UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2014

List of Courses

Paper 3, Section I

2F Analysis II

Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be a differentiable function on U such that $\| Df|_x \| \leqslant M$ for some constant M and all $x \in U$, where $\| Df|_x \|$ denotes the operator norm of the linear map $Df|_x$. Let $[a,b] = \{ta + (1-t)b : 0 \leqslant t \leqslant 1\}$ $(a, b, \in \mathbb{R}^n)$ be a straight-line segment contained in U. Prove that $|f(b) - f(a)| \leqslant M\|b - a\|$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Prove that if U is an open ball and $Df|_x = 0$ for each $x \in U$, then f is constant on U.

Paper 4, Section I

3F Analysis II

Define a *contraction mapping* and state the contraction mapping theorem.

Let $C[0,1]$ be the space of continuous real-valued functions on $[0,1]$ endowed with the uniform norm. Show that the map $A : C[0,1] \to C[0,1]$ defined by

$$Af(x) = \int_0^x f(t)dt$$

is not a contraction mapping, but that $A \circ A$ is.

Paper 2, Section I

3F Analysis II

Define what is meant by a *uniformly continuous* function on a set $E \subset \mathbb{R}$.

If f and g are uniformly continuous functions on $\mathbb{R}$, is the (pointwise) product fg necessarily uniformly continuous on $\mathbb{R}$?

Is a uniformly continuous function on $(0,1)$ necessarily bounded?

Is $\cos(1/x)$ uniformly continuous on $(0,1)$?

Justify your answers.

Paper 1, Section II

11F Analysis II

Define what it means for two norms on a real vector space V to be Lipschitz equivalent. Show that if two norms on V are Lipschitz equivalent and $F \subset V$, then F is closed in one norm if and only if F is closed in the other norm.

Show that if V is finite-dimensional, then any two norms on V are Lipschitz equivalent.

Show that $\|f\|_1 = \int_0^1 |f(x)|dx$ is a norm on the space $C[0,1]$ of continuous real-valued functions on $[0,1]$. Is the set $S = \{f \in C[0,1] : f(1/2) = 0\}$ closed in the norm $\|\cdot\|_1$?

Determine whether or not the norm $\|\cdot\|_1$ is Lipschitz equivalent to the uniform norm $\|\cdot\|_\infty$ on $C[0,1]$.

[You may assume the Bolzano–Weierstrass theorem for sequences in $\mathbb{R}^n$.]

Paper 4, Section II

12F Analysis II

Let $U \subset \mathbb{R}^2$ be an open set. Define what it means for a function $f : U \to \mathbb{R}$ to be *differentiable* at a point $(x_0, y_0) \in U$.

Prove that if the partial derivatives $D_1 f$ and $D_2 f$ exist on U and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

If f is differentiable on U must $D_1 f$, $D_2 f$ be continuous at (x_0, y_0)? Give a proof or counterexample as appropriate.

The function $h : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$h(x,y) = xy\sin(1/x) \quad \text{for } x \neq 0, \qquad h(0,y) = 0.$$

Determine all the points (x,y) at which h is differentiable.

Paper 3, Section II
12F Analysis II

Let f_n, $n = 1, 2, \ldots$, be continuous functions on an open interval (a, b). Prove that if the sequence (f_n) converges to f uniformly on (a, b) then the function f is continuous on (a, b).

If instead (f_n) is only known to converge pointwise to f and f is continuous, must (f_n) be uniformly convergent? Justify your answer.

Suppose that a function f has a continuous derivative on (a, b) and let

$$g_n(x) = n\left(f(x + \frac{1}{n}) - f(x)\right).$$

Stating clearly any standard results that you require, show that the functions g_n converge uniformly to f' on each interval $[\alpha, \beta] \subset (a, b)$.

Paper 2, Section II
12F Analysis II

Let X, Y be subsets of $\mathbb{R}^n$ and define $X + Y = \{x + y : x \in X, \ y \in Y\}$. For each of the following statements give a proof or a counterexample (with justification) as appropriate.

(i) If each of X, Y is bounded and closed, then $X + Y$ is bounded and closed.

(ii) If X is bounded and closed and Y is closed, then $X + Y$ is closed.

(iii) If X, Y are both closed, then $X + Y$ is closed.

(iv) If X is open and Y is closed, then $X + Y$ is open.

[The Bolzano–Weierstrass theorem in $\mathbb{R}^n$ may be assumed without proof.]

Paper 4, Section I

4G Complex Analysis

Let f be an entire function. State Cauchy's Integral Formula, relating the nth derivative of f at a point z with the values of f on a circle around z.

State Liouville's Theorem, and deduce it from Cauchy's Integral Formula.

Let f be an entire function, and suppose that for some k we have that $|f(z)| \leqslant |z|^k$ for all z. Prove that f is a polynomial.

Paper 3, Section II

13G Complex Analysis

State the Residue Theorem precisely.

Let D be a star-domain, and let γ be a closed path in D. Suppose that f is a holomorphic function on D, having no zeros on γ. Let N be the number of zeros of f inside γ, counted with multiplicity (i.e. order of zero and winding number). Show that

$$N = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz\ .$$

[The Residue Theorem may be used without proof.]

Now suppose that g is another holomorphic function on D, also having no zeros on γ and with $|g(z)| < |f(z)|$ on γ. Explain why, for any $0 \leqslant t \leqslant 1$, the expression

$$I(t) = \int_\gamma \frac{f'(z)+tg'(z)}{f(z)+tg(z)}\,dz$$

is well-defined. By considering the behaviour of the function $I(t)$ as t varies, deduce Rouché's Theorem.

For each n, let p_n be the polynomial $\sum_{k=0}^{n}\frac{z^k}{k!}$. Show that, as n tends to infinity, the smallest modulus of the roots of p_n also tends to infinity.

[You may assume any results on convergence of power series, provided that they are stated clearly.]

Paper 1, Section I

2B Complex Analysis or Complex Methods

Let $f(z)$ be an analytic/holomorphic function defined on an open set D, and let $z_0 \in D$ be a point such that $f'(z_0) \neq 0$. Show that the transformation $w = f(z)$ preserves the angle between smooth curves intersecting at z_0. Find such a transformation $w = f(z)$ that maps the second quadrant of the unit disc (i.e. $|z| < 1$, $\pi/2 < \arg(z) < \pi$) to the region in the first quadrant of the complex plane where $|w| > 1$ (i.e. the region in the first quadrant *outside* the unit circle).

Paper 1, Section II

13B Complex Analysis or Complex Methods

By choice of a suitable contour show that for $a > b > 0$

$$\int_0^{2\pi} \frac{\sin^2\theta d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}\left[a - \sqrt{a^2 - b^2}\right].$$

Hence evaluate

$$\int_0^1 \frac{(1-x^2)^{1/2}x^2 dx}{1+x^2}$$

using the substitution $x = \cos(\theta/2)$.

Paper 2, Section II

13B Complex Analysis or Complex Methods

By considering a rectangular contour, show that for $0 < a < 1$ we have

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx = \frac{\pi}{\sin \pi a}.$$

Hence evaluate

$$\int_0^\infty \frac{dt}{t^{5/6}(1+t)}.$$

Paper 3, Section I

4B Complex Methods

Find the most general cubic form

$$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$

which satisfies Laplace's equation, where a, b, c and d are all real. Hence find an analytic function $f(z) = f(x+iy)$ which has such a u as its real part.

Paper 4, Section II

14B Complex Methods

Find the Laplace transforms of t^n for n a positive integer and $H(t-a)$ where $a > 0$ and $H(t)$ is the Heaviside step function.

Consider a semi-infinite string which is initially at rest and is fixed at one end. The string can support wave-like motions, and for $t > 0$ it is allowed to fall under gravity. Therefore the deflection $y(x,t)$ from its initial location satisfies

$$\frac{\partial^2}{\partial t^2} y = c^2 \frac{\partial^2}{\partial x^2} y + g \quad \text{for} \quad x > 0,\ t > 0$$

with

$$y(0,t) = y(x,0) = \frac{\partial}{\partial t} y(x,0) = 0 \quad \text{and} \quad y(x,t) \to \frac{gt^2}{2} \text{ as } x \to \infty,$$

where g is a constant. Use Laplace transforms to find $y(x,t)$.

[The convolution theorem for Laplace transforms may be quoted without proof.]

Paper 2, Section I

6A Electromagnetism

Starting from Maxwell's equations, deduce that

$$\frac{d\Phi}{dt} = -\mathcal{E},$$

for a moving circuit C, where Φ is the flux of $\mathbf{B}$ through the circuit and where the electromotive force $\mathcal{E}$ is defined to be

$$\mathcal{E} = \oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{dr}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{r})$ denotes the velocity of a point $\mathbf{r}$ on C.

[*Hint: Consider the closed surface consisting of the surface $S(t)$ bounded by $C(t)$, the surface $S(t + \delta t)$ bounded by $C(t + \delta t)$ and the surface S' stretching from $C(t)$ to $C(t + \delta t)$. Show that the flux of $\mathbf{B}$ through S' is $-\delta t \oint_C \mathbf{B} \cdot (\mathbf{v} \times \mathbf{dr})$.*]

Paper 4, Section I

7A Electromagnetism

A continuous wire of resistance R is wound around a very long right circular cylinder of radius a, and length l (long enough so that end effects can be ignored). There are $N \gg 1$ turns of wire per unit length, wound in a spiral of very small pitch. Initially, the magnetic field $\mathbf{B}$ is $\mathbf{0}$.

Both ends of the coil are attached to a battery of electromotance $\mathcal{E}_0$ at $t = 0$, which induces a current $I(t)$. Use Ampère's law to derive $\mathbf{B}$ inside and outside the cylinder when the displacement current may be neglected. Write the self-inductance of the coil L in terms of the quantities given above. Using Ohm's law and Faraday's law of induction, find $I(t)$ explicitly in terms of $\mathcal{E}_0$, R, L and t.

Paper 1, Section II

16A Electromagnetism

The region $z < 0$ is occupied by an ideal earthed conductor and a point charge q with mass m is held above it at $(0,\ 0,\ d)$.

(i) What are the boundary conditions satisfied by the electric field $\mathbf{E}$ on the surface of the conductor?

(ii) Consider now a system without the conductor mentioned above. A point charge q with mass m is held at $(0,\ 0,\ d)$, and one of charge $-q$ is held at $(0,\ 0,\ -d)$. Show that the boundary condition on $\mathbf{E}$ at $z = 0$ is identical to the answer to (i). Explain why this represents the electric field due to the charge at $(0,\ 0,\ d)$ under the influence of the conducting boundary.

(iii) The original point charge in (i) is released with zero initial velocity. Find the time taken for the point charge to reach the plane (ignoring gravity).

[You may assume that the force on the point charge is equal to $m\, d^2\mathbf{x}/dt^2$, where $\mathbf{x}$ is the position vector of the charge, and t is time.]

Paper 3, Section II

17A Electromagnetism

(i) Consider charges $-q$ at $\pm\mathbf{d}$ and $2q$ at $(0,\ 0,\ 0)$. Write down the electric potential.

(ii) Take $\mathbf{d} = (0,\ 0,\ d)$. A *quadrupole* is defined in the limit that $q \to \infty$, $d \to 0$ such that qd^2 tends to a constant p. Find the quadrupole's potential, showing that it is of the form

$$\phi(\mathbf{r}) = A\frac{(r^2 + Cz^D)}{r^B},$$

where $r = |\mathbf{r}|$. Determine the constants A, B, C and D.

(iii) The quadrupole is fixed at the origin. At time $t = 0$ a particle of charge $-Q$ (Q has the same sign as q) and mass m is at $(1,\ 0,\ 0)$ travelling with velocity $d\mathbf{r}/dt = (-\kappa,\ 0,\ 0)$, where

$$\kappa = \sqrt{\frac{Q\,p}{2\pi\epsilon_0\, m}}\,.$$

Neglecting gravity, find the time taken for the particle to reach the quadrupole in terms of κ, given that the force on the particle is equal to $m\, d^2\mathbf{r}/dt^2$.

Paper 2, Section II

18A Electromagnetism

What is the relationship between the electric field $\mathbf{E}$ and the charge per unit area σ on the surface of a perfect conductor?

Consider a charge distribution $\rho(\mathbf{r})$ distributed with potential $\phi(\mathbf{r})$ over a finite volume V within which there is a set of perfect conductors with charges Q_i, each at a potential ϕ_i (normalised such that the potential at infinity is zero). Using Maxwell's equations and the divergence theorem, derive a relationship between the electrostatic energy W and a volume integral of an explicit function of the electric field $\mathbf{E}$, where

$$W = \frac{1}{2}\int_V \rho\phi \, d\tau + \frac{1}{2}\sum_i Q_i\phi_i \, .$$

Consider N concentric perfectly conducting spherical shells. Shell n has radius r_n (where $r_n > r_{n-1}$) and charge q for $n = 1$, and charge $2(-1)^{(n+1)}q$ for $n > 1$. Show that

$$W \propto \frac{1}{r_1},$$

and determine the constant of proportionality.

Paper 1, Section I

5B Fluid Dynamics

Constant density viscous fluid with dynamic viscosity μ flows in a two-dimensional horizontal channel of depth h. There is a constant pressure gradient $G > 0$ in the horizontal $x-$direction. The upper horizontal boundary at $y = h$ is driven at constant horizontal speed $U > 0$, with the lower boundary being held at rest. Show that the steady fluid velocity u in the $x-$direction is

$$u = \frac{-G}{2\mu}y(h-y) + \frac{Uy}{h}.$$

Show that it is possible to have $du/dy < 0$ at some point in the flow for sufficiently large pressure gradient. Derive a relationship between G and U so that there is no net volume flux along the channel. For the flow with no net volume flux, sketch the velocity profile.

Paper 2, Section I

7B Fluid Dynamics

Consider the steady two-dimensional fluid velocity field

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \epsilon & -\gamma \\ \gamma & -\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\epsilon \geqslant 0$ and $\gamma \geqslant 0$. Show that the fluid is incompressible. The streamfunction ψ is defined by $\mathbf{u} = \boldsymbol{\nabla} \times \boldsymbol{\Psi}$, where $\boldsymbol{\Psi} = (0, 0, \psi)$. Show that ψ is given by

$$\psi = \epsilon xy - \frac{\gamma}{2}(x^2 + y^2).$$

Hence show that the streamlines are defined by

$$(\epsilon - \gamma)(x+y)^2 - (\epsilon + \gamma)(x-y)^2 = C,$$

for C a constant. For each of the three cases below, sketch the streamlines and briefly describe the flow.

(i) $\epsilon = 1$, $\gamma = 0$,

(ii) $\epsilon = 0$, $\gamma = 1$,

(iii) $\epsilon = 1$, $\gamma = 1$.

Paper 1, Section II

17B Fluid Dynamics

Consider the purely two-dimensional steady flow of an inviscid incompressible constant density fluid in the absence of body forces. For velocity $\mathbf{u}$, the vorticity is $\boldsymbol{\nabla} \times \mathbf{u} = \boldsymbol{\omega} = (0, 0, \omega)$. Show that

$$\mathbf{u} \times \boldsymbol{\omega} = \boldsymbol{\nabla} \left[\frac{p}{\rho} + \frac{1}{2} |\mathbf{u}|^2 \right],$$

where p is the pressure and ρ is the fluid density. Hence show that, if ω is a constant in both space and time,

$$\frac{1}{2} |\mathbf{u}|^2 + \omega \psi + \frac{p}{\rho} = C,$$

where C is a constant and ψ is the streamfunction. Here, ψ is defined by $\mathbf{u} = \boldsymbol{\nabla} \times \boldsymbol{\Psi}$, where $\boldsymbol{\Psi} = (0, 0, \psi)$.

Fluid in the annular region $a < r < 2a$ has constant (in both space and time) vorticity ω. The streamlines are concentric circles, with the fluid speed zero on $r = 2a$ and $V > 0$ on $r = a$. Calculate the velocity field, and hence show that

$$\omega = \frac{-2V}{3a}.$$

Deduce that the pressure difference between the outer and inner edges of the annular region is

$$\Delta p = \left(\frac{15 - 16 \ln 2}{18} \right) \rho V^2.$$

[*Hint: Note that in cylindrical polar coordinates* (r, ϕ, z), *the curl of a vector field* $\mathbf{A}(r, \phi) = [a(r, \phi), b(r, \phi), c(r, \phi)]$ *is*

$$\boldsymbol{\nabla} \times \mathbf{A} = \left[\frac{1}{r} \frac{\partial c}{\partial \phi}, -\frac{\partial c}{\partial r}, \frac{1}{r} \left(\frac{\partial (rb)}{\partial r} - \frac{\partial a}{\partial \phi} \right) \right].$$]

Paper 4, Section II

18B Fluid Dynamics

Consider a layer of fluid of constant density ρ and equilibrium depth h_0 in a rotating frame of reference, rotating at constant angular velocity Ω about the vertical z-axis. The equations of motion are

$$\begin{aligned}\frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y},\\ 0 &= -\frac{\partial p}{\partial z} - \rho g,\end{aligned}$$

where p is the fluid pressure, u and v are the fluid velocities in the x-direction and y-direction respectively, $f = 2\Omega$, and g is the constant acceleration due to gravity. You may also assume that the horizontal extent of the layer is sufficiently large so that the layer may be considered to be shallow, such that vertical velocities may be neglected.

By considering mass conservation, show that the depth $h(x, y, t)$ of the layer satisfies

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0.$$

Now assume that $h = h_0 + \eta(x, y, t)$, where $|\eta| \ll h_0$. Show that the (linearised) potential vorticity $\mathbf{Q} = Q\hat{\mathbf{z}}$, defined by

$$Q = \zeta - \eta\frac{f}{h_0}, \text{ where } \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

and $\hat{\mathbf{z}}$ is the unit vector in the vertical z-direction, is a constant in time, i.e. $Q = Q_0(x, y)$.

When $Q_0 = 0$ everywhere, establish that the surface perturbation η satisfies

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) + f^2\eta = 0,$$

and show that this equation has wave-like solutions $\eta = \eta_0 \cos[k(x - ct)]$ when c and k are related through a dispersion relation to be determined. Show that, to leading order, the trajectories of fluid particles for these waves are ellipses. Assuming that $\eta_0 > 0$, $k > 0$, $c > 0$ and $f > 0$, sketch the fluid velocity when $k(x - ct) = n\pi/2$ for $n = 0, 1, 2, 3$.

Paper 3, Section II

18B Fluid Dynamics

A bubble of gas occupies the spherical region $r \leqslant R(t)$, and an incompressible irrotational liquid of constant density ρ occupies the outer region $r \geqslant R$, such that as $r \to \infty$ the liquid is at rest with constant pressure p_∞. Briefly explain why it is appropriate to use a velocity potential $\phi(r,t)$ to describe the liquid velocity $\mathbf{u}$.

By applying continuity of velocity across the gas-liquid interface, show that the liquid pressure (for $r \geqslant R$) satisfies

$$\frac{p}{\rho} + \frac{1}{2}\left(\frac{R^2\dot{R}}{r^2}\right)^2 - \frac{1}{r}\frac{d}{dt}\left(R^2\dot{R}\right) = \frac{p_\infty}{\rho}, \quad \text{where } \dot{R} = \frac{dR}{dt}.$$

Show that the excess pressure $p_s - p_\infty$ at the bubble surface $r = R$ is

$$p_s - p_\infty = \frac{\rho}{2}\left(3\dot{R}^2 + 2R\ddot{R}\right), \quad \text{where } \ddot{R} = \frac{d^2R}{dt^2},$$

and hence that

$$p_s - p_\infty = \frac{\rho}{2R^2}\frac{d}{dR}\left(R^3\dot{R}^2\right).$$

The pressure $p_g(t)$ inside the gas bubble satisfies the equation of state

$$p_g V^{4/3} = C,$$

where C is a constant, and $V(t)$ is the bubble volume. At time $t = 0$ the bubble is at rest with radius $R = a$. If the bubble then expands and comes to rest at $R = 2a$, determine the required gas pressure p_0 at $t = 0$ in terms of p_∞.

[You may assume that there is contact between liquid and gas for all time, that all motion is spherically symmetric about the origin $r = 0$, and that there is no body force. You may also assume Bernoulli's integral of the equation of motion to determine the liquid pressure p:

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 = A(t),$$

where $\phi(r,t)$ is the velocity potential.]

Paper 1, Section I

3F Geometry

Determine the second fundamental form of a surface in $\mathbb{R}^3$ defined by the parametrisation

$$\sigma(u,v) = \Big((a + b\cos u)\cos v,\ (a + b\cos u)\sin v,\ b\sin u\Big),$$

for $0 < u < 2\pi,\ 0 < v < 2\pi$, with some fixed $a > b > 0$. Show that the Gaussian curvature $K(u,v)$ of this surface takes both positive and negative values.

Paper 3, Section I

5F Geometry

Let $f(x) = Ax + b$ be an isometry $\mathbb{R}^n \to \mathbb{R}^n$, where A is an $n \times n$ matrix and $b \in \mathbb{R}^n$. What are the possible values of $\det A$?

Let I denote the $n \times n$ identity matrix. Show that if $n = 2$ and $\det A > 0$, but $A \neq I$, then f has a fixed point. Must f have a fixed point if $n = 3$ and $\det A > 0$, but $A \neq I$? Justify your answer.

Paper 3, Section II

14F Geometry

Let $\mathcal{T}$ be a decomposition of the two-dimensional sphere into polygonal domains, with every polygon having at least three edges. Let V, E, and F denote the numbers of vertices, edges and faces of $\mathcal{T}$, respectively. State Euler's formula. Prove that $2E \geqslant 3F$.

Suppose that at least three edges meet at every vertex of $\mathcal{T}$. Let F_n be the number of faces of $\mathcal{T}$ that have exactly n edges ($n \geqslant 3$) and let V_m be the number of vertices at which exactly m edges meet ($m \geqslant 3$). Is it possible for $\mathcal{T}$ to have $V_3 = F_3 = 0$? Justify your answer.

By expressing $6F - \sum_n nF_n$ in terms of the V_j, or otherwise, show that $\mathcal{T}$ has at least four faces that are triangles, quadrilaterals and/or pentagons.

Paper 2, Section II
14F Geometry

Let $H = \{x + iy : x, y \in \mathbb{R},\ y > 0\} \subset \mathbb{C}$ be the upper half-plane with a hyperbolic metric $g = \frac{dx^2+dy^2}{y^2}$. Prove that every hyperbolic circle C in H is also a Euclidean circle. Is the centre of C as a hyperbolic circle always the same point as the centre of C as a Euclidean circle? Give a proof or counterexample as appropriate.

Let ABC and $A'B'C'$ be two hyperbolic triangles and denote the hyperbolic lengths of their sides by a, b, c and a', b', c', respectively. Show that if $a = a'$, $b = b'$ and $c = c'$, then there is a hyperbolic isometry taking ABC to $A'B'C'$. Is there always such an isometry if instead the triangles have one angle the same and $a = a'$, $b = b'$? Justify your answer.

[Standard results on hyperbolic isometries may be assumed, provided they are clearly stated.]

Paper 4, Section II
15F Geometry

Define an embedded parametrised surface in $\mathbb{R}^3$. What is the Riemannian metric induced by a parametrisation? State, in terms of the Riemannian metric, the equations defining a geodesic curve $\gamma : (0,1) \to S$, assuming that γ is parametrised by arc-length.

Let S be a conical surface

$$S = \{(x, y, z) \in \mathbb{R}^3 :\ 3(x^2 + y^2) = z^2,\ \ z > 0\}.$$

Using an appropriate smooth parametrisation, or otherwise, prove that S is locally isometric to the Euclidean plane. Show that any two points on S can be joined by a geodesic. Is this geodesic always unique (up to a reparametrisation)? Justify your answer.

[The expression for the Euclidean metric in polar coordinates on $\mathbb{R}^2$ may be used without proof.]

Paper 3, Section I

1E Groups, Rings and Modules

State and prove Hilbert's Basis Theorem.

Paper 4, Section I

2E Groups, Rings and Modules

Let G be the abelian group generated by elements a, b and c subject to the relations: $3a + 6b + 3c = 0$, $9b + 9c = 0$ and $-3a + 3b + 6c = 0$. Express G as a product of cyclic groups. Hence determine the number of elements of G of order 3.

Paper 2, Section I

2E Groups, Rings and Modules

List the conjugacy classes of A_6 and determine their sizes. Hence prove that A_6 is simple.

Paper 1, Section II

10E Groups, Rings and Modules

Let G be a finite group and p a prime divisor of the order of G. Give the definition of a Sylow p-subgroup of G, and state Sylow's theorems.

Let p and q be distinct primes. Prove that a group of order p^2q is not simple.

Let G be a finite group, H a normal subgroup of G and P a Sylow p-subgroup of H. Let $N_G(P)$ denote the normaliser of P in G. Prove that if $g \in G$ then there exist $k \in N_G(P)$ and $h \in H$ such that $g = kh$.

Paper 4, Section II

11E Groups, Rings and Modules

(a) Consider the four following types of rings: Principal Ideal Domains, Integral Domains, Fields, and Unique Factorisation Domains. Arrange them in the form $A \implies B \implies C \implies D$ (where $A \implies B$ means if a ring is of type A then it is of type B).

Prove that these implications hold. [You may assume that irreducibles in a Principal Ideal Domain are prime.] Provide examples, with brief justification, to show that these implications cannot be reversed.

(b) Let R be a ring with ideals I and J satisfying $I \subseteq J$. Define K to be the set $\{r \in R : rJ \subseteq I\}$. Prove that K is an ideal of R. If J and K are principal, prove that I is principal.

Paper 3, Section II

11E Groups, Rings and Modules

Let R be a ring, M an R-module and $S = \{m_1, \ldots, m_k\}$ a subset of M. Define what it means to say S *spans* M. Define what it means to say S is an *independent* set.

We say S is a *basis* for M if S spans M and S is an independent set. Prove that the following two statements are equivalent.

1. S is a basis for M.
2. Every element of M is uniquely expressible in the form $r_1m_1 + \cdots + r_km_k$ for some $r_1, \ldots, r_k \in R$.

We say S *generates* M *freely* if S spans M and any map $\Phi : S \to N$, where N is an R-module, can be extended to an R-module homomorphism $\Theta : M \to N$. Prove that S generates M freely if and only if S is a basis for M.

Let M be an R-module. Are the following statements true or false? Give reasons.

(i) If S spans M then S necessarily contains an independent spanning set for M.

(ii) If S is an independent subset of M then S can always be extended to a basis for M.

Paper 2, Section II

11E Groups, Rings and Modules

Prove that every finite integral domain is a field.

Let F be a field and f an irreducible polynomial in the polynomial ring $F[X]$. Prove that $F[X]/(f)$ is a field, where (f) denotes the ideal generated by f.

Hence construct a field of 4 elements, and write down its multiplication table.

Construct a field of order 9.

Paper 4, Section I

1G Linear Algebra

Let V denote the vector space of all real polynomials of degree at most 2. Show that

$$(f,g) \;=\; \int_{-1}^{1} f(x)g(x)\,dx$$

defines an inner product on V.

Find an orthonormal basis for V.

Paper 2, Section I

1G Linear Algebra

State and prove the Rank–Nullity Theorem.

Let α be a linear map from $\mathbb{R}^5$ to $\mathbb{R}^3$. What are the possible dimensions of the kernel of α? Justify your answer.

Paper 1, Section I

1G Linear Algebra

State and prove the Steinitz Exchange Lemma. Use it to prove that, in a finite-dimensional vector space: any two bases have the same size, and every linearly independent set extends to a basis.

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. Is e_1+e_2, e_2+e_3, e_3+e_1 a basis for $\mathbb{R}^3$? Is e_1+e_2, e_2+e_3, e_3+e_4, e_4+e_1 a basis for $\mathbb{R}^4$? Justify your answers.

Paper 1, Section II

9G Linear Algebra

Let V be an n-dimensional real vector space, and let T be an endomorphism of V. We say that T *acts* on a subspace W if $T(W) \subset W$.

(i) For any $x \in V$, show that T acts on the linear span of $\{x, T(x), T^2(x), \ldots, T^{n-1}(x)\}$.

(ii) If $\{x, T(x), T^2(x), \ldots, T^{n-1}(x)\}$ spans V, show directly (i.e. without using the Cayley–Hamilton Theorem) that T satisfies its own characteristic equation.

(iii) Suppose that T acts on a subspace W with $W \neq \{0\}$ and $W \neq V$. Let $e_1, \ldots, e_k$ be a basis for W, and extend to a basis $e_1, \ldots, e_n$ for V. Describe the matrix of T with respect to this basis.

(iv) Using (i), (ii) and (iii) and induction, give a proof of the Cayley–Hamilton Theorem.

[Simple properties of determinants may be assumed without proof.]

Paper 4, Section II

10G Linear Algebra

Let V be a real vector space. What is the *dual* V^* of V? If $e_1, \ldots, e_n$ is a basis for V, define the *dual basis* $e_1^*, \ldots, e_n^*$ for V^*, and show that it is indeed a basis for V^*.

[No result about dimensions of dual spaces may be assumed.]

For a subspace U of V, what is the *annihilator* of U? If V is n-dimensional, how does the dimension of the annihilator of U relate to the dimension of U?

Let $\alpha : V \to W$ be a linear map between finite-dimensional real vector spaces. What is the *dual map* α^*? Explain why the rank of α^* is equal to the rank of α. Prove that the kernel of α^* is the annihilator of the image of α, and also that the image of α^* is the annihilator of the kernel of α.

[Results about the matrices representing a map and its dual may be used without proof, provided they are stated clearly.]

Now let V be the vector space of all real polynomials, and define elements $L_0, L_1, \ldots$ of V^* by setting $L_i(p)$ to be the coefficient of X^i in p (for each $p \in V$). Do the L_i form a basis for V^*?

Paper 3, Section II

10G Linear Algebra

Let q be a nonsingular quadratic form on a finite-dimensional real vector space V. Prove that we may write $V = P \bigoplus N$, where the restriction of q to P is positive definite, the restriction of q to N is negative definite, and $q(x+y) = q(x) + q(y)$ for all $x \in P$ and $y \in N$. [No result on diagonalisability may be assumed.]

Show that the dimensions of P and N are independent of the choice of P and N. Give an example to show that P and N are not themselves uniquely defined.

Find such a decomposition $V = P \bigoplus N$ when $V = \mathbb{R}^3$ and q is the quadratic form $q((x, y, z)) \;=\; x^2 + 2y^2 - 2xy - 2xz$.

Paper 2, Section II

10G Linear Algebra

Define the *determinant* of an $n \times n$ complex matrix A. Explain, with justification, how the determinant of A changes when we perform row and column operations on A.

Let A, B, C be complex $n \times n$ matrices. Prove the following statements.

(i) $\det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det A \ \det B$.

(ii) $\det \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det(A + iB) \ \det(A - iB)$.

Paper 4, Section I

9H Markov Chains

Let $(X_n : n \geqslant 0)$ be a homogeneous Markov chain with state space S and transition matrix $P = (p_{i,j} : i, j \in S)$.

(a) Let $W_n = X_{2n}$, $n = 0, 1, 2, \ldots$. Show that $(W_n : n \geqslant 0)$ is a Markov chain and give its transition matrix. If $\lambda_i = \mathbb{P}(X_0 = i)$, $i \in S$, find $\mathbb{P}(W_1 = 0)$ in terms of the λ_i and the $p_{i,j}$.

[Results from the course may be quoted without proof, provided they are clearly stated.]

(b) Suppose that $S = \{-1, 0, 1\}$, $p_{0,1} = p_{-1,-1} = 0$ and $p_{-1,0} \neq p_{1,0}$. Let $Y_n = |X_n|$, $n = 0, 1, 2, \ldots$. In terms of the $p_{i,j}$, find

(i) $\mathbb{P}(Y_{n+1} = 0 \mid Y_n = 1, Y_{n-1} = 0)$ and

(ii) $\mathbb{P}(Y_{n+1} = 0 \mid Y_n = 1, Y_{n-1} = 1, Y_{n-2} = 0)$.

What can you conclude about whether or not $(Y_n : n \geqslant 0)$ is a Markov chain?

Paper 3, Section I

9H Markov Chains

Let $(X_n : n \geqslant 0)$ be a homogeneous Markov chain with state space S. For i, j in S let $p_{i,j}(n)$ denote the n-step transition probability $\mathbb{P}(X_n = j \mid X_0 = i)$.

(i) Express the $(m+n)$-step transition probability $p_{i,j}(m+n)$ in terms of the n-step and m-step transition probabilities.

(ii) Write $i \to j$ if there exists $n \geqslant 0$ such that $p_{i,j}(n) > 0$, and $i \leftrightarrow j$ if $i \to j$ and $j \to i$. Prove that if $i \leftrightarrow j$ and $i \neq j$ then either both i and j are recurrent or both i and j are transient. [You may assume that a state i is recurrent if and only if $\sum_{n=0}^{\infty} p_{i,i}(n) = \infty$, and otherwise i is transient.]

(iii) A Markov chain has state space $\{0, 1, 2, 3\}$ and transition matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

For each state i, determine whether i is recurrent or transient. [Results from the course may be quoted without proof, provided they are clearly stated.]

Paper 1, Section II
20H Markov Chains
Consider a homogeneous Markov chain $(X_n : n \geqslant 0)$ with state space S and transition matrix $P = (p_{i,j} : i, j \in S)$. For a state i, define the terms *aperiodic*, *positive recurrent* and *ergodic*.

Let $S = \{0, 1, 2, \ldots\}$ and suppose that for $i \geqslant 1$ we have $p_{i,i-1} = 1$ and

$$p_{0,0} = 0,\ p_{0,j} = pq^{j-1},\ j = 1, 2, \ldots,$$

where $p = 1 - q \in (0, 1)$. Show that this Markov chain is irreducible.

Let $T_0 = \inf\{n \geqslant 1 : X_n = 0\}$ be the first passage time to 0. Find $\mathbb{P}(T_0 = n \mid X_0 = 0)$ and show that state 0 is ergodic.

Find the invariant distribution π for this Markov chain. Write down:

(i) the mean recurrence time for state i, $i \geqslant 1$;

(ii) $\lim_{n\to\infty} \mathbb{P}(X_n \neq 0 \mid X_0 = 0)$.

[Results from the course may be quoted without proof, provided they are clearly stated.]

Paper 2, Section II
20H Markov Chains
Let $(X_n : n \geqslant 0)$ be a homogeneous Markov chain with state space S and transition matrix $P = (p_{i,j} : i, j \in S)$. For $A \subseteq S$, let

$$H^A = \inf\{n \geqslant 0 : X_n \in A\} \text{ and } h_i^A = \mathbb{P}(H^A < \infty \mid X_0 = i),\ i \in S.$$

Prove that $h^A = (h_i^A : i \in S)$ is the minimal non-negative solution to the equations

$$h_i^A = \begin{cases} 1 & \text{for } i \in A \\ \sum_{j \in S} p_{i,j} h_j^A & \text{otherwise.} \end{cases}$$

Three people A, B and C play a series of two-player games. In the first game, two people play and the third person sits out. Any subsequent game is played between the winner of the previous game and the person sitting out the previous game. The overall winner of the series is the first person to win two consecutive games. The players are evenly matched so that in any game each of the two players has probability $\frac{1}{2}$ of winning the game, independently of all other games. For $n = 1, 2, \ldots$, let X_n be the ordered pair consisting of the winners of games n and $n + 1$. Thus the state space is $\{AA, AB, AC, BA, BB, BC, CA, CB, CC\}$, and, for example, $X_1 = AC$ if A wins the first game and C wins the second.

The first game is between A and B. Treating AA, BB and CC as absorbing states, or otherwise, find the probability of winning the series for each of the three players.

Paper 4, Section I
5D Methods
Consider the ordinary differential equation

$$\frac{d^2\psi}{dz^2} - \left[\frac{15k^2}{4(k|z|+1)^2} - 3k\delta(z)\right]\psi = 0\,, \qquad (\dagger)$$

where k is a positive constant and δ denotes the Dirac delta function. Physically relevant solutions for ψ are bounded over the entire range $z \in \mathbb{R}$.

(i) Find piecewise bounded solutions to this differential equations in the ranges $z > 0$ and $z < 0$, respectively. [*Hint: The equation* $\frac{d^2y}{dx^2} - \frac{c}{x^2}y = 0$ *for a constant c may be solved using the Ansatz* $y = x^\alpha$.]

(ii) Derive a matching condition at $z = 0$ by integrating ($\dagger$) over the interval $(-\epsilon, \epsilon)$ with $\epsilon \to 0$ and use this condition together with the requirement that ψ be continuous at $z = 0$ to determine the solution over the entire range $z \in \mathbb{R}$.

Paper 2, Section I
5D Methods
(i) Calculate the Fourier series for the periodic extension on $\mathbb{R}$ of the function

$$f(x) = x(1-x)\,,$$

defined on the interval $[0, 1)$.

(ii) Explain why the Fourier series for the periodic extension of $f'(x)$ can be obtained by term-by-term differentiation of the series for $f(x)$.

(iii) Let $G(x)$ be the Fourier series for the periodic extension of $f'(x)$. Determine the value of $G(0)$ and explain briefly how it is related to the values of f'.

Paper 3, Section I
7D Methods
Using the method of characteristics, solve the differential equation

$$-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0\,,$$

where x, $y \in \mathbb{R}$ and $u = \cos y^2$ on $x = 0$, $y \geqslant 0$.

Paper 1, Section II

14D Methods

(a) Legendre's differential equation may be written

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0\,, \qquad y(1) = 1\,.$$

Show that for non-negative integer n, this equation has a solution $P_n(x)$ that is a polynomial of degree n. Find P_0, P_1 and P_2 explicitly.

(b) Laplace's equation in spherical coordinates for an axisymmetric function $U(r,\theta)$ (i.e. no ϕ dependence) is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) = 0\,.$$

Use separation of variables to find the general solution for $U(r,\theta)$.

Find the solution $U(r,\theta)$ that satisfies the boundary conditions

$$U(r,\theta) \to v_0\, r\, \cos\theta \quad \text{ as } r\to\infty\,,$$

$$\frac{\partial U}{\partial r} = 0 \quad \text{ at } r = r_0\,,$$

where v_0 and r_0 are constants.

Paper 3, Section II

15D Methods

Let $\mathcal{L}$ be a linear second-order differential operator on the interval $[0, \pi/2]$. Consider the problem

$$\mathcal{L}y(x) = f(x)\,; \qquad y(0) = y(\pi/2) = 0\,,$$

with $f(x)$ bounded in $[0, \pi/2]$.

(i) How is a Green's function for this problem defined?

(ii) How is a solution $y(x)$ for this problem constructed from the Green's function?

(iii) Describe the continuity and jump conditions used in the construction of the Green's function.

(iv) Use the continuity and jump conditions to construct the Green's function for the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{5}{4}y = f(x)$$

on the interval $[0, \pi/2]$ with the boundary conditions $y(0) = 0$, $y(\pi/2)$=0 and an arbitrary bounded function $f(x)$. Use the Green's function to construct a solution $y(x)$ for the particular case $f(x) = e^{x/2}$.

Paper 2, Section II

16D Methods

The Fourier transform $\tilde{f}$ of a function f is defined as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \,, \quad \text{so that } \; f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \,.$$

A Green's function $G(t, t', x, x')$ for the diffusion equation in one spatial dimension satisfies

$$\frac{\partial G}{\partial t} - D \frac{\partial^2 G}{\partial x^2} = \delta(t - t') \, \delta(x - x') \,.$$

(a) By applying a Fourier transform, show that the Fourier transform $\tilde{G}$ of this Green's function and the Green's function G are

$$\begin{aligned} \tilde{G}(t, t', k, x') &= H(t - t') \, e^{-ikx'} \, e^{-Dk^2(t-t')} \,, \\ G(t, t', x, x') &= \frac{H(t - t')}{\sqrt{4\pi D(t - t')}} e^{-\frac{(x-x')^2}{4D(t-t')}} \,, \end{aligned}$$

where H is the Heaviside function. [*Hint: The Fourier transform* $\tilde{F}$ *of a Gaussian* $F(x) = \dfrac{1}{\sqrt{4\pi a}} e^{-\frac{x^2}{4a}}$, $a = \text{const}$, *is given by* $\tilde{F}(k) = e^{-ak^2}$.]

(b) The analogous result for the Green's function for the diffusion equation in two spatial dimensions is

$$G(t, t', x, x', y, y') = \frac{H(t - t')}{4\pi D(t - t')} e^{-\frac{1}{4D(t-t')} \left[(x-x')^2 + (y-y')^2 \right]} \,.$$

Use this Green's function to construct a solution for $t \geqslant 0$ to the diffusion equation

$$\frac{\partial \Psi}{\partial t} - D \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) = p(t) \, \delta(x) \, \delta(y) \,,$$

with the initial condition $\Psi(0, x, y) = 0$.

Now set

$$p(t) = \begin{cases} p_0 = \text{const} & \text{for } \; 0 \leqslant t \leqslant t_0 \\ 0 & \text{for } \; t > t_0 \end{cases}$$

Find the solution $\Psi(t, x, y)$ for $t > t_0$ in terms of the exponential integral defined by

$$Ei(-\eta) = -\int_{\eta}^{\infty} \frac{e^{-\lambda}}{\lambda} d\lambda \,.$$

Use the approximation $Ei(-\eta) \approx \ln \eta + C$, valid for $\eta \ll 1$, to simplify this solution $\Psi(t, x, y)$. Here $C \approx 0.577$ is Euler's constant.

Paper 4, Section II

17D Methods

Let $f(x)$ be a complex-valued function defined on the interval $[-L, L]$ and periodically extended to $x \in \mathbb{R}$.

(i) Express $f(x)$ as a complex Fourier series with coefficients c_n, $n \in \mathbb{Z}$. How are the coefficients c_n obtained from $f(x)$?

(ii) State Parseval's theorem for complex Fourier series.

(iii) Consider the function $f(x) = \cos(\alpha x)$ on the interval $[-\pi, \pi]$ and periodically extended to $x \in \mathbb{R}$ for a complex but non-integer constant α. Calculate the complex Fourier series of $f(x)$.

(iv) Prove the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \tan(\alpha\pi)} .$$

(v) Now consider the case where α is a real, non-integer constant. Use Parseval's theorem to obtain a formula for

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2 - \alpha^2)^2} .$$

What value do you obtain for this series for $\alpha = 5/2$?

Paper 3, Section I

3E Metric and Topological Spaces

Suppose (X, d) is a metric space. Do the following necessarily define a metric on X? Give proofs or counterexamples.

(i) $d_1(x, y) = kd(x, y)$ for some constant $k > 0$, for all $x, y \in X$.

(ii) $d_2(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$.

(iii) $d_3(x, y) = (d(x, y))^2$ for all $x, y \in X$.

Paper 2, Section I

4E Metric and Topological Spaces

Let X and Y be topological spaces. What does it mean to say that a function $f : X \to Y$ is *continuous*?

Are the following statements true or false? Give proofs or counterexamples.

(i) Every continuous function $f : X \to Y$ is an open map, i.e. if U is open in X then $f(U)$ is open in Y.

(ii) If $f : X \to Y$ is continuous and bijective then f is a homeomorphism.

(iii) If $f : X \to Y$ is continuous, open and bijective then f is a homeomorphism.

Paper 1, Section II

12E Metric and Topological Spaces

Define what it means for a topological space to be *compact*. Define what it means for a topological space to be *Hausdorff*.

Prove that a compact subspace of a Hausdorff space is closed. Hence prove that if C_1 and C_2 are compact subspaces of a Hausdorff space X then $C_1 \cap C_2$ is compact.

A subset U of $\mathbb{R}$ is open in the *cocountable topology* if U is empty or its complement in $\mathbb{R}$ is countable. Is $\mathbb{R}$ Hausdorff in the cocountable topology? Which subsets of $\mathbb{R}$ are compact in the cocountable topology?

Paper 4, Section II

13E Metric and Topological Spaces

Explain what it means for a metric space to be *complete.*

Let X be a metric space. We say the subsets A_i of X, with $i \in \mathbb{N}$, form a *descending sequence* in X if $A_1 \supset A_2 \supset A_3 \supset \cdots$.

Prove that the metric space X is complete if and only if any descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed subsets of X, such that the diameters of the subsets A_i converge to zero, has an intersection $\bigcap_{i=1}^{\infty} A_i$ that is non-empty.

[Recall that the diameter $\operatorname{diam}(S)$ of a set S is the supremum of the set $\{d(x,y) : x, y \in S\}$.]

Give examples of

(i) a metric space X, and a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed subsets of X, with $\operatorname{diam}(A_i)$ converging to 0 but $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

(ii) a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty sets in $\mathbb{R}$ with $\operatorname{diam}(A_i)$ converging to 0 but $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

(iii) a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed sets in $\mathbb{R}$ with $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

Paper 1, Section I

6C Numerical Analysis

(i) A general multistep method for the numerical approximation to the scalar differential equation $y' = f(t, y)$ is given by

$$\sum_{\ell=0}^{s} \rho_\ell \, y_{n+\ell} = h \sum_{\ell=0}^{s} \sigma_\ell f_{n+\ell}, \qquad n = 0, 1, \ldots$$

where $f_{n+\ell} = f(t_{n+\ell}, y_{n+\ell})$. Show that this method is of order $p \geqslant 1$ if and only if

$$\rho(\mathrm{e}^z) - z\sigma(\mathrm{e}^z) = \mathcal{O}(z^{p+1}) \quad \text{as} \quad z \to 0$$

where

$$\rho(w) = \sum_{\ell=0}^{s} \rho_\ell \, w^\ell \quad \text{and} \quad \sigma(w) = \sum_{\ell=0}^{s} \sigma_\ell \, w^\ell \,.$$

(ii) A particular three-step implicit method is given by

$$y_{n+3} + (a-1)y_{n+1} - a y_n = h \left(f_{n+3} + \sum_{\ell=0}^{2} \sigma_\ell f_{n+\ell} \right) .$$

where the σ_ℓ are chosen to make the method third order. [The σ_ℓ need not be found.] For what values of a is the method convergent?

Paper 4, Section I

8C Numerical Analysis

Consider the quadrature given by

$$\int_0^\pi w(x) f(x) dx \approx \sum_{k=1}^{\nu} b_k f(c_k)$$

for $\nu \in \mathbb{N}$, disjoint $c_k \in (0, \pi)$ and $w > 0$. Show that it is not possible to make this quadrature exact for all polynomials of order 2ν.

For the case that $\nu = 2$ and $w(x) = \sin x$, by considering orthogonal polynomials find suitable b_k and c_k that make the quadrature exact on cubic polynomials.

[*Hint:* $\int_0^\pi x^2 \sin x \, dx = \pi^2 - 4$ *and* $\int_0^\pi x^3 \sin x \, dx = \pi^3 - 6\pi$.]

Paper 1, Section II

18C Numerical Analysis

Define a Householder transformation H and show that it is an orthogonal matrix. Briefly explain how these transformations can be used for QR factorisation of an $m \times n$ matrix.

Using Householder transformations, find a QR factorisation of

$$\mathsf{A} = \begin{bmatrix} 2 & 5 & 4 \\ 2 & 5 & 1 \\ -2 & 1 & 5 \\ 2 & -1 & 16 \end{bmatrix}.$$

Using this factorisation, find the value of λ for which

$$\mathsf{A}\,x = \begin{bmatrix} 1+\lambda \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

has a unique solution $x \in \mathbb{R}^3$.

Paper 3, Section II

19C Numerical Analysis

A Runge–Kutta scheme is given by

$$k_1 = hf(y_n), \quad k_2 = hf(y_n + [(1-a)k_1 + ak_2]), \quad y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

for the solution of an autonomous differential equation $y' = f(y)$, where a is a real parameter. What is the order of the scheme? Identify all values of a for which the scheme is A-stable. Determine the linear stability domain for this range.

Paper 2, Section II

19C Numerical Analysis

A linear functional acting on $f \in C^{k+1}[a,b]$ is approximated using a linear scheme $L(f)$. The approximation is exact when f is a polynomial of degree k. The error is given by $\lambda(f)$. Starting from the Taylor formula for $f(x)$ with an integral remainder term, show that the error can be written in the form

$$\lambda(f) = \frac{1}{k!}\int_a^b K(\theta) f^{(k+1)}(\theta) d\theta$$

subject to a condition on λ that you should specify. Give an expression for $K(\theta)$.

Find c_0, c_1 and c_3 such that the approximation scheme

$$f''(2) \approx c_0 f(0) + c_1 f(1) + c_3 f(3)$$

is exact for all f that are polynomials of degree 2. Assuming $f \in C^3[0,3]$, apply the Peano kernel theorem to the error. Find and sketch $K(\theta)$ for $k=2$.

Find the minimum values for the constants r and s for which

$$|\lambda(f)| \leqslant r\|f^{(3)}\|_1 \quad \text{and} \quad |\lambda(f)| \leqslant s\|f^{(3)}\|_\infty$$

and show explicitly that both error bounds hold for $f(x) = x^3$.

Paper 1, Section I

8H Optimization

State and prove the Lagrangian sufficiency theorem.

Use the Lagrangian sufficiency theorem to find the minimum of $2x_1^2 + 2x_2^2 + x_3^2$ subject to $x_1 + x_2 + x_3 = 1$ (where x_1, x_2 and x_3 are real).

Paper 2, Section I

9H Optimization

Explain what is meant by a two-player zero-sum game with $m \times n$ pay-off matrix $P = (p_{ij})$, and state the optimal strategies for each player.

Find these optimal strategies when

$$P = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}.$$

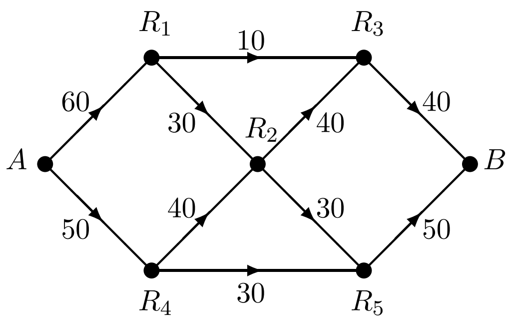
Paper 4, Section II
20H Optimization

Consider a network with a single source and a single sink, where all the edge capacities are finite. Write down the maximum flow problem, and state the max-flow min-cut theorem.

Describe the Ford–Fulkerson algorithm. If all edge capacities are integers, explain why, starting from a suitable initial flow, the algorithm is guaranteed to end after a finite number of iterations.

The graph in the diagram below represents a one-way road network taking traffic from point A to point B via five roundabouts R_i, $i = 1, \ldots, 5$. The capacity of each road is shown on the diagram in terms of vehicles per minute. Assuming that all roundabouts can deal with arbitrary amounts of flow of traffic, find the maximum flow of traffic (in vehicles per minute) through this network of roads. Show that this flow is indeed optimal.

After a heavy storm, roundabout R_2 is flooded and only able to deal with at most 20 vehicles per minute. Find a suitable new network for the situation after the storm. Apply the Ford–Fulkerson algorithm to the new network, starting with the zero flow and explaining each step, to determine the maximum flow and the associated flows on each road.



Paper 3, Section II
21H Optimization

Use the two-phase simplex method to maximise $2x_1 + x_2 + x_3$ subject to the constraints

$$x_1 + x_2 \geqslant 1, \quad x_1 + x_2 + 2x_3 \leqslant 4, \quad x_i \geqslant 0 \text{ for } i = 1, 2, 3.$$

Derive the dual of this linear programming problem and find the optimal solution of the dual.

Paper 4, Section I

6A Quantum Mechanics

For some quantum mechanical observable Q, prove that its uncertainty (ΔQ) satisfies

$$(\Delta Q)^2 = \langle Q^2 \rangle - \langle Q \rangle^2 .$$

A quantum mechanical harmonic oscillator has Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2},$$

where $m > 0$. Show that (in a stationary state of energy E)

$$E \geqslant \frac{(\Delta p)^2}{2m} + \frac{m\omega^2 (\Delta x)^2}{2}.$$

Write down the Heisenberg uncertainty relation. Then, use it to show that

$$E \geqslant \frac{1}{2}\hbar\omega$$

for our stationary state.

Paper 3, Section I

8A Quantum Mechanics

The wavefunction of a normalised Gaussian wavepacket for a particle of mass m in one dimension with potential $V(x) = 0$ is given by

$$\psi(x,t) = B\sqrt{A(t)}\exp\left(\frac{-x^2 A(t)}{2}\right),$$

where $A(0) = 1$. Given that $\psi(x,t)$ is a solution of the time-dependent Schrödinger equation, find the complex-valued function $A(t)$ and the real constant B.

[You may assume that $\int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx = \sqrt{\pi}/\sqrt{\lambda}$.]

Paper 1, Section II

15A Quantum Mechanics

Consider a particle confined in a one-dimensional infinite potential well: $V(x) = \infty$ for $|x| \geqslant a$ and $V(x) = 0$ for $|x| < a$. The normalised stationary states are

$$\psi_n(x) = \begin{cases} \alpha_n \sin\left(\dfrac{\pi n(x+a)}{2a}\right) & \text{for} \quad |x| < a \\ 0 & \text{for} \quad |x| \geqslant a \end{cases}$$

where $n = 1, 2, \ldots$.

(i) Determine the α_n and the stationary states' energies E_n.

(ii) A state is prepared within this potential well: $\psi(x) \propto x$ for $0 < x < a$, but $\psi(x) = 0$ for $x \leqslant 0$ or $x \geqslant a$. Find an explicit expansion of $\psi(x)$ in terms of $\psi_n(x)$.

(iii) If the energy of the state is then immediately measured, show that the probability that it is *greater* than $\frac{\hbar^2\pi^2}{ma^2}$ is

$$\sum_{n=0}^{4} \frac{b_n}{\pi^n},$$

where the b_n are integers which you should find.

(iv) By considering the normalisation condition for $\psi(x)$ in terms of the expansion in $\psi_n(x)$, show that

$$\frac{\pi^2}{3} = \sum_{p=1}^{\infty} \frac{A}{p^2} + \frac{B}{(2p-1)^2}\left(1 + \frac{C(-1)^p}{(2p-1)\pi}\right)^2,$$

where A, B and C are integers which you should find.

Paper 3, Section II

16A Quantum Mechanics

The Hamiltonian of a two-dimensional isotropic harmonic oscillator is given by

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2)\,,$$

where x and y denote position operators and p_x and p_y the corresponding momentum operators.

State without proof the commutation relations between the operators x, y, p_x, p_y. From these commutation relations, write $[x^2,\ p_x]$ and $[x,\ p_x^2]$ in terms of a single operator. Now consider the observable

$$L = xp_y - yp_x.$$

Ehrenfest's theorem states that, for some observable Q with expectation value $\langle Q\rangle$,

$$\frac{d\langle Q\rangle}{dt} = \frac{1}{i\hbar}\langle [Q,\ H]\rangle + \langle \frac{\partial Q}{\partial t}\rangle.$$

Use it to show that the expectation value of L is constant with time.

Given two states

$$\psi_1 = \alpha x \exp\left(-\beta(x^2+y^2)\right) \text{ and } \psi_2 = \alpha y \exp\left(-\beta(x^2+y^2)\right)\,,$$

where α and β are constants, find a normalised linear combination of ψ_1 and ψ_2 that is an eigenstate of L, and the corresponding L eigenvalue. [You may assume that α correctly normalises both ψ_1 and ψ_2.] If a quantum state is prepared in the linear combination you have found at time $t = 0$, what is the expectation value of L at a later time t?

Paper 2, Section II

17A Quantum Mechanics

For an electron of mass m in a hydrogen atom, the time-independent Schrödinger equation may be written as

$$-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{2mr^2}L^2\psi-\frac{e^2}{4\pi\epsilon_0 r}\psi=E\psi.$$

Consider normalised energy eigenstates of the form

$$\psi_{lm}(r,\ \theta,\ \phi)=R(r)Y_{lm}(\theta,\ \phi)$$

where Y_{lm} are orbital angular momentum eigenstates:

$$L^2Y_{lm}=\hbar^2 l(l+1)Y_{lm},\qquad L_3Y_{lm}=\hbar mY_{lm},$$

where $l=1,\ 2,\ \ldots$ and $m=0,\ \pm1,\ \pm2,\ \ldots\pm l$. The Y_{lm} functions are normalised with $\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}|Y_{lm}|^2\ \sin\theta\ d\theta\ d\phi=1$.

(i) Write down the resulting equation satisfied by $R(r)$, for fixed l. Show that it has solutions of the form

$$R(r)=Ar^l\exp\left(-\frac{r}{a(l+1)}\right),$$

where a is a constant which you should determine. Show that

$$E=-\frac{e^2}{D\pi\epsilon_0 a},$$

where D is an integer which you should find (in terms of l). Also, show that

$$|A|^2=\frac{2^{2l+3}}{a^F G!(l+1)^H},$$

where F, G and H are integers that you should find in terms of l.

(ii) Given the radius of the proton $r_p\ll a$, show that the probability of the electron being found within the proton is approximately

$$\frac{2^{2l+3}}{C}\left(\frac{r_p}{a}\right)^{2l+3}\left[1+\mathcal{O}\left(\frac{r_p}{a}\right)\right],$$

finding the integer C in terms of l.

[You may assume that $\int_0^\infty t^l e^{-t}dt=l!$.]

Paper 1, Section I

7H Statistics

Consider an estimator $\hat{\theta}$ of an unknown parameter θ, and assume that $\mathbb{E}_\theta\big(\hat{\theta}^2\big) < \infty$ for all θ. Define the *bias* and *mean squared error* of $\hat{\theta}$.

Show that the mean squared error of $\hat{\theta}$ is the sum of its variance and the square of its bias.

Suppose that $X_1, \ldots, X_n$ are independent identically distributed random variables with mean θ and variance θ^2, and consider estimators of θ of the form $k\bar{X}$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

(i) Find the value of k that gives an unbiased estimator, and show that the mean squared error of this unbiased estimator is θ^2/n.

(ii) Find the range of values of k for which the mean squared error of $k\bar{X}$ is smaller than θ^2/n.

Paper 2, Section I

8H Statistics

There are 100 patients taking part in a trial of a new surgical procedure for a particular medical condition. Of these, 50 patients are randomly selected to receive the new procedure and the remaining 50 receive the old procedure. Six months later, a doctor assesses whether or not each patient has fully recovered. The results are shown below:

	Fully recovered	Not fully recovered
Old procedure	25	25
New procedure	31	19

The doctor is interested in whether there is a difference in full recovery rates for patients receiving the two procedures. Carry out an appropriate 5% significance level test, stating your hypotheses carefully. [You do not need to derive the test.] What conclusion should be reported to the doctor?

[*Hint: Let* $\chi^2_k(\alpha)$ *denote the upper* 100α *percentage point of a* χ^2_k *distribution. Then*

$$\chi^2_1(0.05) = 3.84,\ \chi^2_2(0.05) = 5.99,\ \chi^2_3(0.05) = 7.82,\ \chi^2_4(0.05) = 9.49.]$$

Paper 4, Section II

19H Statistics

Consider a linear model

$$\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \qquad (\dagger)$$

where X is a known $n \times p$ matrix, $\boldsymbol{\beta}$ is a $p \times 1$ $(p < n)$ vector of unknown parameters and $\boldsymbol{\varepsilon}$ is an $n \times 1$ vector of independent $N(0, \sigma^2)$ random variables with σ^2 unknown. Assume that X has full rank p. Find the least squares estimator $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ and derive its distribution. Define the residual sum of squares RSS and write down an unbiased estimator $\hat{\sigma}^2$ of σ^2.

Suppose that $V_i = a + bu_i + \delta_i$ and $Z_i = c + dw_i + \eta_i$, for $i = 1, \ldots, m$, where u_i and w_i are known with $\sum_{i=1}^m u_i = \sum_{i=1}^m w_i = 0$, and $\delta_1, \ldots, \delta_m, \eta_1, \ldots, \eta_m$ are independent $N(0, \sigma^2)$ random variables. Assume that at least two of the u_i are distinct and at least two of the w_i are distinct. Show that $\mathbf{Y} = (V_1, \ldots, V_m, Z_1, \ldots, Z_m)^T$ (where T denotes transpose) may be written as in $(\dagger)$ and identify X and $\boldsymbol{\beta}$. Find $\hat{\boldsymbol{\beta}}$ in terms of the V_i, Z_i, u_i and w_i. Find the distribution of $\hat{b} - \hat{d}$ and derive a 95% confidence interval for $b - d$.

[*Hint: You may assume that* $\frac{RSS}{\sigma^2}$ *has a* χ^2_{n-p} *distribution, and that* $\hat{\beta}$ *and the residual sum of squares are independent. Properties of* χ^2 *distributions may be used without proof.*]

Paper 1, Section II

19H Statistics

Suppose that X_1, X_2, and X_3 are independent identically distributed Poisson random variables with expectation θ, so that

$$\mathbb{P}(X_i = x) = \frac{e^{-\theta}\theta^x}{x!} \quad x = 0, 1, \ldots,$$

and consider testing $H_0 : \theta = 1$ against $H_1 : \theta = \theta_1$, where θ_1 is a known value greater than 1. Show that the test with critical region $\{(x_1, x_2, x_3) : \sum_{i=1}^{3} x_i > 5\}$ is a likelihood ratio test of H_0 against H_1. What is the size of this test? Write down an expression for its power.

A scientist counts the number of bird territories in n randomly selected sections of a large park. Let Y_i be the number of bird territories in the ith section, and suppose that $Y_1, \ldots, Y_n$ are independent Poisson random variables with expectations $\theta_1, \ldots, \theta_n$ respectively. Let a_i be the area of the ith section. Suppose that $n = 2m$, $a_1 = \cdots = a_m = a(> 0)$ and $a_{m+1} = \cdots = a_{2m} = 2a$. Derive the generalised likelihood ratio Λ for testing

$$H_0 : \theta_i = \lambda a_i \text{ against } H_1 : \theta_i = \begin{cases} \lambda_1 & i = 1, \ldots, m \\ \lambda_2 & i = m+1, \ldots, 2m. \end{cases}$$

What should the scientist conclude about the number of bird territories if $2\log_e(\Lambda)$ is 15.67?

[*Hint: Let $F_\theta(x)$ be $\mathbb{P}(W \leqslant x)$ where W has a Poisson distribution with expectation θ. Then*

$$F_1(3) = 0.998, \quad F_3(5) = 0.916, \quad F_3(6) = 0.966, \quad F_5(3) = 0.433\,.]$$

Paper 3, Section II

20H Statistics

Suppose that $X_1, \ldots, X_n$ are independent identically distributed random variables with

$$\mathbb{P}(X_i = x) = \binom{k}{x} \theta^x (1-\theta)^{k-x}, \quad x = 0, \ldots, k,$$

where k is known and θ $(0 < \theta < 1)$ is an unknown parameter. Find the maximum likelihood estimator $\hat{\theta}$ of θ.

Statistician 1 has prior density for θ given by $\pi_1(\theta) = \alpha\theta^{\alpha-1}$, $0 < \theta < 1$, where $\alpha > 1$. Find the posterior distribution for θ after observing data $X_1 = x_1, \ldots, X_n = x_n$. Write down the posterior mean $\hat{\theta}_1^{(B)}$, and show that

$$\hat{\theta}_1^{(B)} = c\,\hat{\theta} + (1-c)\tilde{\theta}_1,$$

where $\tilde{\theta}_1$ depends only on the prior distribution and c is a constant in $(0,1)$ that is to be specified.

Statistician 2 has prior density for θ given by $\pi_2(\theta) = \alpha(1-\theta)^{\alpha-1}$, $0 < \theta < 1$. Briefly describe the prior beliefs that the two statisticians hold about θ. Find the posterior mean $\hat{\theta}_2^{(B)}$ and show that $\hat{\theta}_2^{(B)} < \hat{\theta}_1^{(B)}$.

Suppose that α increases (but n, k and the x_i remain unchanged). How do the prior beliefs of the two statisticians change? How does c vary? Explain briefly what happens to $\hat{\theta}_1^{(B)}$ and $\hat{\theta}_2^{(B)}$.

[*Hint: The Beta*(α, β) $(\alpha > 0,\ \beta > 0)$ *distribution has density*

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1,$$

with expectation $\frac{\alpha}{\alpha+\beta}$ *and variance* $\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$. *Here,* $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, $\alpha > 0$, *is the Gamma function.*]

Paper 1, Section I

4C Variational Principles

Define the Legendre transform $f^*(\mathbf{p})$ of a function $f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$.

Show that for $g(\mathbf{x}) = \lambda\, f(\mathbf{x} - \mathbf{x}_0) - \mu$,

$$g^*(\mathbf{p}) = \lambda f^*\left(\frac{\mathbf{p}}{\lambda}\right) + \mathbf{p}^{\mathbf{T}}\mathbf{x}_0 + \mu.$$

Show that for $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\,\mathbf{x}$ where $\mathbf{A}$ is a real, symmetric, invertible matrix with positive eigenvalues,

$$f^*(\mathbf{p}) = \tfrac{1}{2}\mathbf{p}^{\mathbf{T}}\mathbf{A}^{-\mathbf{1}}\,\mathbf{p}.$$

Paper 3, Section I

6C Variational Principles

Let $f(x, y, z) = xz + yz$. Using Lagrange multipliers, find the location(s) and value of the maximum of f on the intersection of the unit sphere ($x^2 + y^2 + z^2 = 1$) and the ellipsoid given by $\frac{1}{4}x^2 + \frac{1}{4}y^2 + 4z^2 = 1$.

Paper 2, Section II

15C Variational Principles

Write down the Euler–Lagrange equation for the integral

$$\int f(y, y', x)dx.$$

An ant is walking on the surface of a sphere, which is parameterised by $\theta \in [0, \pi]$ (angle from top of sphere) and $\phi \in [0, 2\pi)$ (azimuthal angle). The sphere is sticky towards the top and the bottom and so the ant's speed is proportional to $\sin\theta$. Show that the ant's fastest route between two points will be of the form

$$\sinh(A\phi + B) = \cot\theta$$

for some constants A and B. [A, B need not be determined.]

Paper 4, Section II

16C Variational Principles

Consider the integral

$$I = \int f(y, y')dx.$$

Show that if f satisfies the Euler–Lagrange equation, then

$$f - y'\frac{\partial f}{\partial y'} = \text{constant.}$$

An axisymmetric soap film $y(x)$ is formed between two circular wires at $x = \pm l$. The wires both have radius r. Show that the shape that minimises the surface area takes the form

$$y(x) = k \cosh \frac{x}{k}.$$

Show that there exist two possible k that satisfy the boundary conditions for r/l sufficiently large.

Show that for these solutions the second variation is given by

$$\delta^2 I = \pi \int_{-l}^{+l} \left(k\eta'^2 - \frac{1}{k}\eta^2\right) \text{sech}^2\left(\frac{x}{k}\right) dx$$

where η is an axisymmetric perturbation with $\eta(\pm l) = 0$.