UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 5 June, 2014 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

Use Euclid's algorithm to determine d, the greatest common divisor of 203 and 147, and to express it in the form $203x + 147y$ for integers x, y. Hence find all solutions in integers x, y of the equation $203x + 147y = d$.

How many integers n are there with $1 \leqslant n \leqslant 2014$ and $21n \equiv 25 \pmod{29}$?

2E Numbers and Sets

Define the binomial coefficients $\binom{n}{k}$, for integers n, k satisfying $n \geqslant k \geqslant 0$. Prove directly from your definition that if $n > k \geqslant 0$ then

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

and that for every $m \geqslant 0$ and $n \geqslant 0$,

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}.$$

3C Dynamics and Relativity

A particle of mass m has charge q and moves in a constant magnetic field $\mathbf{B}$. Show that the particle's path describes a helix. In which direction is the axis of the helix, and what is the particle's rotational angular frequency about that axis?

4C Dynamics and Relativity

What is a *4-vector*? Define the inner product of two 4-vectors and give the meanings of the terms *timelike*, *null* and *spacelike*. How do the four components of a 4-vector change under a Lorentz transformation of speed v? [Without loss of generality, you may take the velocity of the transformation to be along the positive x-axis.]

Show that a 4-vector that is timelike in one frame of reference is also timelike in a second frame of reference related by a Lorentz transformation. [Again, you may without loss of generality take the velocity of the transformation to be along the positive x-axis.]

Show that any null 4-vector may be written in the form $a(1, \hat{\mathbf{n}})$ where a is real and $\hat{\mathbf{n}}$ is a unit 3-vector. Given any two null 4-vectors that are *future-pointing*, that is, which have positive time-components, show that their sum is either null or timelike.

SECTION II

5E Numbers and Sets

What does it mean to say that the sequence of real numbers (x_n) converges to the limit x? What does it mean to say that the series $\sum_{n=1}^{\infty} x_n$ converges to s?

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series of positive real numbers. Suppose that (x_n) is a sequence of positive real numbers such that for every $n \geqslant 1$, either $x_n \leqslant a_n$ or $x_n \leqslant b_n$. Show that $\sum_{n=1}^{\infty} x_n$ is convergent.

Show that $\sum_{n=1}^{\infty} 1/n^2$ is convergent, and that $\sum_{n=1}^{\infty} 1/n^{\alpha}$ is divergent if $\alpha \leqslant 1$.

Let (x_n) be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} n^2 x_n^2$ is convergent. Show that $\sum_{n=1}^{\infty} x_n$ is convergent. Determine (with proof or counterexample) whether or not the converse statement holds.

6E Numbers and Sets

(i) State and prove the Fermat–Euler Theorem.

(ii) Let p be an odd prime number, and x an integer coprime to p. Show that $x^{(p-1)/2} \equiv \pm 1 \pmod{p}$, and that if the congruence $y^2 \equiv x \pmod{p}$ has a solution then $x^{(p-1)/2} \equiv 1 \pmod{p}$.

(iii) By arranging the residue classes coprime to p into pairs $\{a, bx\}$ with $ab \equiv 1 \pmod{p}$, or otherwise, show that if the congruence $y^2 \equiv x \pmod{p}$ has no solution then $x^{(p-1)/2} \equiv -1 \pmod{p}$.

(iv) Show that $5^{5^5} \equiv 5 \pmod{23}$.

7E Numbers and Sets

(i) What does it mean to say that a set X is countable? Show directly that the set of sequences $(x_n)_{n\in\mathbb{N}}$, with $x_n \in \{0,1\}$ for all n, is uncountable.

(ii) Let S be any subset of $\mathbb{N}$. Show that there exists a bijection $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(S) = 2\mathbb{N}$ (the set of even natural numbers) if and only if both S and its complement are infinite.

(iii) Let $\sqrt{2} = 1{\cdot}a_1a_2a_3\ldots$ be the binary expansion of $\sqrt{2}$. Let X be the set of all sequences (x_n) with $x_n \in \{0,1\}$ such that for infinitely many n, $x_n = 0$. Let Y be the set of all $(x_n) \in X$ such that for infinitely many n, $x_n = a_n$. Show that Y is uncountable.

8E Numbers and Sets

(i) State and prove the Inclusion–Exclusion Principle.

(ii) Let $n > 1$ be an integer. Denote by $\mathbb{Z}/n\mathbb{Z}$ the integers modulo n. Let X be the set of all functions $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ such that for every $j \in \mathbb{Z}/n\mathbb{Z}$, $f(j) - f(j-1) \not\equiv j \pmod{n}$. Show that

$$|X| = \begin{cases} (n-1)^n + 1 - n & \text{if } n \text{ is odd,} \\ (n-1)^n - 1 & \text{if } n \text{ is even.} \end{cases}$$

9C Dynamics and Relativity

A rocket of mass $m(t)$, which includes the mass of its fuel and everything on board, moves through free space in a straight line at speed $v(t)$. When its engines are operational, they burn fuel at a constant mass rate α and eject the waste gases behind the rocket at a constant speed u relative to the rocket. Obtain the rocket equation

$$m\frac{dv}{dt} - \alpha u = 0.$$

The rocket is initially at rest in a cloud of space dust which is also at rest. The engines are started and, as the rocket travels through the cloud, it collects dust which it stores on board for research purposes. The mass of dust collected in a time δt is given by $\beta\,\delta x$, where δx is the distance travelled in that time and β is a constant. Obtain the new equations

$$\begin{aligned} \frac{dm}{dt} &= \beta v - \alpha, \\ m\frac{dv}{dt} &= \alpha u - \beta v^2. \end{aligned}$$

By eliminating t, or otherwise, obtain the relationship

$$m = \lambda m_0 u \sqrt{\frac{(\lambda u - v)^{\lambda - 1}}{(\lambda u + v)^{\lambda + 1}}},$$

where m_0 is the initial mass of the rocket and $\lambda = \sqrt{\alpha/\beta u}$.

If $\lambda > 1$, show that the fuel will be exhausted before the speed of the rocket can reach λu. Comment on the case when $\lambda < 1$, giving a physical interpretation of your answer.

10C Dynamics and Relativity

A reference frame S' rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame S that has the same origin as S'. A particle of mass m at position vector $\mathbf{x}$ is subject to a force $\mathbf{F}$. Derive the equation of motion for the particle in S'.

A marble moves on a smooth plane which is inclined at an angle θ to the horizontal. The whole plane rotates at constant angular speed ω about a vertical axis through a point O fixed in the plane. Coordinates (ξ, η) are defined with respect to axes fixed in the plane: $O\xi$ horizontal and $O\eta$ up the line of greatest slope in the plane. Ensuring that you account for the normal reaction force, show that the motion of the marble obeys

$$\ddot{\xi} = \omega^2 \xi + 2\omega\dot{\eta}\cos\theta,$$
$$\ddot{\eta} = \omega^2 \eta \cos^2\theta - 2\omega\dot{\xi}\cos\theta - g\sin\theta.$$

By considering the marble's kinetic energy as measured on the plane in the rotating frame, or otherwise, find a constant of the motion.

[You may assume that the marble never leaves the plane.]

11C Dynamics and Relativity

A thin flat disc of radius a has density (mass per unit area) $\rho(r,\theta) = \rho_0(a-r)$ where (r,θ) are plane polar coordinates on the disc and ρ_0 is a constant. The disc is free to rotate about a light, thin rod that is rigidly fixed in space, passing through the centre of the disc orthogonal to it. Find the moment of inertia of the disc about the rod.

The section of the disc lying in $r \geqslant \frac{1}{2}a$, $-\frac{\pi}{13} \leqslant \theta \leqslant \frac{\pi}{13}$ is cut out and removed. Starting from rest, a constant torque τ is applied to the remaining part of the disc until its angular speed about the axis reaches Ω. Show that this takes a time

$$\frac{3\pi\rho_0 a^5 \Omega}{32\tau}.$$

After this time, no further torque is applied and the partial disc continues to rotate at constant angular speed Ω. Given that the total mass of the partial disc is $k\rho_0 a^3$, where k is a constant that you need not determine, find the position of the centre of mass, and hence its acceleration. From where does the force required to produce this acceleration arise?

12C Dynamics and Relativity

Define the *4-momentum* of a particle and describe briefly the principle of conservation of 4-momentum.

A photon of angular frequency ω is absorbed by a particle of rest mass m that is stationary in the laboratory frame of reference. The particle then splits into two equal particles, each of rest mass αm.

Find the maximum possible value of α as a function of $\mu = \hbar\omega/mc^2$. Verify that as $\mu \to 0$, this maximum value tends to $\frac{1}{2}$. For general μ, show that when the maximum value of α is achieved, the resulting particles are each travelling at speed $c/(1+\mu^{-1})$ in the laboratory frame.

END OF PAPER