UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday, 7 June, 2013 9:00 am to 12:00 noon

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *…,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. Define the Riemann zeta function $\zeta(s)$ for $\sigma > 1$. Show that for $\sigma > 1$,

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1},$$

where the product is taken over all primes. Deduce that there are infinitely many primes.

2F Topics in Analysis

State the Baire Category Theorem. A set $X \subseteq \mathbb{R}$ is said to be a G_δ-set if it is the intersection of countably many open sets. Show that the set $\mathbb{Q}$ of rationals is not a G_δ-set.

[You may assume that the rationals are countable and that $\mathbb{R}$ is complete.]

3G Geometry and Groups

Let Δ_1, Δ_2 be two disjoint closed discs in the Riemann sphere with bounding circles Γ_1, Γ_2 respectively. Let J_k be inversion in the circle Γ_k and let T be the Möbius transformation $J_2 \circ J_1$.

Show that, if $w \notin \Delta_1$, then $T(w) \in \Delta_2$ and so $T^n(w) \in \Delta_2$ for $n = 1, 2, 3, \ldots$. Deduce that T has a fixed point in Δ_2 and a second in Δ_1.

Deduce that there is a Möbius transformation A with

$$A(\Delta_1) = \{z : |z| \leqslant 1\} \qquad \text{and} \qquad A(\Delta_2) = \{z : |z| \geqslant R\}$$

for some $R > 1$.

4H Coding and Cryptography

Describe how a stream cipher works. What is a one-time pad?

A one-time pad is used to send the message $x_1x_2x_3x_4x_5x_6y_7$ which is encoded as 0101011. In error, it is reused to send the message $y_0x_1x_2x_3x_4x_5x_6$ which is encoded as 0100010. Show that there are two possibilities for the substring $x_1x_2x_3x_4x_5x_6$, and find them.

5J Statistical Modelling

The output X of a process depends on the levels of two adjustable variables: A, a factor with four levels, and B, a factor with two levels. For each combination of a level of A and a level of B, nine independent values of X are observed.

Explain and interpret the R commands and (abbreviated) output below. In particular, describe the model being fitted, and describe and comment on the hypothesis tests performed under the summary and anova commands.

```
> fit1 <- lm(x~a+b)

> summary(fit1)

Coefficients:
              Estimate  Std. Error  t value  Pr(>|t|)
(Intercept)     2.5445      0.2449    10.39  6.66e-16  ***
a2             -5.6704      0.4859   -11.67   < 2e-16  ***
a3              4.3254      0.3480    12.43   < 2e-16  ***
a4             -0.5003      0.3734    -1.34    0.0923
b2             -3.5689      0.2275   -15.69   < 2e-16  ***

> anova(fit1)

Response:  x

           Df  Sum Sq  mean Sq  F value    Pr(>F)
a          3    71.51    23.84    17.79   1.34e-8  ***
b          1   105.11   105.11    78.44  6.91e-13  ***
Residuals  67   89.56     1.34
```

6A Mathematical Biology

A model of two populations competing for resources takes the form

$$\begin{aligned}\frac{dn_1}{dt} &= r_1 n_1(1 - n_1 - a_{12}n_2)\,,\\ \frac{dn_2}{dt} &= r_2 n_2(1 - n_2 - a_{21}n_1)\,,\end{aligned}$$

where all parameters are positive. Give a brief biological interpretation of a_{12}, a_{21}, r_1 and r_2. Briefly describe the dynamics of each population in the absence of the other.

Give conditions for there to exist a steady-state solution with both populations present (that is, $n_1 > 0$ and $n_2 > 0$), and give conditions for this solution to be stable.

In the case where there exists a solution with both populations present but the solution is not stable, what is the likely long-term outcome for the biological system? Explain your answer with the aid of a phase diagram in the (n_1, n_2) plane.

7C Dynamical Systems

Consider the system

$$\begin{aligned} \dot{x} &= y + ax + bx^3 \,, \\ \dot{y} &= -x \,. \end{aligned}$$

What is the Poincaré index of the single fixed point? If there is a closed orbit, why must it enclose the origin?

By writing $\dot{x} = \partial H/\partial y + g(x)$ and $\dot{y} = -\partial H/\partial x$ for suitable functions $H(x, y)$ and $g(x)$, show that if there is a closed orbit $\mathcal{C}$ then

$$\oint_{\mathcal{C}} (ax + bx^3)x \, dt = 0 \,.$$

Deduce that there is no closed orbit when $ab > 0$.

If $ab < 0$ and a and b are both $O(\epsilon)$, where ϵ is a small parameter, then there is a single closed orbit that is to within $O(\epsilon)$ a circle of radius R centred on the origin. Deduce a relation between a, b and R.

8E Further Complex Methods

Let the function $f(z)$ be analytic in the upper half-plane and such that $|f(z)| \to 0$ as $|z| \to \infty$. Show that

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x} dx = i\pi f(0) \,,$$

where $\mathcal{P}$ denotes the Cauchy principal value.

Use the Cauchy integral theorem to show that

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{u(x,0)}{x-t} dx = -\pi v(t,0) \,, \quad \mathcal{P} \int_{-\infty}^{\infty} \frac{v(x,0)}{x-t} dx = \pi u(t,0) \,,$$

where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of $f(z)$.

9B Classical Dynamics

The Lagrangian for a heavy symmetric top of mass M, pinned at point O which is a distance l from the centre of mass, is

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 - Mgl \cos \theta .$$

(i) Starting with the fixed space frame $(\tilde{\mathbf{e}}_{\mathbf{1}}, \tilde{\mathbf{e}}_{\mathbf{2}}, \tilde{\mathbf{e}}_{\mathbf{3}})$ and choosing O at its origin, sketch the top with embedded body frame axis $\mathbf{e_3}$ being the symmetry axis. Clearly identify the Euler angles (θ, ϕ, ψ).

(ii) Obtain the momenta p_θ, p_ϕ and p_ψ and the Hamiltonian $H(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi)$. Derive Hamilton's equations. Identify the three conserved quantities.

10D Cosmology

List the relativistic species of bosons and fermions from the standard model of particle physics that are present in the early universe when the temperature falls to $1\, MeV/k_B$.

Which of the particles above will be interacting when the temperature is above $1\, MeV/k_B$ and between $1\, MeV/k_B \gtrsim T \gtrsim 0.51\, MeV/k_B$, respectively?

Explain what happens to the populations of particles present when the temperature falls to $0.51\, MeV/k_B$.

The entropy density of fermion and boson species with temperature T is $s \propto g_s T^3$, where g_s is the number of relativistic spin degrees of freedom, that is,

$$g_s = \sum_{\text{bosons}} g_i + \frac{7}{8} \sum_{\text{fermions}} g_i \, .$$

Show that when the temperature of the universe falls below $0.51\, MeV/k_B$ the ratio of the neutrino and photon temperatures will be given by

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11} \right)^{1/3} .$$

SECTION II

11I Number Theory

(i) What is meant by the continued fraction expansion of a real number θ? Suppose that θ has continued fraction $[a_0, a_1, a_2, \ldots]$. Define the convergents p_n/q_n to θ and give the recurrence relations satisfied by the p_n and q_n. Show that the convergents p_n/q_n do indeed converge to θ.
[You need not justify the basic order properties of finite continued fractions.]

(ii) Find two solutions in strictly positive integers to each of the equations

$$x^2 - 10y^2 = 1 \qquad \text{and} \qquad x^2 - 11y^2 = 1\,.$$

12G Geometry and Groups

Define the *limit set* for a Kleinian group. If your definition of the limit set requires an arbitrary choice of a base point, you should prove that the limit set does not depend on this choice.

Let $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ be the four discs $\{z \in \mathbb{C} : |z - c| \leqslant 1\}$ where c is the point $1+i, 1-i, -1-i, -1+i$ respectively. Show that there is a parabolic Möbius transformation A that maps the interior of Δ_1 onto the exterior of Δ_2 and fixes the point where Δ_1 and Δ_2 touch. Show further that we can choose A so that it maps the unit disc onto itself.

Let B be the similar parabolic transformation that maps the interior of Δ_3 onto the exterior of Δ_4, fixes the point where Δ_3 and Δ_4 touch, and maps the unit disc onto itself. Explain why the group generated by A and B is a Kleinian group G. Find the limit set for the group G and justify your answer.

13J Statistical Modelling

Let f_0 be a probability density function, with cumulant generating function K. Define what it means for a random variable Y to have a model function of exponential dispersion family form, generated by f_0.

A random variable Y is said to have an *inverse Gaussian distribution*, with parameters ϕ and λ (both positive), if its density function is

$$f(y;\phi,\lambda) = \frac{\sqrt{\lambda}}{\sqrt{2\pi y^3}}\, e^{\sqrt{\lambda\phi}} \exp\left\{-\frac{1}{2}\left(\frac{\lambda}{y} + \phi y\right)\right\} \qquad (y > 0).$$

Show that the family of all inverse Gaussian distributions for Y is of exponential dispersion family form. Deduce directly the corresponding expressions for $E(Y)$ and $\mathrm{Var}(Y)$ in terms of ϕ and λ. What are the corresponding canonical link function and variance function?

Consider a generalized linear model, M, for independent variables Y_i $(i = 1, \ldots, n)$, whose random component is defined by the inverse Gaussian distribution with link function $g(\mu) = \log(\mu)$: thus $g(\mu_i) = x_i^{\mathrm{T}}\beta$, where $\beta = (\beta_1, \ldots, \beta_p)^{\mathrm{T}}$ is the vector of unknown regression coefficients and $x_i = (x_{i1}, \ldots, x_{ip})^{\mathrm{T}}$ is the vector of known values of the explanatory variables for the i^{th} observation. The vectors x_i $(i = 1, \ldots, n)$ are linearly independent. Assuming that the dispersion parameter is known, obtain expressions for the score function and Fisher information matrix for β. Explain how these can be used to compute the maximum likelihood estimate $\widehat{\beta}$ of β.

14C Dynamical Systems

Consider the dynamical system

$$\begin{aligned} \dot{x} &= (x+y+a)(x-y+a)\,, \\ \dot{y} &= y - x^2 - b\,, \end{aligned}$$

where $a > 0$.

Find the fixed points of the dynamical system. Show that for any fixed value of a there exist three values $b_1 > b_2 \geqslant b_3$ of b where a bifurcation occurs. Show that $b_2 = b_3$ when $a = 1/2$.

In the remainder of this question set $a = 1/2$.

(i) Being careful to explain your reasoning, show that the extended centre manifold for the bifurcation at $b = b_1$ can be written in the form $X = \alpha Y + \beta\mu + p(Y, \mu)$, where X and Y denote the departures from the values of x and y at the fixed point, $b = b_1 + \mu$, α and β are suitable constants (to be determined) and p is quadratic to leading order. Derive a suitable approximate form for p, and deduce the nature of the bifurcation and the stability of the different branches of the steady state solution near the bifurcation.

(ii) Repeat the calculations of part (i) for the bifurcation at $b = b_2$.

(iii) Sketch the x values of the fixed points as functions of b, indicating the nature of the bifurcations and where each branch is stable.

15B Classical Dynamics

The motion of a particle of charge q and mass m in an electromagnetic field with scalar potential $\phi(\mathbf{r},t)$ and vector potential $\mathbf{A}(\mathbf{r},t)$ is characterized by the Lagrangian

$$L = \frac{m\dot{\mathbf{r}}^2}{2} - q(\phi - \dot{\mathbf{r}} \cdot \mathbf{A}) \ .$$

(i) Write down the Hamiltonian of the particle.

(ii) Write down Hamilton's equations of motion for the particle.

(iii) Show that Hamilton's equations are invariant under the gauge transformation

$$\phi \to \phi - \frac{\partial \Lambda}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla \Lambda,$$

for an arbitrary function $\Lambda(\mathbf{r},t)$.

(iv) The particle moves in the presence of a field such that $\phi = 0$ and $\mathbf{A} = (-\frac{1}{2}yB, \frac{1}{2}xB, 0)$, where (x,y,z) are Cartesian coordinates and B is a constant.

(a) Find a gauge transformation such that only one component of $\mathbf{A}(x,y,z)$ remains non-zero.

(b) Determine the motion of the particle.

(v) Now assume that B varies very slowly with time on a time-scale much longer than $(qB/m)^{-1}$. Find the quantity which remains approximately constant throughout the motion.
[You may use the expression for the action variable $I = \frac{1}{2\pi} \oint p_i dq_i$.]

16G Logic and Set Theory

State the Axiom of Foundation and the Principle of $\in$-Induction, and show that they are equivalent in the presence of the other axioms of ZF set theory. [You may assume the existence of transitive closures.]

Given a model $(V, \in)$ for all the axioms of ZF except Foundation, show how to define a transitive class R which, with the restriction of the given relation $\in$, is a model of ZF.

Given a model $(V, \in)$ of ZF, indicate briefly how one may modify the relation $\in$ so that the resulting structure $(V, \in')$ fails to satisfy Foundation, but satisfies all the other axioms of ZF. [You need not verify that all the other axioms hold in $(V, \in')$.]

17F Graph Theory

Define the *maximum degree* $\Delta(G)$ and the *chromatic index* $\chi'(G)$ of the graph G.

State and prove Vizing's theorem relating $\Delta(G)$ and $\chi'(G)$.

Let G be a connected graph such that $\chi'(G) = \Delta(G) + 1$ but, for every subgraph H of G, $\chi'(H) = \Delta(H)$ holds. Show that G is a circuit of odd length.

18I Galois Theory

(i) Let $\zeta_N = e^{2\pi i/N} \in \mathbb{C}$ for $N \geqslant 1$. For the cases $N = 11, 13$, is it possible to express ζ_N, starting with integers and using rational functions and (possibly nested) radicals? If it is possible, briefly explain how this is done, assuming standard facts in Galois Theory.

(ii) Let $F = \mathbb{C}(X, Y, Z)$ be the rational function field in three variables over $\mathbb{C}$, and for integers $a, b, c \geqslant 1$ let $K = \mathbb{C}(X^a, Y^b, Z^c)$ be the subfield of F consisting of all rational functions in X^a, Y^b, Z^c with coefficients in $\mathbb{C}$. Show that F/K is Galois, and determine its Galois group. [*Hint: For* $\alpha, \beta, \gamma \in \mathbb{C}^\times$, *the map* $(X, Y, Z) \longmapsto (\alpha X, \beta Y, \gamma Z)$ *is an automorphism of* F.]

19G Representation Theory

State and prove Burnside's $p^a q^b$-theorem.

20H Number Fields

State Dedekind's criterion. Use it to factor the primes up to 5 in the ring of integers $\mathcal{O}_K$ of $K = \mathbb{Q}(\sqrt{65})$. Show that every ideal in $\mathcal{O}_K$ of norm 10 is principal, and compute the class group of K.

21G Algebraic Topology

(i) State, but do not prove, the Lefschetz fixed point theorem.

(ii) Show that if n is even, then for every map $f : S^n \to S^n$ there is a point $x \in S^n$ such that $f(x) = \pm x$. Is this true if n is odd? [Standard results on the homology groups for the n-sphere may be assumed without proof, provided they are stated clearly.]

22F Linear Analysis

Let $T\colon X \to X$ be a bounded linear operator on a complex Banach space X. Define the *spectrum* $\sigma(T)$ of T. What is an *approximate eigenvalue* of T? What does it mean to say that T is *compact*?

Assume now that T is compact. Show that if λ is in the boundary of $\sigma(T)$ and $\lambda \neq 0$, then λ is an eigenvalue of T. [You may use without proof the result that every λ in the boundary of $\sigma(T)$ is an approximate eigenvalue of T.]

Let $T\colon H \to H$ be a compact Hermitian operator on a complex Hilbert space H. Prove the following:

(a) If $\lambda \in \sigma(T)$ and $\lambda \neq 0$, then λ is an eigenvalue of T.

(b) $\sigma(T)$ is countable.

23H Algebraic Geometry

Let C be a nonsingular projective curve, and D a divisor on C of degree d.

(i) State the Riemann–Roch theorem for D, giving a brief explanation of each term. Deduce that if $d > 2g-2$ then $\ell(D) = 1 - g + d$.

(ii) Show that, for every $P \in C$,

$$\ell(D-P) \geqslant \ell(D) - 1.$$

Deduce that $\ell(D) \leqslant 1 + d$. Show also that if $\ell(D) > 1$, then $\ell(D-P) = \ell(D) - 1$ for all but finitely many $P \in C$.

(iii) Deduce that for every $d \geqslant g-1$ there exists a divisor D of degree d with $\ell(D) = 1 - g + d$.

24H Differential Geometry

Define what is meant by the *geodesic curvature* k_g of a regular curve $\alpha : I \to S$ parametrized by arc length on a smooth oriented surface $S \subset \mathbf{R}^3$. If S is the unit sphere in $\mathbf{R}^3$ and $\alpha : I \to S$ is a parametrized geodesic circle of radius ϕ, with $0 < \phi < \pi/2$, justify the fact that $|k_g| = \cot\phi$.

State the general form of the Gauss–Bonnet theorem with boundary on an oriented surface S, explaining briefly the terms which occur.

Let $S \subset \mathbf{R}^3$ now denote the circular cone given by $z > 0$ and $x^2 + y^2 = z^2 \tan^2\phi$, for a fixed choice of ϕ with $0 < \phi < \pi/2$, and with a fixed choice of orientation. Let $\alpha : I \to S$ be a simple closed piecewise regular curve on S, with (signed) exterior angles $\theta_1, \ldots, \theta_N$ at the vertices (that is, θ_i is the angle between limits of tangent directions, with sign determined by the orientation). Suppose furthermore that the smooth segments of α are geodesic curves. What possible values can $\theta_1 + \cdots + \theta_N$ take? Justify your answer.

[You may assume that a simple closed curve in $\mathbf{R}^2$ bounds a region which is homeomorphic to a disc. Given another simple closed curve in the interior of this region, you may assume that the two curves bound a region which is homeomorphic to an annulus.]

25K Probability and Measure

State Birkhoff's almost-everywhere ergodic theorem.

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables such that

$$\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = 1/2\,.$$

Define for $k \in \mathbb{N}$

$$Y_k = \sum_{n=1}^{\infty} X_{k+n-1}/2^n\,.$$

What is the distribution of Y_k? Show that the random variables Y_1 and Y_2 are not independent.

Set $S_n = Y_1 + \cdots + Y_n$. Show that S_n/n converges as $n \to \infty$ almost surely and determine the limit. [You may use without proof any standard theorem provided you state it clearly.]

26J Applied Probability

(i) Define an $M/M/1$ queue. Justifying briefly your answer, specify when this queue has a stationary distribution, and identify that distribution. State and prove Burke's theorem for this queue.

(ii) Let $(L_1(t), \ldots, L_N(t), t \geqslant 0)$ denote a Jackson network of N queues, where the entrance and service rates for queue i are respectively λ_i and μ_i, and each customer leaving queue i moves to queue j with probability p_{ij} after service. We assume $\sum_j p_{ij} < 1$ for each $i = 1, \ldots, N$; with probability $1 - \sum_j p_{ij}$ a customer leaving queue i departs from the system. State Jackson's theorem for this network. [You are not required to prove it.] Are the processes $(L_1(t), \ldots, L_N(t), t \geqslant 0)$ independent at equilibrium? Justify your answer.

(iii) Let $D_i(t)$ be the process of final departures from queue i. Show that, at equilibrium, $(L_1(t), \ldots, L_N(t))$ is independent of $(D_i(s), 1 \leqslant i \leqslant N, 0 \leqslant s \leqslant t)$. Show that, for each fixed $i = 1, \ldots, N$, $(D_i(t), t \geqslant 0)$ is a Poisson process, and specify its rate.

27K Principles of Statistics

Assuming only the existence and properties of the univariate normal distribution, define $\mathcal{N}_p(\underline{\mu}, \Sigma)$, the *multivariate normal* distribution with mean (row-)vector $\underline{\mu}$ and dispersion matrix Σ; and $W_p(\nu; \Sigma)$, the *Wishart* distribution on integer $\nu > 1$ degrees of freedom and with scale parameter Σ. Show that, if $\underline{X} \sim \mathcal{N}_p(\underline{\mu}, \Sigma)$, $S \sim W_p(\nu; \Sigma)$, and $\underline{b}$ $(1 \times q)$, A $(p \times q)$ are fixed, then $\underline{b} + \underline{X}A \sim \mathcal{N}_q(\underline{b} + \underline{\mu}A, \Phi)$, $A^{\mathrm{T}}SA \sim W_p(\nu; \Phi)$, where $\Phi = A^{\mathrm{T}}\Sigma A$.

The random $(n \times p)$ matrix X has rows that are independently distributed as $\mathcal{N}_p(\underline{\mathrm{M}}, \Sigma)$, where both parameters $\underline{\mathrm{M}}$ and Σ are unknown. Let $\overline{\underline{X}} := n^{-1}\mathbf{1}^{\mathrm{T}}X$, where $\mathbf{1}$ is the $(n \times 1)$ vector of 1s; and $S^c := X^{\mathrm{T}}\Pi X$, with $\Pi := I_n - n^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}}$. State the joint distribution of $\overline{\underline{X}}$ and S^c given the parameters.

Now suppose $n > p$ and Σ is positive definite. *Hotelling's* T^2 is defined as

$$T^2 := n(\overline{\underline{X}} - \underline{\mathrm{M}})\left(\overline{S}^c\right)^{-1}(\overline{\underline{X}} - \underline{\mathrm{M}})^{\mathrm{T}}$$

where $\overline{S}^c := S^c/\nu$ with $\nu := (n-1)$. Show that, for any values of $\underline{\mathrm{M}}$ and Σ,

$$\left(\frac{\nu - p + 1}{\nu p}\right) T^2 \sim F^p_{\nu-p+1},$$

the F distribution on p and $\nu - p + 1$ degrees of freedom.

[You may assume that:

1. If $S \sim W_p(\nu; \Sigma)$ and $\mathbf{a}$ is a fixed $(p \times 1)$ vector, then
$$\frac{\mathbf{a}^{\mathrm{T}}\Sigma^{-1}\mathbf{a}}{\mathbf{a}^{\mathrm{T}}S^{-1}\mathbf{a}} \sim \chi^2_{\nu-p+1}.$$

2. If $V \sim \chi^2_p$, $W \sim \chi^2_\lambda$ are independent, then
$$\frac{V/p}{W/\lambda} \sim F^p_\lambda.$$
]

28K Optimization and Control

Given r, ρ, μ, T, all positive, it is desired to choose $u(t) > 0$ to maximize

$$\mu x(T) + \int_0^T e^{-\rho t} \log u(t)\, dt$$

subject to $\dot{x}(t) = rx(t) - u(t)$, $x(0) = 10$.

Explain what Pontryagin's maximum principle guarantees about a solution to this problem.

Show that no matter whether $x(T)$ is constrained or unconstrained there is a constant α such that the optimal control is of the form $u(t) = \alpha e^{-(\rho - r)t}$. Find an expression for α under the constraint $x(T) = 5$.

Show that if $x(T)$ is unconstrained then $\alpha = (1/\mu)e^{-rT}$.

29J Stochastic Financial Models

Let $S_t := (S_t^1, S_t^2, \ldots, S_t^n)^{\mathrm{T}}$ denote the time-t prices of n risky assets in which an agent may invest, $t = 0, 1$. He may also invest his money in a bank account, which will return interest at rate $r > 0$. At time 0, he knows S_0 and r, and he knows that $S_1 \sim N(\mu, V)$. If he chooses at time 0 to invest cash value θ_i in risky asset i, express his wealth w_1 at time 1 in terms of his initial wealth $w_0 > 0$, the choices $\theta := (\theta_1, \ldots, \theta_n)^{\mathrm{T}}$, the value of S_1, and r.

Suppose that his goal is to minimize the variance of w_1 subject to the requirement that the mean $E(w_1)$ should be at least m, where $m \geqslant (1 + r)w_0$ is given. What portfolio θ should he choose to achieve this?

Suppose instead that his goal is to minimize $E(w_1^2)$ subject to the same constraint. Show that his optimal portfolio is unchanged.

30C Partial Differential Equations

(i) Show that an arbitrary C^2 solution of the one-dimensional wave equation $u_{tt} - u_{xx} = 0$ can be written in the form $u = F(x-t) + G(x+t)$.

Hence, deduce the formula for the solution at arbitrary $t > 0$ of the Cauchy problem

$$u_{tt} - u_{xx} = 0\,, \qquad u(0,x) = u_0(x)\,, \qquad u_t(0,x) = u_1(x)\,, \tag{$*$}$$

where u_0, u_1 are arbitrary Schwartz functions.

Deduce from this formula a theorem on finite propagation speed for the one-dimensional wave equation.

(ii) Define the Fourier transform of a tempered distribution. Compute the Fourier transform of the tempered distribution $T_t \in \mathcal{S}'(\mathbb{R})$ defined for all $t > 0$ by the function

$$T_t(y) = \begin{cases} \frac{1}{2} & \text{if } |y| \leqslant t, \\ 0 & \text{if } |y| > t, \end{cases}$$

that is, $\langle T_t\,,\, f\,\rangle = \frac{1}{2}\int_{-t}^{+t} f(y)\,dy$ for all $f \in \mathcal{S}(\mathbb{R})$. By considering the Fourier transform in x, deduce from this the formula for the solution of $(*)$ that you obtained in part (i) in the case $u_0 = 0$.

31B Asymptotic Methods

Show that the equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \left(\frac{1}{x^2} - 1\right) y = 0$$

has an irregular singular point at infinity. Using the Liouville–Green method, show that one solution has the asymptotic expansion

$$y(x) \sim \frac{1}{x} e^x \left(1 + \frac{1}{2x} + \ldots\right)$$

as $x \to \infty$.

32E Principles of Quantum Mechanics

(i) The creation and annihilation operators for a harmonic oscillator of angular frequency ω satisfy the commutation relation $[a, a^\dagger] = 1$. Write down an expression for the Hamiltonian H and number operator N in terms of a and $a^\dagger$. Explain how the space of eigenstates $|n\rangle$, $n = 0, 1, 2, \ldots$, of H is formed, and deduce the eigenenergies for these states. Show that

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,.$$

(ii) The operator K_r is defined to be

$$K_r = \frac{(a^\dagger)^r a^r}{r!}\,,$$

for $r = 0, 1, 2, \ldots$. Show that K_r commutes with N. Show that if $r \leqslant n$, then

$$K_r|n\rangle = \frac{n!}{(n-r)!r!}|n\rangle\,,$$

and $K_r|n\rangle = 0$ otherwise. By considering the action of K_r on the state $|n\rangle$, deduce that

$$\sum_{r=0}^{\infty}(-1)^r K_r = |0\rangle\langle 0|\,.$$

33D Applications of Quantum Mechanics

Define the *Floquet matrix* for a particle moving in a periodic potential in one dimension and explain how it determines the allowed energy bands of the system.

A potential barrier in one dimension has the form

$$V(x) = \begin{cases} V_0(x)\,, & |x| < a/4\,, \\ 0\,, & |x| > a/4\,, \end{cases}$$

where $V_0(x)$ is a smooth, positive function of x. The reflection and transmission amplitudes for a particle of wavenumber $k > 0$, incident from the left, are $r(k)$ and $t(k)$ respectively. For a particle of wavenumber $-k$, incident from the right, the corresponding amplitudes are $r'(k)$ and $t'(k) = t(k)$. In the following, for brevity, we will suppress the k-dependence of these quantities.

Consider the periodic potential $\widetilde{V}$, defined by $\widetilde{V}(x) = V(x)$ for $|x| < a/2$ and by $\widetilde{V}(x+a) = \widetilde{V}(x)$ elsewhere. Write down two linearly independent solutions of the corresponding Schrödinger equation in the region $-3a/4 < x < -a/4$. Using the scattering data given above, extend these solutions to the region $a/4 < x < 3a/4$. Hence find the Floquet matrix of the system in terms of the amplitudes r, r' and t defined above.

Show that the edges of the allowed energy bands for this potential lie at $E = \hbar^2 k^2/2m$, where

$$ka \;=\; i \log\left(t \pm \sqrt{rr'}\right)\,.$$

34A Statistical Physics

A classical particle of mass m moving non-relativistically in two-dimensional space is enclosed inside a circle of radius R and attached by a spring with constant κ to the centre of the circle. The particle thus moves in a potential

$$V(r) = \begin{cases} \frac{1}{2}\kappa r^2 & \text{for } r < R\,, \\ \infty & \text{for } r \geqslant R\,, \end{cases}$$

where $r^2 = x^2 + y^2$. Let the particle be coupled to a heat reservoir at temperature T.

(i) Which of the ensembles of statistical physics should be used to model the system?

(ii) Calculate the partition function for the particle.

(iii) Calculate the average energy $\langle E \rangle$ and the average potential energy $\langle V \rangle$ of the particle.

(iv) What is the average energy in:

(a) the limit $\frac{1}{2}\kappa R^2 \gg k_{\mathrm{B}}T$ (strong coupling)?

(b) the limit $\frac{1}{2}\kappa R^2 \ll k_{\mathrm{B}}T$ (weak coupling)?

Compare the two results with the values expected from equipartition of energy.

35B Electrodynamics

(i) For a time-dependent source, confined within a domain D, show that the time derivative $\dot{\mathbf{d}}$ of the dipole moment $\mathbf{d}$ satisfies

$$\dot{\mathbf{d}} = \int_D d^3y\, \mathbf{J}(\mathbf{y})\,,$$

where $\mathbf{J}$ is the current density.

(ii) The vector potential $\mathbf{A}(\mathbf{x}, t)$ due to a time-dependent source is given by

$$\mathbf{A} = \frac{1}{r} f\,(t - r/c)\,\mathbf{k}\,,$$

where $r = |\mathbf{x}| \neq 0$, and $\mathbf{k}$ is the unit vector in the z direction. Calculate the resulting magnetic field $\mathbf{B}(\mathbf{x}, t)$. By considering the magnetic field for small r show that the dipole moment of the effective source satisfies

$$\frac{\mu_0}{4\pi}\dot{\mathbf{d}} = f(t)\mathbf{k}\,.$$

Calculate the asymptotic form of the magnetic field $\mathbf{B}$ at very large r.

(iii) Using the equation

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \mathbf{B}\,,$$

calculate $\mathbf{E}$ at very large r. Show that $\mathbf{E}, \mathbf{B}$ and $\hat{\mathbf{r}} = \mathbf{x}/|\mathbf{x}|$ form a right-handed triad, and moreover $|\mathbf{E}| = c|\mathbf{B}|$. How do $|\mathbf{E}|$ and $|\mathbf{B}|$ depend on r? What is the significance of this?

(iv) Calculate the power $P(\theta, \phi)$ emitted per unit solid angle and sketch its dependence on θ. Show that the emitted radiation is polarised and describe how the plane of polarisation (that is, the plane in which $\mathbf{E}$ and $\hat{\mathbf{r}}$ lie) depends on the direction of the dipole. Suppose the dipole moment has constant amplitude and constant frequency and so the radiation is monochromatic with wavelength λ. How does the emitted power depend on λ?

36D General Relativity

Consider the metric describing the interior of a star,

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) ,$$

defined for $0 \leqslant r \leqslant r_0$ by

$$e^{\alpha(r)} = \frac{3}{2}e^{-\beta_0} - \frac{1}{2}e^{-\beta(r)} ,$$

with

$$e^{-2\beta(r)} = 1 - Ar^2 .$$

Here $A = 2M/r_0^3$, where M is the mass of the star, $\beta_0 = \beta(r_0)$, and we have taken units in which we have set $G = c = 1$.

(i) The star is made of a perfect fluid with energy-momentum tensor

$$T_{ab} = (p+\rho)u_a u_b + p\, g_{ab} .$$

Here u^a is the 4-velocity of the fluid which is at rest, the density ρ is constant throughout the star ($0 \leqslant r \leqslant r_0$) and the pressure $p = p(r)$ depends only on the radial coordinate. Write down the Einstein field equations and show that they may be written as

$$R_{ab} = 8\pi(p+\rho)u_a u_b + 4\pi(\rho - p)g_{ab} .$$

(ii) Using the formulae given below, or otherwise, show that for $0 \leqslant r \leqslant r_0$, one has

$$\begin{aligned} 4\pi(\rho + p) &= \frac{(\alpha' + \beta')}{r}e^{-2\beta(r)} , \\ 4\pi(\rho - p) &= \left(\frac{\beta' - \alpha'}{r} - \frac{1}{r^2}\right)e^{-2\beta(r)} + \frac{1}{r^2} , \end{aligned}$$

where primes denote differentiation with respect to r. Hence show that

$$\rho = \frac{3A}{8\pi} \quad , \quad p(r) = \frac{3A}{8\pi}\left(\frac{e^{-\beta(r)} - e^{-\beta_0}}{3e^{-\beta_0} - e^{-\beta(r)}}\right) .$$

[The non-zero components of the Ricci tensor are

$$\begin{aligned} R_{00} &= e^{2\alpha - 2\beta}\left(\alpha'' - \alpha'\beta' + \alpha'^2 + \frac{2\alpha'}{r}\right) \\ R_{11} &= -\alpha'' + \alpha'\beta' - \alpha'^2 + \frac{2\beta'}{r} \\ R_{22} &= 1 + e^{-2\beta}\left[(\beta' - \alpha')r - 1\right] \\ R_{33} &= \sin^2\theta R_{22} . \end{aligned}$$

Note that

$$\alpha' = \frac{1}{2}A\, r\, e^{\beta - \alpha} \quad , \quad \beta' = A\, r\, e^{2\beta} .]$$

37A Fluid Dynamics II

Consider the flow of an incompressible fluid of uniform density ρ and dynamic viscosity μ. Show that the rate of viscous dissipation per unit volume is given by

$$\Phi = 2\mu e_{ij} e_{ij},$$

where e_{ij} is the strain rate.

Determine expressions for e_{ij} and Φ when the flow is irrotational with velocity potential ϕ.

In deep water a linearised wave with a surface displacement $\eta = a\cos(kx - \omega t)$ has a velocity potential $\phi = -(\omega a/k)e^{-kz}\sin(kx - \omega t)$. Hence determine the rate of the viscous dissipation, averaged over a wave period $2\pi/\omega$, for an irrotational surface wave of wavenumber k and small amplitude $a \ll 1/k$ in a fluid with very small viscosity $\mu \ll \rho\omega/k^2$ and great depth $H \gg 1/k$.

Calculate the depth-integrated kinetic energy per unit wavelength. Assuming that the average potential energy is equal to the average kinetic energy, show that the total wave energy decreases to leading order as $e^{-\gamma t}$, where γ should be found.

38C Waves

A wave disturbance satisfies the equation

$$\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} + c^2\psi = 0\,,$$

where c is a positive constant. Find the dispersion relation, and write down the solution to the initial-value problem for which $\partial\psi/\partial t(x,0) = 0$ for all x, and $\psi(x,0)$ is given in the form

$$\psi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk\ ,$$

where $A(k)$ is a real function with $A(k) = A(-k)$, so that $\psi(x,0)$ is real and even.

Use the method of stationary phase to obtain an approximation to $\psi(x,t)$ for large t, with x/t taking the constant value V, and $0 \leqslant V < c$. Explain briefly why your answer is inappropriate if $V > c$.

[You are given that

$$\int_{-\infty}^{\infty} \exp(iu^2)\,du = \pi^{1/2}e^{i\pi/4}\,.]$$

39C Numerical Analysis

Consider the solution of the two-point boundary value problem

$$(2 - \sin \pi x)u'' + u = 1, \quad -1 \leqslant x \leqslant 1,$$

with periodic boundary conditions at $x = -1$ and $x = 1$. Construct explicitly the linear algebraic system that arises from the application of a spectral method to the above equation.

The Fourier coefficients of u are defined by

$$\hat{u}_n = \frac{1}{2} \int_{-1}^{1} u(\tau) e^{-i\pi n\tau} \, d\tau.$$

Prove that the computation of the Fourier coefficients for the truncated system with $-N/2 + 1 \leqslant n \leqslant N/2$ (where N is an even and positive integer, and assuming that $\hat{u}_n = 0$ outside this range of n) reduces to the solution of a tridiagonal system of algebraic equations, which you should specify.

Explain the term *convergence with spectral speed* and justify its validity for the derived approximation of u.

END OF PAPER