UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday, 30 May, 2014 9:00 am to 12:00 noon

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A****,* ***B****,* ***C****, ...,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Number Theory

Define what it means for a number N to be a *pseudoprime* to the base b.

Show that if there is a base b to which N is not a pseudoprime, then N is a pseudoprime to at most half of all possible bases.

Let n be an integer greater than 1 such that $F_n = 2^{2^n} + 1$ is composite. Show that F_n is a pseudoprime to the base 2.

2G Topics in Analysis

(i) State Brouwer's fixed point theorem in the plane and an equivalent theorem concerning mapping a triangle T to its boundary ∂T.

(ii) Let A be a 3×3 matrix with positive real entries. Use the theorems you stated in (i) to prove that A has an eigenvector with positive entries.

3F Geometry and Groups

Let $G \leqslant SO(3)$ be a finite group. Suppose G does not preserve any plane in $\mathbb{R}^3$. Show that for any point p in the unit sphere $S^2 \subset \mathbb{R}^3$, the stabiliser $\mathrm{Stab}_G(p)$ contains at most 5 elements.

4I Coding and Cryptography

State and prove Gibbs' inequality.

Show that, for a pair of discrete random variables X and Y, each taking finitely many values, the joint entropy $H(X, Y)$ satisfies

$$H(X, Y) \leqslant H(X) + H(Y) \ ,$$

with equality precisely when X and Y are independent.

5K Statistical Modelling

Write down the model being fitted by the following R command, where $y \in \{0, 1, 2, \ldots\}^n$ and X is an $n \times p$ matrix with real-valued entries.

```
fit <- glm(y ~ X, family = poisson)
```

Write down the log-likelihood for the model. Explain why the command

```
sum(y) - sum(predict(fit, type = "response"))
```

gives the answer 0, by arguing based on the log-likelihood you have written down. [*Hint: Recall that if $Z \sim Pois(\mu)$ then*

$$\mathbb{P}(Z = k) = \frac{\mu^k e^{-\mu}}{k!}$$

for $k \in \{0, 1, 2, \ldots\}$.]

6B Mathematical Biology

A population model for two species is given by

$$\begin{aligned} \frac{dN}{dt} &= aN - bNP - kN^2\,, \\ \frac{dP}{dt} &= -dP + cNP\,, \end{aligned}$$

where a, b, c, d and k are positive parameters. Show that this may be rescaled to

$$\begin{aligned} \frac{du}{d\tau} &= u(1 - v - \beta u)\,, \\ \frac{dv}{d\tau} &= -\alpha v(1 - u)\,, \end{aligned}$$

and give α and β in terms of the original parameters.

For $\beta < 1$ find all fixed points in $u \geqslant 0$, $v \geqslant 0$, and analyse their stability. Assuming that both populations are present initially, what does this suggest will be the long-term outcome?

7D Dynamical Systems

Consider the system

$$\begin{aligned} \dot{x} &= y + xy\,, \\ \dot{y} &= x - \tfrac{3}{2}y + x^2 . \end{aligned}$$

Show that the origin is a hyperbolic fixed point and find the stable and unstable invariant subspaces of the linearised system.

Calculate the stable and unstable manifolds correct to quadratic order, expressing y as a function of x for each.

8B Further Complex Methods

Show that the Cauchy–Riemann equations for $f : \mathbb{C} \to \mathbb{C}$ are equivalent to

$$\frac{\partial f}{\partial \bar{z}} = 0\,,$$

where $z = x + iy$, and $\partial/\partial\bar{z}$ should be defined in terms of $\partial/\partial x$ and $\partial/\partial y$. Use Green's theorem, together with the formula $dz\,d\bar{z} = -2i\,dx\,dy$, to establish the generalised Cauchy formula

$$\oint_\gamma f(z,\bar{z})\,dz = -\int\!\!\int_D \frac{\partial f}{\partial \bar{z}}\,dz\,d\bar{z}\,,$$

where γ is a contour in the complex plane enclosing the region D and f is sufficiently differentiable.

9A Classical Dynamics

Consider a one-dimensional dynamical system with generalized coordinate and momentum (q, p).

(a) Define the *Poisson bracket* $\{f, g\}$ of two functions $f(q, p, t)$ and $g(q, p, t)$.

(b) Verify the Leibniz rule

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

(c) Explain what is meant by a *canonical transformation* $(q, p) \to (Q, P)$.

(d) State the condition for a transformation $(q, p) \to (Q, P)$ to be canonical in terms of the Poisson bracket $\{Q, P\}$. Use this to determine whether or not the following transformations are canonical:

(i) $Q = \dfrac{q^2}{2}\,,\ P = \dfrac{p}{q}\,,$

(ii) $Q = \tan q\,,\ P = p\cos q\,,$

(iii) $Q = \sqrt{2q}\,e^{t}\cos p\,,\ P = \sqrt{2q}\,e^{-t}\sin p\,.$

10E Cosmology

Which particle states are expected to be relativistic and which interacting when the temperature T of the early universe satisfies

(i) $10^{10}\,\mathrm{K} < T < 5\times 10^{10}\,\mathrm{K}$,

(ii) $5\times 10^{9}\,\mathrm{K} < T < 10^{10}\,\mathrm{K}$,

(iii) $T < 5\times 10^{9}\,\mathrm{K}$?

Calculate the total spin weight factor, g_*, of the relativistic particles and the total spin weight factor, g_I, of the interacting particles, in each of the three temperature intervals.

What happens when the temperature falls below $5\times 10^{9}\,\mathrm{K}$? Calculate the ratio of the temperatures of neutrinos and photons. Find the effective value of g_* after the universe cools below this temperature. [*Note that the equilibrium entropy density is given by $s = (\rho c^2 + P)/T$, where ρ is the density and P is the pressure.*]

SECTION II

11F Geometry and Groups

Prove that an orientation-preserving isometry of the ball-model of hyperbolic space $\mathbb{H}^3$ which fixes the origin is an element of $SO(3)$. Hence, or otherwise, prove that a finite subgroup of the group of orientation-preserving isometries of hyperbolic space $\mathbb{H}^3$ has a common fixed point.

Can an infinite non-cyclic subgroup of the isometry group of $\mathbb{H}^3$ have a common fixed point? Can any such group be a Kleinian group? Justify your answers.

12I Coding and Cryptography

Describe, briefly, either the RSA or the Elgamal public key cipher. You should explain, without proof, why it is believed to be difficult to break the cipher you describe.

How can such a cipher be used to sign messages? You should explain how the intended recipient of the message can (a) know from whom it came; (b) know that the message has not been changed; and (c) demonstrate that the sender must have signed it.

Let $I_0, I_1, \ldots, I_N$ be friendly individuals each of whom has a public key cipher. I_0 wishes to send a message to I_N by passing it first to I_1, then I_1 passes it to I_2, I_2 to I_3, until finally it is received by I_N. At each stage the message can be modified to show from whom it was received and to whom it is sent. Devise a way in which these modifications can be made so that I_N can be confident both of the content of the original message and that the message has been passed through the intermediaries $I_1, I_2, \ldots, I_{N-1}$ in that order and has not been modified by an enemy agent. Assume that it takes a negligible time to transmit a message from I_k to I_{k+1} for each k, but the time needed to modify a message is not negligible.

13K Statistical Modelling

Consider the normal linear model where the n-vector of responses Y satisfies $Y = X\beta + \varepsilon$ with $\varepsilon \sim N_n(0, \sigma^2 I)$. Here X is an $n \times p$ matrix of predictors with full column rank where $n \geqslant p+3$, and $\beta \in \mathbb{R}^p$ is an unknown vector of regression coefficients. Let X_0 be the matrix formed from the first $p_0 < p$ columns of X, and partition β as $\beta = (\beta_0^T, \beta_1^T)^T$ where $\beta_0 \in \mathbb{R}^{p_0}$ and $\beta_1 \in \mathbb{R}^{p-p_0}$. Denote the orthogonal projections onto the column spaces of X and X_0 by P and P_0 respectively.

It is desired to test the null hypothesis $H_0 : \beta_1 = 0$ against the alternative hypothesis $H_1 : \beta_1 \neq 0$. Recall that the F-test for testing H_0 against H_1 rejects H_0 for large values of

$$F = \frac{\|(P - P_0)Y\|^2/(p - p_0)}{\|(I - P)Y\|^2/(n - p)}.$$

Show that $(I - P)(P - P_0) = 0$, and hence prove that the numerator and denominator of F are independent under either hypothesis.

Show that

$$\mathbb{E}_{\beta,\sigma^2}(F) = \frac{(n - p)(\tau^2 + 1)}{n - p - 2},$$

where $\tau^2 = \dfrac{\|(P - P_0)X\beta\|^2}{(p - p_0)\sigma^2}$.

[*In this question you may use the following facts without proof:* $P - P_0$ is an orthogonal projection with rank $p - p_0$; any $n \times n$ orthogonal projection matrix Π satisfies $\|\Pi\varepsilon\|^2 \sim \sigma^2\chi^2_\nu$, where $\nu = \operatorname{rank}(\Pi)$; and if $Z \sim \chi^2_\nu$ then $\mathbb{E}(Z^{-1}) = (\nu - 2)^{-1}$ when $\nu > 2$.]

14B Further Complex Methods

Obtain solutions of the second-order ordinary differential equation

$$zw'' - w = 0$$

in the form

$$w(z) = \int_\gamma f(t)e^{-zt}\,dt,$$

where the function f and the choice of contour γ should be determined from the differential equation.

Show that a non-trivial solution can be obtained by choosing γ to be a suitable closed contour, and find the resulting solution in this case, expressing your answer in the form of a power series.

Describe a contour γ that would provide a second linearly independent solution for the case $\operatorname{Re}(z) > 0$.

15E Cosmology

What are the cosmological *flatness* and *horizon* problems? Explain what form of time evolution of the cosmological expansion scale factor $a(t)$ must occur during a period of inflationary expansion in a Friedmann universe. How can inflation solve the horizon and flatness problems? [You may assume an equation of state where pressure P is proportional to density ρ.]

The universe has Hubble expansion rate $H = \dot{a}/a$ and contains only a scalar field ϕ with self-interaction potential $V(\phi) > 0$. The density and pressure are given by

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \\ P &= \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \end{aligned}$$

in units where $c = \hbar = 1$. Show that the conservation equation

$$\dot{\rho} + 3H(\rho + P) = 0$$

requires

$$\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0\,.$$

If the Friedmann equation has the form

$$3H^2 = 8\pi G\rho$$

and the scalar-field potential has the form

$$V(\phi) = V_0 e^{-\lambda\phi}\,,$$

where V_0 and λ are positive constants, show that there is an exact cosmological solution with

$$\begin{aligned} a(t) &\propto t^{16\pi G/\lambda^2}\,, \\ \phi(t) &= \phi_0 + \frac{2}{\lambda}\ln(t)\,, \end{aligned}$$

where ϕ_0 is a constant. Find the algebraic relation between λ, V_0 and ϕ_0. Show that a solution only exists when $0 < \lambda^2 < 48\pi G$. For what range of values of λ^2 does inflation occur? Comment on what happens when $\lambda \to 0$.

16I Logic and Set Theory

Explain what is meant by saying that a binary relation $r \subseteq a \times a$ is *well-founded.* Show that r is well-founded if and only if, for any set b and any function $f\colon \mathcal{P}b \to b$, there exists a unique function $g\colon a \to b$ satisfying

$$g(x) = f(\{g(y) \mid \langle y, x\rangle \in r\})$$

for all $x \in a$. [*Hint: For 'if', it suffices to take* $b = \{0, 1\}$*, with* $f : \mathcal{P}b \to b$ *defined by* $f(b') = 1 \Leftrightarrow 1 \in b'$.]

17I Graph Theory

Show that a graph is bipartite if and only if all of its cycles are of even length.

Show that a bridgeless plane graph is bipartite if and only if all of its faces are of even length.

Let G be an Eulerian plane graph. Show that the faces of G can be coloured with two colours so that no two contiguous faces have the same colour. Deduce that it is possible to assign a direction to each edge of G in such a way that the edges around each face form a directed cycle.

18H Galois Theory

What is meant by the statement that L is a *splitting field* for $f \in K[X]$?

Show that if $f \in K[X]$, then there exists a splitting field for f over K. Explain the sense in which a splitting field for f over K is unique.

Determine the degree $[L : K]$ of a splitting field L of the polynomial $f = X^4 - 4X^2 + 2$ over K in the cases (i) $K = \mathbb{Q}$, (ii) $K = \mathbb{F}_5$, and (iii) $K = \mathbb{F}_7$.

19H Representation Theory

(i) Let K be any field and let $\lambda \in K$. Let $J_{\lambda,n}$ be the $n \times n$ Jordan block

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

Compute $J_{\lambda,n}^r$ for each $r \geqslant 0$.

(ii) Let G be a cyclic group of order N, and let K be an algebraically closed field of characteristic $p \geqslant 0$. Determine all the representations of G on vector spaces over K, up to equivalence. Which are irreducible? Which do not split as a direct sum $W \oplus W'$, with $W \neq 0$ and $W' \neq 0$?

20F Number Fields

State a result involving the discriminant of a number field that implies that the class group is finite.

Use Dedekind's theorem to factor 2, 3, 5 and 7 into prime ideals in $K = \mathbb{Q}(\sqrt{-34})$. By factoring $1 + \sqrt{-34}$ and $4 + \sqrt{-34}$, or otherwise, prove that the class group of K is cyclic, and determine its order.

21F Algebraic Topology

Define what it means for a map $p : \widetilde{X} \to X$ to be a *covering space*. State the homotopy lifting lemma.

Let $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a based covering space and let $f : (Y, y_0) \to (X, x_0)$ be a based map from a path-connected and locally path-connected space. Show that there is a based lift $\tilde{f} : (Y, y_0) \to (\widetilde{X}, \tilde{x}_0)$ of f if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$.

22G Linear Analysis

Let X and Y be normed spaces. What is an *isomorphism* between X and Y? Show that a bounded linear map $T\colon X \to Y$ is an isomorphism if and only if T is surjective and there is a constant $c > 0$ such that $\|Tx\| \geqslant c\|x\|$ for all $x \in X$. Show that if there is an isomorphism $T\colon X \to Y$ and X is complete, then Y is complete.

Show that two normed spaces of the same finite dimension are isomorphic. [You may assume without proof that any two norms on a finite-dimensional space are equivalent.] *Briefly* explain why this implies that every finite-dimensional space is complete, and every closed and bounded subset of a finite-dimensional space is compact.

Let Z and F be subspaces of a normed space X with $Z \cap F = \{0\}$. Assume that Z is closed in X and F is finite-dimensional. Prove that $Z + F$ is closed in X. [*Hint: First show that the function* $x \mapsto d(x, Z) = \inf\{\|x - z\| : z \in Z\}$ *restricted to the unit sphere of* F *achieves its minimum.*]

23H Riemann Surfaces

If X is a Riemann surface and $p : Y \to X$ is a covering map of topological spaces, show that there is a conformal structure on Y such that $p : Y \to X$ is analytic.

Let $f(z)$ be the complex polynomial $z^5 - 1$. Consider the subspace R of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ given by the equation $w^2 = f(z)$, where (z, w) denotes coordinates in $\mathbb{C}^2$, and let $\pi : R \to \mathbb{C}$ be the restriction of the projection map onto the first factor. Show that R has the structure of a Riemann surface which makes π an analytic map. If τ denotes projection onto the second factor, show that τ is also analytic. [You may assume that R is connected.]

Find the ramification points and the branch points of both π and τ. Compute also the ramification indices at the ramification points.

Assuming that it is possible to add a point P to R so that $X = R \cup \{P\}$ is a compact Riemann surface and τ extends to a holomorphic map $\tau : X \to \mathbb{C}_\infty$ such that $\tau^{-1}(\infty) = \{P\}$, compute the genus of X.

24H Algebraic Geometry

Let k be an algebraically closed field and $n \geqslant 1$. We say that $f \in k[x_1, \ldots, x_n]$ is *singular* at $p \in \mathbf{A}^n$ if either p is a singularity of the hypersurface $\{f = 0\}$ or f has an irreducible factor h of multiplicity strictly greater than one with $h(p) = 0$. Given $d \geqslant 1$, let $X = \{f \in k[x_1, \ldots, x_n] \mid \deg f \leqslant d\}$ and let

$$Y = \{(f,p) \in X \times \mathbf{A}^n \mid f \text{ is singular at } p\}.$$

(i) Show that $X \simeq \mathbf{A}^N$ for some N (you need not determine N) and that Y is a Zariski closed subvariety of $X \times \mathbf{A}^n$.

(ii) Show that the fibres of the projection map $Y \to \mathbf{A}^n$ are linear subspaces of dimension $N - (n+1)$. Conclude that $\dim Y < \dim X$.

(iii) Hence show that $\{f \in X \mid \deg f = d,\ Z(f) \text{ smooth}\}$ is dense in X.

[You may use standard results from lectures if they are accurately quoted.]

25G Differential Geometry

Define the concepts of (smooth) *manifold* and *manifold with boundary* for subsets of $\mathbf{R}^N$.

Let $X \subset \mathbf{R}^6$ be the subset defined by the equations

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1, \quad x_4^2 - x_5^2 - x_6^2 = -1.$$

Prove that X is a manifold of dimension four.

For $a > 0$, let $B(a) \subset \mathbf{R}^6$ denote the spherical ball $x_1^2 + \ldots + x_6^2 \leqslant a$. Prove that $X \cap B(a)$ is empty if $a < 2$, is a manifold diffeomorphic to $S^2 \times S^1$ if $a = 2$, and is a manifold with boundary if $a > 2$, with each component of the boundary diffeomorphic to $S^2 \times S^1$.

[You may quote without proof any general results from lectures that you may need.]

26K Probability and Measure

What is meant by the *Borel σ-algebra* on the real line $\mathbb{R}$?

Define the *Lebesgue measure* of a Borel subset of $\mathbb{R}$ using the concept of outer measure.

Let μ be the Lebesgue measure on $\mathbb{R}$. Show that, for any Borel set B which is contained in the interval $[0,1]$, and for any $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and disjoint intervals $I_1, \ldots, I_n$ contained in $[0,1]$ such that, for $A = I_1 \cup \cdots \cup I_n$, we have

$$\mu(A \triangle B) \leqslant \varepsilon,$$

where $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$.

Show that there does not exist a Borel set B contained in $[0,1]$ such that, for all intervals I contained in $[0,1]$,

$$\mu(B \cap I) = \mu(I)/2.$$

27J Applied Probability

(i) Explain what a Q-matrix is. Let Q be a Q-matrix. Define the notion of a *Markov chain* $(X_t, t \geqslant 0)$ in continuous time with Q-matrix given by Q, and give a construction of $(X_t, t \geqslant 0)$. [You are not required to justify this construction.]

(ii) A population consists of N_t individuals at time $t \geqslant 0$. We assume that each individual gives birth to a new individual at constant rate $\lambda > 0$. As the population is competing for resources, we assume that for each $n \geqslant 1$, if $N_t = n$, then any individual in the population at time t dies in the time interval $[t, t+h)$ with probability $\delta_n h + o(h)$, where $(\delta_n)_{n=1}^{\infty}$ is a given sequence satisfying $\delta_1 = 0$, $\delta_n > 0$ for $n \geqslant 2$. Formulate a Markov chain model for $(N_t, t \geqslant 0)$ and write down the Q-matrix explicitly. Then find a necessary and sufficient condition on $(\delta_n)_{n=1}^{\infty}$ so that the Markov chain has an invariant distribution. Compute the invariant distribution in the case where $\delta_n = \mu(n-1)$ and $\mu > 0$.

28J Principles of Statistics

State without proof the inequality known as the Cramér–Rao lower bound in a parametric model $\{f(\cdot,\theta) : \theta \in \Theta\}, \Theta \subseteq \mathbb{R}$. Give an example of a maximum likelihood estimator that attains this lower bound, and justify your answer.

Give an example of a parametric model where the maximum likelihood estimator based on observations $X_1, \ldots, X_n$ is biased. State without proof an analogue of the Cramér–Rao inequality for biased estimators.

Define the concept of a *minimax decision rule*, and show that the maximum likelihood estimator $\hat{\theta}_{MLE}$ based on $X_1, \ldots, X_n$ in a $N(\theta, 1)$ model is minimax for estimating $\theta \in \Theta = \mathbb{R}$ in quadratic risk.

29K Stochastic Financial Models

Suppose that $\bar{S}_t \equiv (S_t^0, \ldots, S_t^d)^T$ denotes the vector of prices of $d+1$ assets at times $t = 0, 1, \ldots$, and that $\bar{\theta}_t \equiv (\theta_t^0, \ldots, \theta_t^d)^T$ denotes the vector of the numbers of the $d+1$ different assets held by an investor from time $t-1$ to time t. Assuming that asset 0 is a bank account paying zero interest, that is, $S_t^0 = 1$ for all $t \geqslant 0$, explain what is meant by the statement that the portfolio process $(\bar{\theta}_t)_{t \geqslant 0}$ is *self-financing*. If the portfolio process is self-financing, prove that for any $t > 0$

$$\bar{\theta}_t \cdot \bar{S}_t - \bar{\theta}_0 \cdot \bar{S}_0 = \sum_{j=1}^{t} \theta_j \cdot \Delta S_j,$$

where $S_j \equiv (S_j^1, \ldots, S_j^d)^T$, $\Delta S_j = S_j - S_{j-1}$, and $\theta_j \equiv (\theta_j^1, \ldots, \theta_j^d)^T$.

Suppose now that the ΔS_t are independent with common $N(0, V)$ distribution. Let

$$F(z) = \inf E\left[\sum_{t \geqslant 1} (1-\beta)\beta^t \left\{ (\bar{\theta}_t \cdot \bar{S}_t - \bar{\theta}_0 \cdot \bar{S}_0)^2 + \sum_{j=1}^{t} |\Delta \theta_j|^2 \right\} \,\middle|\, \theta_0 = z \right],$$

where $\beta \in (0,1)$ and the infimum is taken over all self-financing portfolio processes $(\bar{\theta}_t)_{t \geqslant 0}$ with $\theta_0^0 = 0$. Explain why F should satisfy the equation

$$F(z) = \beta \inf_y \left[\, y \cdot Vy + |y - z|^2 + F(y) \, \right]. \qquad (*)$$

If Q is a positive-definite symmetric matrix satisfying the equation

$$Q = \beta (V + I + Q)^{-1}(V + Q),$$

show that $(*)$ has a solution of the form $F(z) = z \cdot Qz$.

30D Partial Differential Equations

State the Cauchy–Kovalevskaya theorem, including a definition of the term *non-characteristic.*

For which values of the real number a, and for which functions f, does the Cauchy–Kovalevskaya theorem ensure that the Cauchy problem

$$u_{tt} = u_{xx} + a u_{xxxx}\,, \quad u(x,0) = 0\,,\ u_t(x,0) = f(x) \tag{1}$$

has a local solution?

Now consider the Cauchy problem (1) in the case that $f(x) = \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{imx}$ is a smooth 2π-periodic function.

(i) Show that if $a \leqslant 0$ there exists a unique smooth solution u for all times, and show that for all $T \geqslant 0$ there exists a number $C = C(T) > 0$, independent of f, such that

$$\int_{-\pi}^{+\pi} |u(x,t)|^2 dx \leqslant C \int_{-\pi}^{+\pi} |f(x)|^2 dx \tag{2}$$

for all $t : |t| \leqslant T$.

(ii) If $a = 1$ does there exist a choice of $C = C(T)$ for which (2) holds? Give a full justification for your answer.

31C Asymptotic Methods

(a) Consider the integral

$$I(k) = \int_0^\infty f(t) e^{-kt} \, dt, \quad k > 0.$$

Suppose that $f(t)$ possesses an asymptotic expansion for $t \to 0^+$ of the form

$$f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n}, \qquad \alpha > -1, \;\; \beta > 0,$$

where a_n are constants. Derive an asymptotic expansion for $I(k)$ as $k \to \infty$ in the form

$$I(k) \sim \sum_{n=0}^{\infty} \frac{A_n}{k^{\gamma + \beta n}},$$

giving expressions for A_n and γ in terms of α, β, n and the gamma function. Hence establish the asymptotic approximation as $k \to \infty$

$$I_1(k) = \int_0^1 e^{kt} t^{-a} (1 - t^2)^{-b} dt \sim 2^{-b} \Gamma(1-b) e^k k^{b-1} \left(1 + \frac{(a + b/2)(1-b)}{k} \right),$$

where $a < 1, b < 1$.

(b) Using Laplace's method, or otherwise, find the leading-order asymptotic approximation as $k \to \infty$ for

$$I_2(k) = \int_0^\infty e^{-(2k^2/t + t^2/k)} \, dt \, .$$

[*You may assume that* $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$ *for* $\operatorname{Re} z > 0$,

and that $\int_{-\infty}^{\infty} e^{-qt^2} dt = \sqrt{\pi/q}$ *for* $q > 0$.]

32D Integrable Systems

Consider the coordinate transformation

$$g^\epsilon : (x,u) \mapsto (\tilde{x},\tilde{u}) = (x\cos\epsilon - u\sin\epsilon, x\sin\epsilon + u\cos\epsilon)\,.$$

Show that g^ϵ defines a one-parameter group of transformations. Define what is meant by the *generator* V of a one-parameter group of transformations and compute it for the above case.

Now suppose $u = u(x)$. Explain what is meant by the *first prolongation* $\mathrm{pr}^{(1)}g^\epsilon$ of g^ϵ. Compute $\mathrm{pr}^{(1)}g^\epsilon$ in this case and deduce that

$$\mathrm{pr}^{(1)}V = V + (1+u_x^2)\frac{\partial}{\partial u_x}\,. \qquad (\star)$$

Similarly find $\mathrm{pr}^{(2)}V$.

Define what is meant by a *Lie point symmetry* of the first-order differential equation $\Delta[x,u,u_x] = 0$. Describe this condition in terms of the vector field that generates the Lie point symmetry. Consider the case

$$\Delta[x,u,u_x] \equiv u_x - \frac{u + xf(x^2+u^2)}{x - uf(x^2+u^2)}\,,$$

where f is an arbitrary smooth function of one variable. Using $(\star)$, show that g^ϵ generates a Lie point symmetry of the corresponding differential equation.

33A Principles of Quantum Mechanics

Let $\hat{x}, \hat{p}$ and $H(\hat{x}, \hat{p}) = \hat{p}^2/2m + V(\hat{x})$ be the position operator, momentum operator and Hamiltonian for a particle moving in one dimension. Let $|\psi\rangle$ be the state vector for the particle. The position and momentum eigenstates have inner products

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp(ipx/\hbar)\,, \qquad \langle x|x'\rangle = \delta(x-x') \qquad \text{and} \qquad \langle p|p'\rangle = \delta(p-p')\,.$$

Show that

$$\langle x|\hat{p}|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\psi(x) \quad \text{and} \quad \langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\tilde{\psi}(p)\,,$$

where $\psi(x)$ and $\tilde{\psi}(p)$ are the wavefunctions in the position representation and momentum representation, respectively. Show how $\psi(x)$ and $\tilde{\psi}(p)$ may be expressed in terms of each other.

For general $V(\hat{x})$, express $\langle p|V(\hat{x})|\psi\rangle$ in terms of $\tilde{\psi}(p)$, and hence write down the time-independent Schrödinger equation in the momentum representation satisfied by $\tilde{\psi}(p)$.

Consider now the case $V(x) = -(\hbar^2\lambda/m)\delta(x),\ \lambda > 0$. Show that there is a bound state with energy $E = -\varepsilon,\ \varepsilon > 0$, with wavefunction $\tilde{\psi}(p)$ satisfying

$$\tilde{\psi}(p) = \frac{\hbar\lambda}{\pi}\frac{1}{2m\varepsilon + p^2}\int_{-\infty}^{\infty} \tilde{\psi}(p')\,dp'\,.$$

Hence show that there is a unique value for ε and determine this value.

34A Applications of Quantum Mechanics

A particle of mass m scatters on a localised potential well $V(x)$ in one dimension. With reference to the asymptotic behaviour of the wavefunction as $x \to \pm\infty$, define the *reflection* and *transmission amplitudes*, r and t, for a right-moving incident particle of wave number k. Define also the corresponding amplitudes, r' and t', for a left-moving incident particle of wave number k. Derive expressions for r' and t' in terms of r and t.

(a) Define the *S-matrix*, giving its elements in terms of r and t. Using the relation

$$|r|^2 + |t|^2 = 1$$

(which you need not derive), show that the S-matrix is unitary. How does the S-matrix simplify if the potential well satisfies $V(-x) = V(x)$?

(b) Consider the potential well

$$V(x) = -\frac{3\hbar^2}{m}\frac{1}{\cosh^2(x)} .$$

The corresponding Schrödinger equation has an exact solution

$$\psi_k(x) = \exp(ikx)\left[3\tanh^2(x) - 3ik\tanh(x) - (1+k^2)\right] ,$$

with energy $E = \hbar^2 k^2/2m$, for every real value of k. [You do not need to verify this.] Find the S-matrix for scattering on this potential. What special feature does the scattering have in this case?

(c) Explain the connection between singularities of the S-matrix and bound states of the potential well. By analytic continuation of the solution $\psi_k(x)$ to appropriate complex values of k, find the wavefunctions and energies of the bound states of the well. [You do not need to normalise the wavefunctions.]

35E Statistical Physics

Write down the equation of state and the internal energy of a monatomic ideal gas.

Describe the meaning of an adiabatic process. Derive the equation for an adiabatic process in the pressure–volume (P, V) plane for a monatomic ideal gas.

Briefly describe the Carnot cycle. Sketch the Carnot cycle in the (P, V) plane and in the temperature–entropy (T, S) plane.

The Diesel cycle is an idealised version of the process realised in the Diesel engine. It consists of the following four reversible steps:

$A \rightarrow B$: Adiabatic compression
$B \rightarrow C$: Expansion at constant pressure
$C \rightarrow D$: Adiabatic expansion
$D \rightarrow A$: Cooling at constant volume.

Sketch the Diesel cycle for a monatomic gas in the (P, V) plane and the (T, S) plane. Determine the equations for the curves $B \rightarrow C$ and $D \rightarrow A$ in the (T, S) plane.

The efficiency η of the cycle is defined as

$$\eta = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}} \, ,$$

where Q_{in} is the heat entering the gas in step $B \rightarrow C$ and Q_{out} is the heat leaving the gas in step $D \rightarrow A$. Calculate η as a function of the temperatures at points A, B, C and D.

36C Electrodynamics

(i) Starting from the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A^\mu = (\phi/c, \mathbf{A})$ is the 4-vector potential with components such that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}\,,$$

derive the transformation laws for the components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ under the standard Lorentz boost $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ with

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(ii) Two point charges, each with electric charge q, are at rest and separated by a distance d in some inertial frame S. By transforming the fields from the rest frame S, calculate the magnitude and direction of the force between the two charges in an inertial frame in which the charges are moving with speed βc in a direction perpendicular to their separation.

(iii) The 4-force for a particle with 4-momentum p^μ is $F^\mu = dp^\mu/d\tau$, where τ is proper time. Show that the components of F^μ in an inertial frame in which the particle has 3-velocity $\mathbf{v}$ are

$$F^\mu = \gamma\left(\mathbf{F}\cdot\mathbf{v}/c, \mathbf{F}\right)\,,$$

where $\gamma = (1 - \mathbf{v}\cdot\mathbf{v}/c^2)^{-1/2}$ and $\mathbf{F}$ is the 3-force acting on the particle. Hence verify that your result in (ii) above is consistent with Lorentz transforming the electromagnetic 3-force from the rest frame S.

37E General Relativity

For a timelike geodesic in the equatorial plane ($\theta = \frac{1}{2}\pi$) of the Schwarzschild space-time with line element

$$ds^2 = -(1 - r_s/r)c^2dt^2 + (1 - r_s/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,,$$

derive the equation

$$\tfrac{1}{2}\dot{r}^2 + V(r) = \tfrac{1}{2}(E/c)^2\ ,$$

where

$$\frac{2V(r)}{c^2} = 1 - \frac{r_s}{r} + \frac{h^2}{c^2r^2} - \frac{h^2r_s}{c^2r^3}$$

and h and E are constants. The dot denotes the derivative with respect to an affine parameter τ satisfying $c^2d\tau^2 = -ds^2$.

Given that there is a stable circular orbit at $r = R$, show that

$$\frac{h^2}{c^2} = \frac{R^2\epsilon}{2 - 3\epsilon}\ ,$$

where $\epsilon = r_s/R$.

Compute Ω, the orbital angular frequency (with respect to τ).

Show that the angular frequency ω of small radial perturbations is given by

$$\frac{\omega^2R^2}{c^2} = \frac{\epsilon(1 - 3\epsilon)}{2 - 3\epsilon}\,.$$

Deduce that the rate of precession of the perihelion of the Earth's orbit, $\Omega - \omega$, is approximately $3\Omega^3T^2$, where T is the time taken for light to travel from the Sun to the Earth. [You should assume that the Earth's orbit is approximately circular, with $r_s/R \ll 1$ and $E \simeq c^2$.]

38B Fluid Dynamics II

A particle of arbitrary shape and volume $4\pi a^3/3$ moves at velocity $\mathbf{U}(t)$ through an unbounded incompressible fluid of density ρ and viscosity μ. The Reynolds number of the flow is very small so that the inertia of the fluid can be neglected. Show that the particle experiences a force $\mathbf{F}(t)$ due to the surface stresses given by

$$F_i(t) = -\mu a A_{ij} U_j(t),$$

where A_{ij} is a dimensionless second-rank tensor determined solely by the shape and orientation of the particle. State the reason why A_{ij} must be positive definite.

Show further that, if the particle has the same reflectional symmetries as a cube, then

$$A_{ij} = \lambda \delta_{ij}.$$

Let b be the radius of the smallest sphere that contains the particle (still assuming cubic symmetry). By considering the Stokes flow associated with this sphere, suitably extended, and using the minimum dissipation theorem (which should be stated carefully), show that

$$\lambda \leqslant 6\pi b/a.$$

[You may assume the expression for the Stokes drag on a sphere.]

39C Waves

State the equations that relate strain to displacement and stress to strain in a uniform, linear, isotropic elastic solid with Lamé moduli λ and μ. In the absence of body forces, the Cauchy momentum equation for the infinitesimal displacements $\mathbf{u}(\mathbf{x}, t)$ is

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \ ,$$

where ρ is the density and $\boldsymbol{\sigma}$ the stress tensor. Show that both the dilatation $\boldsymbol{\nabla} \cdot \mathbf{u}$ and the rotation $\boldsymbol{\nabla} \wedge \mathbf{u}$ satisfy wave equations, and find the wave-speeds c_P and c_S.

A plane harmonic P-wave with wavevector $\mathbf{k}$ lying in the (x, z) plane is incident from $z < 0$ at an oblique angle on the planar interface $z = 0$ between two elastic solids with different densities and elastic moduli. Show in a diagram the directions of all the reflected and transmitted waves, labelled with their polarisations, assuming that none of these waves are evanescent. State the boundary conditions on components of $\mathbf{u}$ and $\boldsymbol{\sigma}$ that would, in principle, determine the amplitudes.

Now consider a plane harmonic P-wave of unit amplitude incident with $\mathbf{k} = k(\sin\theta, 0, \cos\theta)$ on the interface $z = 0$ between two elastic (and inviscid) *liquids* with wave-speed c_P and modulus λ in $z < 0$ and wave-speed c'_P and modulus λ' in $z > 0$. Obtain solutions for the reflected and transmitted waves. Show that the amplitude of the reflected wave is zero if

$$\sin^2\theta = \frac{Z'^2 - Z^2}{Z'^2 - (c'_P Z/c_P)^2} \ ,$$

where $Z = \lambda/c_P$ and $Z' = \lambda'/c'_P$.

40D Numerical Analysis

(i) Consider the numerical approximation of the boundary-value problem

$$u'' = f\,, \qquad u : [0,1] \to \mathbb{R}\,,$$

$$u(0) = \varphi_0\,, \quad u(1) = \varphi_1\,,$$

where φ_0, φ_1 are given constants and f is a given smooth function on $[0,1]$. A grid $\{x_1, x_2, \ldots, x_N\}$, $N \geqslant 3$, on $[0,1]$ is given by

$$x_1 = \alpha_1 h\,, \qquad x_i = x_{i-1} + h \text{ for } i = 2, \ldots, N-1\,, \qquad x_N = 1 - \alpha_2 h\,,$$

where $0 < \alpha_1, \alpha_2 < 1$, $\alpha_1 + \alpha_2 = 1$ and $h = 1/N$. Derive finite-difference approximations for $u''(x_i)$, for $i = 1, \ldots, N$, using at most one neighbouring grid point of x_i on each side. Hence write down a numerical scheme to solve the problem, displaying explicitly the entries of the system matrix A in the resulting system of linear equations $Au = b$, $A \in \mathbb{R}^{N \times N}$, $u, b \in \mathbb{R}^N$. What is the overall order of this numerical scheme? Explain briefly one strategy by which the order could be improved with the same grid.

(ii) Consider the numerical approximation of the boundary-value problem

$$\nabla^2 u = f\,, \qquad u : \Omega \to \mathbb{R}\,,$$

$$u(x) = 0 \;\text{ for all } x \in \partial\Omega\,,$$

where $\Omega \subset \mathbb{R}^2$ is an arbitrary, simply connected bounded domain with smooth boundary $\partial\Omega$, and f is a given smooth function. Define the *9-point formula* used to approximate the Laplacian. Using this formula and an equidistant grid inside Ω, define a numerical scheme for which the system matrix is symmetric and negative definite. Prove that the system matrix of your scheme has these properties for all choices of ordering of the grid points.

END OF PAPER