UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 2 June, 2014 9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

Let $G = \mathbb{Q}$ be the rational numbers, with addition as the group operation. Let x, y be non-zero elements of G, and let $N \leqslant G$ be the subgroup they generate. Show that N is isomorphic to $\mathbb{Z}$.

Find non-zero elements $x, y \in \mathbb{R}$ which generate a subgroup that is not isomorphic to $\mathbb{Z}$.

2D Groups

Let G be a group, and suppose the centre of G is trivial. If p divides $|G|$, show that G has a non-trivial conjugacy class whose order is prime to p.

3A Vector Calculus

(a) For $\mathbf{x} \in \mathbb{R}^n$ and $r = |\mathbf{x}|$, show that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}.$$

(b) Use index notation and your result in (a), or otherwise, to compute

(i) $\nabla \cdot \big(f(r)\mathbf{x}\big)$, and

(ii) $\nabla \times \big(f(r)\mathbf{x}\big)$ for $n = 3$.

(c) Show that for each $n \in \mathbb{N}$ there is, up to an arbitrary constant, just one vector field $\mathbf{F}(\mathbf{x})$ of the form $f(r)\mathbf{x}$ such that $\nabla \cdot \mathbf{F}(\mathbf{x}) = 0$ everywhere on $\mathbb{R}^n \backslash \{\mathbf{0}\}$, and determine $\mathbf{F}$.

4A Vector Calculus

Let $\mathbf{F}(\mathbf{x})$ be a vector field defined everywhere on the domain $G \subset \mathbb{R}^3$.

(a) Suppose that $\mathbf{F}(\mathbf{x})$ has a potential $\phi(\mathbf{x})$ such that $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$ for $\mathbf{x} \in G$. Show that

$$\int_\gamma \mathbf{F} \cdot \mathbf{dx} = \phi(\mathbf{b}) - \phi(\mathbf{a})$$

for any smooth path γ from $\mathbf{a}$ to $\mathbf{b}$ in G. Show further that necessarily $\nabla \times \mathbf{F} = \mathbf{0}$ on G.

(b) State a condition for G which ensures that $\nabla \times \mathbf{F} = \mathbf{0}$ implies $\int_\gamma \mathbf{F} \cdot \mathbf{dx}$ is path-independent.

(c) Compute the line integral $\oint_\gamma \mathbf{F} \cdot \mathbf{dx}$ for the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{pmatrix},$$

where γ denotes the anti-clockwise path around the unit circle in the (x, y)-plane. Compute $\nabla \times \mathbf{F}$ and comment on your result in the light of (b).

SECTION II

5D Groups

Let S_n be the group of permutations of $\{1, \ldots, n\}$, and suppose n is even, $n \geqslant 4$.

Let $g = (1\,2) \in S_n$, and $h = (1\,2)(3\,4) \ldots (n{-}1\ n) \in S_n$.

(i) Compute the centraliser of g, and the orders of the centraliser of g and of the centraliser of h.

(ii) Now let $n = 6$. Let G be the group of all symmetries of the cube, and X the set of faces of the cube. Show that the action of G on X makes G isomorphic to the centraliser of h in S_6. [*Hint: Show that* $-1 \in G$ *permutes the faces of the cube according to* h.]

Show that G is also isomorphic to the centraliser of g in S_6.

6D Groups

Let p be a prime number. Let G be a group such that every non-identity element of G has order p.

(i) Show that if $|G|$ is finite, then $|G| = p^n$ for some n. [You must prove any theorems that you use.]

(ii) Show that if $H \leqslant G$, and $x \notin H$, then $\langle x \rangle \cap H = \{1\}$.

Hence show that if G is abelian, and $|G|$ is finite, then $G \simeq C_p \times \cdots \times C_p$.

(iii) Let G be the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b, x \in \mathbb{F}_p$ and $\mathbb{F}_p$ is the field of integers modulo p. Show that every non-identity element of G has order p if and only if $p > 2$. [You may assume that G is a subgroup of the group of all 3×3 invertible matrices.]

7D Groups

Let p be a prime number, and $G = GL_2(\mathbb{F}_p)$, the group of 2×2 invertible matrices with entries in the field $\mathbb{F}_p$ of integers modulo p.

The group G acts on $X = \mathbb{F}_p \cup \{\infty\}$ by Möbius transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

(i) Show that given any distinct $x, y, z \in X$ there exists $g \in G$ such that $g \cdot 0 = x$, $g \cdot 1 = y$ and $g \cdot \infty = z$. How many such g are there?

(ii) G acts on $X \times X \times X$ by $g \cdot (x, y, z) = (g \cdot x, g \cdot y, g \cdot z)$. Describe the orbits, and for each orbit, determine its stabiliser, and the orders of the orbit and stabiliser.

8D Groups

(a) Let G be a group, and N a subgroup of G. Define what it means for N to be normal in G, and show that if N is normal then G/N naturally has the structure of a group.

(b) For each of (i)–(iii) below, give an example of a non-trivial finite group G and non-trivial normal subgroup $N \leqslant G$ satisfying the stated properties.

(i) $G/N \times N \simeq G$.

(ii) There is no group homomorphism $G/N \to G$ such that the composite $G/N \to G \to G/N$ is the identity.

(iii) There is a group homomorphism $i \colon G/N \to G$ such that the composite $G/N \to G \to G/N$ is the identity, but the map

$$G/N \times N \to G, \qquad (gN, n) \mapsto i(gN)n$$

is not a group homomorphism.

Show also that for any $N \leqslant G$ satisfying (iii), this map is always a bijection.

9A Vector Calculus

The surface C in $\mathbb{R}^3$ is given by $z^2 = x^2 + y^2$.

(a) Show that the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is tangent to the surface C everywhere.

(b) Show that the surface integral $\int_S \mathbf{F} \cdot \mathbf{dS}$ is a constant independent of S for any surface S which is a subset of C, and determine this constant.

(c) The volume V in $\mathbb{R}^3$ is bounded by the surface C and by the cylinder $x^2 + y^2 = 1$. Sketch V and compute the volume integral

$$\int_V \nabla \cdot \mathbf{F}\, dV$$

directly by integrating over V.

(d) Use the Divergence Theorem to verify the result you obtained in part (b) for the integral $\int_S \mathbf{F} \cdot \mathbf{dS}$, where S is the portion of C lying in $-1 \leqslant z \leqslant 1$.

10A Vector Calculus

(a) State Stokes' Theorem for a surface S with boundary ∂S.

(b) Let S be the surface in $\mathbb{R}^3$ given by $z^2 = 1 + x^2 + y^2$ where $\sqrt{2} \leqslant z \leqslant \sqrt{5}$. Sketch the surface S and find the surface element $\mathbf{dS}$ with respect to the Cartesian coordinates x and y.

(c) Compute $\nabla \times \mathbf{F}$ for the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} -y \\ x \\ xy(x+y) \end{pmatrix}$$

and verify Stokes' Theorem for $\mathbf{F}$ on the surface S.

11A Vector Calculus

(i) Starting with Poisson's equation in $\mathbb{R}^3$,

$$\nabla^2 \phi(\mathbf{x}) = f(\mathbf{x}),$$

derive Gauss' flux theorem

$$\int_V f(\mathbf{x})\, dV = \int_{\partial V} \mathbf{F}(\mathbf{x}) \cdot \mathbf{dS}$$

for $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$ and for any volume $V \subseteq \mathbb{R}^3$.

(ii) Let

$$I = \int_S \frac{\mathbf{x} \cdot \mathbf{dS}}{|\mathbf{x}|^3}.$$

Show that $I = 4\pi$ if S is the sphere $|\mathbf{x}| = R$, and that $I = 0$ if S bounds a volume that does not contain the origin.

(iii) Show that the electric field defined by

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{a}}{|\mathbf{x} - \mathbf{a}|^3}, \quad \mathbf{x} \neq \mathbf{a},$$

satisfies

$$\int_{\partial V} \mathbf{E} \cdot \mathbf{dS} = \begin{cases} 0 & \text{if } \mathbf{a} \notin V \\ \dfrac{q}{\epsilon_0} & \text{if } \mathbf{a} \in V \end{cases}$$

where ∂V is a surface bounding a closed volume V and $\mathbf{a} \notin \partial V$, and where the electric charge q and permittivity of free space ϵ_0 are constants. This is Gauss' law for a point electric charge.

(iv) Assume that $f(\mathbf{x})$ is spherically symmetric around the origin, i.e., it is a function only of $|\mathbf{x}|$. Assume that $\mathbf{F}(\mathbf{x})$ is also spherically symmetric. Show that $\mathbf{F}(\mathbf{x})$ depends only on the values of f inside the sphere with radius $|\mathbf{x}|$ but not on the values of f outside this sphere.

12A Vector Calculus

(a) Show that any rank 2 tensor t_{ij} can be written uniquely as a sum of two rank 2 tensors s_{ij} and a_{ij} where s_{ij} is symmetric and a_{ij} is antisymmetric.

(b) Assume that the rank 2 tensor t_{ij} is invariant under any rotation about the z-axis, as well as under a rotation of angle π about any axis in the (x, y)-plane through the origin.

(i) Show that there exist $\alpha, \beta \in \mathbb{R}$ such that t_{ij} can be written as

$$t_{ij} = \alpha\delta_{ij} + \beta\delta_{i3}\delta_{j3}. \qquad (*)$$

(ii) Is there some proper subgroup of the rotations specified above for which the result $(*)$ still holds if the invariance of t_{ij} is restricted to this subgroup? If so, specify the smallest such subgroup.

(c) The array of numbers d_{ijk} is such that $d_{ijk}s_{ij}$ is a vector for any symmetric matrix s_{ij}.

(i) By writing d_{ijk} as a sum of d^s_{ijk} and d^a_{ijk} with $d^s_{ijk} = d^s_{jik}$ and $d^a_{ijk} = -d^a_{jik}$, show that d^s_{ijk} is a rank 3 tensor. [You may assume without proof the Quotient Theorem for tensors.]

(ii) Does d^a_{ijk} necessarily have to be a tensor? Justify your answer.

END OF PAPER