step size $k > 0$ in time. Define the *Courant number* μ and explain briefly how such a discretization can be used to derive numerical schemes in which solutions $u_m^n \approx u(mh, nk)$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ satisfy equations of the form

$$\sum_{i=r}^{s} a_i u_{m+i}^{n+1} = \sum_{i=r}^{s} b_i u_{m+i}^{n}\,, \tag{1}$$

where the coefficients a_i, b_i are independent of m, n.

(i) Define the *order* of a numerical scheme such as (1). Define what a *convergent* numerical scheme is. Explain the notion of *stability* and state the Lax equivalence theorem that connects convergence and stability of numerical schemes for linear partial differential equations.

(ii) Consider the following example of (1):

$$u_m^{n+1} = u_m^n + \frac{\mu}{2}(u_{m+1}^n - u_{m-1}^n) + \frac{\mu^2}{2}(u_{m+1}^n - 2u_m^n + u_{m-1}^n)\,. \tag{2}$$

Determine conditions on μ such that the scheme (2) is stable and convergent. What is the order of this scheme?

END OF PAPER