UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Wednesday, 5 June, 2013 9:00 am to 12:00 noon

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, …, ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Define Euler's totient function $\phi(n)$, and show that $\sum_{d|n} \phi(d) = n$. Hence or otherwise prove that for any prime p the multiplicative group $\left(\mathbb{Z}/p\mathbb{Z}\right)^{\times}$ is cyclic.

2F Topics in Analysis

(i) Show that for every $\epsilon > 0$ there is a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $|\frac{1}{x} - p(x)| \leqslant \epsilon$ for all $x \in \mathbb{R}$ satisfying $\frac{1}{2} \leqslant |x| \leqslant 2$.

[You may assume standard results provided they are stated clearly.]

(ii) Show that there is no polynomial $p : \mathbb{C} \to \mathbb{C}$ such that $|\frac{1}{z} - p(z)| < 1$ for all $z \in \mathbb{C}$ satisfying $\frac{1}{2} \leqslant |z| \leqslant 2$.

3G Geometry and Groups

Let ℓ_1, ℓ_2 be two straight lines in Euclidean 3-space. Show that there is a rotation about some axis through an angle π that maps ℓ_1 onto ℓ_2. Is this rotation unique?

4H Coding and Cryptography

Let $A(n, d)$ denote the maximum size of a binary code of length n with minimum distance d. For fixed δ with $0 < \delta < 1/2$, let $\alpha(\delta) = \limsup \frac{1}{n} \log_2 A(n, n\delta)$. Show that

$$1 - H(\delta) \leqslant \alpha(\delta) \leqslant 1 - H(\delta/2)$$

where $H(p) = -p \log_2 p - (1 - p) \log_2(1 - p)$.

[You may assume the GSV and Hamming bounds and any form of Stirling's theorem provided you state them clearly.]

5J Statistical Modelling

Consider a linear model $Y = X\beta + \epsilon$, where Y and ϵ are $(n \times 1)$ with $\epsilon \sim N_n(0, \sigma^2 I)$, β is $(p \times 1)$, and X is $(n \times p)$ of full rank $p < n$. Let γ and δ be sub-vectors of β. What is meant by *orthogonality* between γ and δ?

Now suppose

$$Y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \beta_3 P_3(x_i) + \epsilon_i \quad (i = 1, \ldots, n),$$

where $\epsilon_1, \ldots, \epsilon_n$ are independent $N(0, \sigma^2)$ random variables, $x_1, \ldots, x_n$ are real-valued known explanatory variables, and $P_3(x)$ is a cubic polynomial chosen so that β_3 is orthogonal to $(\beta_0, \beta_1, \beta_2)^{\mathrm{T}}$ and β_1 is orthogonal to $(\beta_0, \beta_2)^{\mathrm{T}}$.

Let $\widetilde{\beta} = (\beta_0, \beta_2, \beta_1, \beta_3)^{\mathrm{T}}$. Describe the matrix $\widetilde{X}$ such that $Y = \widetilde{X}\widetilde{\beta} + \epsilon$. Show that $\widetilde{X}^{\mathrm{T}}\widetilde{X}$ is block diagonal. Assuming further that this matrix is non-singular, show that the least-squares estimators of β_1 and β_3 are, respectively,

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2} \qquad \text{and} \qquad \widehat{\beta}_3 = \frac{\sum_{i=1}^n P_3(x_i) Y_i}{\sum_{i=1}^n P_3(x_i)^2}.$$

6A Mathematical Biology

The population density $n(a, t)$ of individuals of age a at time t satisfies

$$\frac{\partial n(a,t)}{\partial t} + \frac{\partial n(a,t)}{\partial a} = -\mu(a) n(a,t),$$

with

$$n(0,t) = \int_0^\infty b(a) n(a,t) da,$$

where $\mu(a)$ is the age-dependent death rate and $b(a)$ is the birth rate per individual of age a.

Seek a similarity solution of the form $n(a,t) = e^{\gamma t} r(a)$ and show that

$$r(a) = r(0) e^{-\gamma a - \int_0^a \mu(s) ds}, \quad r(0) = \int_0^\infty b(s) r(s) ds.$$

Show also that if

$$\phi(\gamma) = \int_0^\infty b(a) e^{-\gamma a - \int_0^a \mu(s) ds} da = 1,$$

then there is such a similarity solution. Give a biological interpretation of $\phi(0)$.

Suppose now that all births happen at age a^*, at which time an individual produces B offspring, and that the death rate is constant with age (i.e. $\mu(a) = \mu$). Find the similarity solution and give the condition for this to represent a growing population.

7C Dynamical Systems

Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a two-dimensional dynamical system with a fixed point at $\mathbf{x} = \mathbf{0}$. Define a Lyapunov function $V(\mathbf{x})$ and explain what it means for $\mathbf{x} = \mathbf{0}$ to be Lyapunov stable.

For the system

$$\begin{aligned} \dot{x} &= -x - 2y + x^3 \,, \\ \dot{y} &= -y + x + \tfrac{1}{2}y^3 + x^2 y \,, \end{aligned}$$

determine the values of C for which $V = x^2 + Cy^2$ is a Lyapunov function in a sufficiently small neighbourhood of the origin.

For the case $C = 2$, find V_1 and V_2 such that $V(\mathbf{x}) < V_1$ at $t = 0$ implies that $V \to 0$ as $t \to \infty$ and $V(\mathbf{x}) > V_2$ at $t = 0$ implies that $V \to \infty$ as $t \to \infty$.

8E Further Complex Methods

(i) Find all branch points of $(z^3 - 1)^{1/4}$ on an extended complex plane.

(ii) Use a branch cut to evaluate the integral

$$\int_{-2}^{2} (4 - x^2)^{1/2} dx \,.$$

9B Classical Dynamics

(i) Consider a rigid body with principal moments of inertia I_1, I_2, I_3. Derive Euler's equations of torque-free motion,

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2, \end{aligned}$$

with components of the angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ given in the body frame.

(ii) Use Euler's equations to show that the energy E and the square of the total angular momentum $\mathbf{L}^2$ of the body are conserved.

(iii) Consider a torque-free motion of a symmetric top with $I_1 = I_2 = \frac{1}{2}I_3$. Show that in the body frame the vector of angular velocity $\boldsymbol{\omega}$ precesses about the body-fixed $\mathbf{e}_3$ axis with constant angular frequency equal to ω_3.

10D Cosmology

The linearised equation for the growth of small inhomogeneous density perturbations $\delta_{\mathbf{k}}$ with comoving wavevector $\mathbf{k}$ in an isotropic and homogeneous universe is

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} + \left(\frac{c_s^2 k^2}{a^2} - 4\pi G\rho\right)\delta_{\mathbf{k}} = 0\,,$$

where ρ is the matter density, $c_s = (dP/d\rho)^{1/2}$ is the sound speed, P is the pressure, $a(t)$ is the expansion scale factor of the unperturbed universe, and overdots denote differentiation with respect to time t.

Define the Jeans wavenumber and explain its physical meaning.

Assume the unperturbed Friedmann universe has zero curvature and cosmological constant and it contains only zero-pressure matter, so that $a(t) = a_0 t^{2/3}$. Show that the solution for the growth of density perturbations is given by

$$\delta_{\mathbf{k}} = A(\mathbf{k})t^{2/3} + B(\mathbf{k})t^{-1}\,.$$

Comment briefly on the cosmological significance of this result.

SECTION II

11F Topics in Analysis

(i) Let $n \geqslant 4$ be an integer. Show that

$$1+\frac{1}{n+\frac{1}{1+\frac{1}{n+\dots}}} \geqslant 1+\frac{1}{2n}\,.$$

(ii) Let us say that an irrational number α is *badly approximable* if there is some constant $c>0$ such that

$$\left|\alpha-\frac{p}{q}\right| \geqslant \frac{c}{q^2}$$

for all $q \geqslant 1$ and for all integers p. Show that if the integers a_n in the continued fraction expansion $\alpha=[a_0,a_1,a_2,\dots]$ are bounded then α is badly approximable.

Give, with proof, an example of an irrational number which is not badly approximable.

[Standard facts about continued fractions may be used without proof provided they are stated clearly.]

12H Coding and Cryptography

Define a BCH code of length n, where n is odd, over the field of 2 elements with design distance δ. Show that the minimum weight of such a code is at least δ. [Results about the van der Monde determinant may be quoted without proof, provided they are stated clearly.]

Consider a BCH code of length 31 over the field of 2 elements with design distance 8. Show that the minimum distance is at least 11. [*Hint: Let α be a primitive element in the field of 2^5 elements, and consider the minimal polynomial for certain powers of α.*]

13A Mathematical Biology

The concentration $c(x,t)$ of insects at position x at time t satisfies the nonlinear diffusion equation

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(c^m \frac{\partial c}{\partial x}\right),$$

with $m > 0$. Find the value of α which allows a similarity solution of the form $c(x,t) = t^\alpha f(\xi)$, with $\xi = t^\alpha x$.

Show that

$$f(\xi) = \begin{cases} \left[\frac{\alpha m}{2}(\xi^2 - {\xi_0}^2)\right]^{1/m} & \text{for } -\xi_0 < \xi < \xi_0\,, \\ 0 & \text{otherwise,} \end{cases}$$

where ξ_0 is a constant. From the original partial differential equation, show that the total number of insects c_0 does not change in time. From this result, find a general expression relating ξ_0 and c_0. Find a closed-form solution for ξ_0 in the case $m = 2$.

14E Further Complex Methods

The Beta function is defined for $\mathrm{Re}(z) > 0$ as

$$B(z,q) = \int_0^1 t^{q-1}(1-t)^{z-1}dt, \quad (\mathrm{Re}(q) > 0)\,,$$

and by analytic continuation elsewhere in the complex z-plane.

Show that:

(i) $(z+q)B(z+1,q) = zB(z,q)$;

(ii) $\Gamma(z)^2 = B(z,z)\Gamma(2z)$.

By considering $\Gamma(z/2^m)$ for all positive integers m, deduce that $\Gamma(z) \neq 0$ for all z with $\mathrm{Re}(z) > 0$.

15B Classical Dynamics

(i) The action for a system with a generalized coordinate q is given by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \,.$$

(a) State the Principle of Least Action and derive the Euler–Lagrange equation.

(b) Consider an arbitrary function $f(q,t)$. Show that $L' = L + df/dt$ leads to the same equation of motion.

(ii) A wire frame ABC in a shape of an equilateral triangle with side a rotates in a horizontal plane with constant angular frequency ω about a vertical axis through A. A bead of mass m is threaded on BC and moves without friction. The bead is connected to B and C by two identical light springs of force constant k and equilibrium length $a/2$.

(a) Introducing the displacement η of the particle from the mid point of BC, determine the Lagrangian $L(\eta, \dot{\eta})$.

(b) Derive the equation of motion. Identify the integral of the motion.

(c) Describe the motion of the bead. Find the condition for there to be a stable equilibrium and find the frequency of small oscillations about it when it exists.

16G Logic and Set Theory

Explain what is meant by a chain-complete poset. State the Bourbaki–Witt fixed-point theorem for such posets.

A poset P is called *directed* if every finite subset of P (including the empty subset) has an upper bound in P; P is called *directed-complete* if every subset of P which is directed (in the induced ordering) has a least upper bound in P. Show that the set of all chains in an arbitrary poset P, ordered by inclusion, is directed-complete.

Given a poset P, let $[P \to P]$ denote the set of all order-preserving maps $P \to P$, ordered pointwise (i.e. $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all x). Show that $[P \to P]$ is directed-complete if P is.

Now suppose P is directed-complete, and that $f : P \to P$ is order-preserving and inflationary. Show that there is a unique smallest set $C \subseteq [P \to P]$ satisfying

(*a*) $f \in C$;

(*b*) C is closed under composition (i.e. $g, h \in C \Rightarrow g \circ h \in C$); and

(*c*) C is closed under joins of directed subsets.

Show that

(i) all maps in C are inflationary;

(ii) C is directed;

(iii) if $g = \bigvee C$, then all values of g are fixed points of f;

(iv) for every $x \in P$, there exists $y \in P$ with $x \leqslant y = f(y)$.

17F Graph Theory

Let G be a graph with $|G| \geqslant 3$. State and prove a necessary and sufficient condition for G to be Eulerian (that is, for G to have an Eulerian circuit).

Prove that if $\delta(G) \geqslant |G|/2$ then G is Hamiltonian (that is, G has a Hamiltonian circuit).

The *line graph* $L(G)$ of G has vertex set $V(L(G)) = E(G)$ and edge set

$$E(L(G)) = \{ef : e, f \in E(G),\ e \text{ and } f \text{ are incident}\}.$$

Show that $L(G)$ is Eulerian if G is regular and connected.

Must $L(G)$ be Hamiltonian if G is Eulerian? Must G be Eulerian if $L(G)$ is Hamiltonian? Justify your answers.

18I Galois Theory

For a positive integer N, let $\mathbb{Q}(\boldsymbol{\mu}_N)$ be the cyclotomic field obtained by adjoining all N-th roots of unity to $\mathbb{Q}$. Let $F = \mathbb{Q}(\boldsymbol{\mu}_{24})$.

(i) Determine the Galois group of F over $\mathbb{Q}$.

(ii) Find all $N > 1$ such that $\mathbb{Q}(\boldsymbol{\mu}_N)$ is contained in F.

(iii) List all quadratic and quartic extensions of $\mathbb{Q}$ which are contained in F, in the form $\mathbb{Q}(\alpha)$ or $\mathbb{Q}(\alpha, \beta)$. Indicate which of these fields occurred in (ii).

[Standard facts on the Galois groups of cyclotomic fields and the fundamental theorem of Galois theory may be used freely without proof.]

19G Representation Theory

Recall that a regular icosahedron has 20 faces, 30 edges and 12 vertices. Let G be the group of rotational symmetries of a regular icosahedron.

Compute the conjugacy classes of G. Hence, or otherwise, construct the character table of G. Using the character table explain why G must be a simple group.

[You may use any general theorems provided that you state them clearly.]

20H Number Fields

(i) State Dirichlet's unit theorem.

(ii) Let K be a number field. Show that if every conjugate of $\alpha \in \mathcal{O}_K$ has absolute value at most 1 then α is either zero or a root of unity.

(iii) Let $k = \mathbb{Q}(\sqrt{3})$ and $K = \mathbb{Q}(\zeta)$ where $\zeta = e^{i\pi/6} = (i + \sqrt{3})/2$. Compute $N_{K/k}(1+\zeta)$. Show that

$$\mathcal{O}_K^* = \{(1+\zeta)^m u : 0 \leqslant m \leqslant 11, u \in \mathcal{O}_k^*\}.$$

Hence or otherwise find fundamental units for k and K.
[You may assume that the only roots of unity in K are powers of ζ.]

21G Algebraic Topology

(i) State the Seifert–van Kampen theorem.

(ii) Assuming any standard results about the fundamental group of a circle that you wish, calculate the fundamental group of the n-sphere, for every $n \geqslant 2$.

(iii) Suppose that $n \geqslant 3$ and that X is a path-connected topological n-manifold. Show that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X - \{P\}, x_0)$ for any $P \in X - \{x_0\}$.

22F Linear Analysis

Let X be a Banach space. Let $T\colon X \to \ell_\infty$ be a bounded linear operator. Show that there is a bounded sequence $(f_n)_{n=1}^\infty$ in X^* such that $Tx = (f_n x)_{n=1}^\infty$ for all $x \in X$.

Fix $1 < p < \infty$. Define the Banach space ℓ_p and *briefly* explain why it is separable. Show that for $x \in \ell_p$ there exists $f \in \ell_p^*$ such that $\|f\| = 1$ and $f(x) = \|x\|_p$. [You may use Hölder's inequality without proof.]

Deduce that ℓ_p embeds isometrically into ℓ_∞.

23I Riemann Surfaces

(i) Show that the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ is biholomorphic to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

(ii) Define the degree of a non-constant holomorphic map between compact connected Riemann surfaces. State the Riemann–Hurwitz formula without proof. Now let X be a complex torus and $f\colon X \to Y$ a holomorphic map of degree 2, where Y is the Riemann sphere. Show that f has exactly four branch points.

(iii) List without proof those Riemann surfaces whose universal cover is the Riemann sphere or $\mathbb{C}$. Now let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic map such that there are two distinct elements $a, b \in \mathbb{C}$ outside the image of f. Assuming the uniformization theorem and the monodromy theorem, show that f is constant.

24H Algebraic Geometry

Let $V \subset \mathbb{P}^3$ be an irreducible quadric surface.

(i) Show that if V is singular, then every nonsingular point lies in exactly one line in V, and that all the lines meet in the singular point, which is unique.

(ii) Show that if V is nonsingular then each point of V lies on exactly two lines of V.

Let V be nonsingular, P_0 a point of V, and $\Pi \subset \mathbb{P}^3$ a plane not containing P_0. Show that the projection from P_0 to Π is a birational map $f\colon V \dashrightarrow \Pi$. At what points does f fail to be regular? At what points does f^{-1} fail to be regular? Justify your answers.

25H Differential Geometry

Let $\alpha : [0, L] \to \mathbf{R}^3$ be a regular curve parametrized by arc length having nowhere-vanishing curvature. State the Frenet relations between the tangent, normal and binormal vectors at a point, and their derivatives.

Let $S \subset \mathbf{R}^3$ be a smooth oriented surface. Define the *Gauss map* $N : S \to S^2$, and show that its derivative at $P \in S$, $dN_P : T_P S \to T_P S$, is self-adjoint. Define the *Gaussian curvature* of S at P.

Now suppose that $\alpha : [0, L] \to \mathbf{R}^3$ has image in S and that its normal curvature is zero for all $s \in [0, L]$. Show that the Gaussian curvature of S at a point $P = \alpha(s)$ of the curve is $K(P) = -\tau(s)^2$, where $\tau(s)$ denotes the torsion of the curve.

If $S \subset \mathbf{R}^3$ is a standard embedded torus, show that there is a curve on S for which the normal curvature vanishes and the Gaussian curvature of S is zero at all points of the curve.

26K Probability and Measure

Let $(f_n : n \in \mathbb{N})$ be a sequence of non-negative measurable functions defined on a measure space $(E, \mathcal{E}, \mu)$. Show that $\liminf_n f_n$ is also a non-negative measurable function.

State the Monotone Convergence Theorem.

State and prove Fatou's Lemma.

Let $(f_n : n \in \mathbb{N})$ be as above. Suppose that $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in E$. Show that

$$\mu(\min\{f_n, f\}) \to \mu(f)\,.$$

Deduce that, if f is integrable and $\mu(f_n) \to \mu(f)$, then f_n converges to f in L^1. [Still assume that f_n and f are as above.]

27J Applied Probability

(i) Define a Poisson process as a Markov chain on the non-negative integers and state three other characterisations.

(ii) Let $\lambda(s)$ $(s \geqslant 0)$ be a continuous positive function. Let $(X_t, t \geqslant 0)$ be a right-continuous process with independent increments, such that

$$\begin{aligned} \mathbb{P}(X_{t+h} = X_t + 1) &= \lambda(t)h + o(h), \\ \mathbb{P}(X_{t+h} = X_t) &= 1 - \lambda(t)h + o(h), \end{aligned}$$

where the $o(h)$ terms are uniform in $t \in [0, \infty)$. Show that X_t is a Poisson random variable with parameter $\Lambda(t) = \int_0^t \lambda(s)ds$.

(iii) Let $X = (X_n : n = 1, 2, \ldots)$ be a sequence of independent and identically distributed positive random variables with continuous density function f. We define the sequence of successive records, $(K_n, n = 0, 1, \ldots)$, by $K_0 := 0$ and, for $n \geqslant 0$,

$$K_{n+1} := \inf\{m > K_n : X_m > X_{K_n}\}.$$

The *record process*, $(R_t, t \geqslant 0)$, is then defined by

$$R_t := \#\{n \geqslant 1 : X_{K_n} \leqslant t\}.$$

Explain why the increments of R are independent. Show that R_t is a Poisson random variable with parameter $-\log\{1 - F(t)\}$ where $F(t) = \int_0^t f(s)ds$.

[You may assume the following without proof: For fixed $t > 0$, let Y (respectively, Z) be the subsequence of X obtained by retaining only those elements that are greater than (respectively, smaller than) t. Then Y (respectively, Z) is a sequence of independent variables each having the distribution of X_1 conditioned on $X_1 > t$ (respectively, $X_1 < t$); and Y and Z are independent.]

28K Principles of Statistics

Describe the *Weak Sufficiency Principle* (WSP) and the *Strong Sufficiency Principle* (SSP). Show that Bayesian inference with a fixed prior distribution respects WSP.

A parameter Φ has a prior distribution which is normal with mean 0 and precision (inverse variance) h_Φ. Given $\Phi = \phi$, further parameters $\boldsymbol{\Theta} := (\Theta_i : i = 1, \ldots, I)$ have independent normal distributions with mean ϕ and precision h_Θ. Finally, given both $\Phi = \phi$ and $\boldsymbol{\Theta} = \boldsymbol{\theta} := (\theta_1, \ldots, \theta_I)$, observables $\boldsymbol{X} := (X_{ij} : i = 1, \ldots, I; j = 1, \ldots, J)$ are independent, X_{ij} being normal with mean θ_i, and precision h_X. The precision parameters (h_Φ, h_Θ, h_X) are all fixed and known. Let $\overline{\boldsymbol{X}} := (\overline{X}_1, \ldots, \overline{X}_I)$, where $\overline{X}_i := \sum_{j=1}^{J} X_{ij}/J$. Show, directly from the definition of sufficiency, that $\overline{\boldsymbol{X}}$ is sufficient for $(\Phi, \boldsymbol{\Theta})$. [You may assume without proof that, if $Y_1, \ldots, Y_n$ have independent normal distributions with the same variance, and $\overline{Y} := n^{-1} \sum_{i=1}^{n} Y_i$, then the vector $(Y_1 - \overline{Y}, \ldots, Y_n - \overline{Y})$ is independent of $\overline{Y}$.]

For data-values $\boldsymbol{x} := (x_{ij} : i = 1, \ldots, I; j = 1, \ldots, J)$, determine the joint distribution, Π_ϕ say, of $\boldsymbol{\Theta}$, given $\boldsymbol{X} = \boldsymbol{x}$ and $\Phi = \phi$. What is the distribution of Φ, given $\boldsymbol{\Theta} = \boldsymbol{\theta}$ and $\boldsymbol{X} = \boldsymbol{x}$?

Using these results, describe clearly how Gibbs sampling combined with Rao–Blackwellisation could be applied to estimate the posterior joint distribution of $\boldsymbol{\Theta}$, given $\boldsymbol{X} = \boldsymbol{x}$.

29K Optimization and Control

Suppose $\{x_t\}_{t\geqslant 0}$ is a Markov chain. Consider the dynamic programming equation

$$F_s(x) = \max\Big\{r(x), \beta E\big[F_{s-1}(x_1) \mid x_0 = x\big]\Big\}, \quad s = 1, 2, \ldots,$$

with $r(x) > 0$, $\beta \in (0,1)$, and $F_0(x) = 0$. Prove that:

(i) $F_s(x)$ is nondecreasing in s;

(ii) $F_s(x) \leqslant F(x)$, where $F(x)$ is the value function of an infinite-horizon problem that you should describe;

(iii) $F_\infty(x) = \lim_{s\to\infty} F_s(x) = F(x)$.

A coin lands heads with probability p. A statistician wishes to choose between: $H_0 : p = 1/3$ and $H_1 : p = 2/3$, one of which is true. Prior probabilities of H_1 and H_0 in the ratio $x : 1$ change after one toss of the coin to ratio $2x : 1$ (if the toss was a head) or to ratio $x : 2$ (if the toss was a tail). What problem is being addressed by the following dynamic programming equation?

$$F(x) = \max\Big\{\tfrac{1}{1+x}, \tfrac{x}{1+x}, \beta\Big[\Big(\tfrac{1}{1+x}\tfrac{2}{3} + \tfrac{x}{1+x}\tfrac{1}{3}\Big)F(x/2) + \Big(\tfrac{1}{1+x}\tfrac{1}{3} + \tfrac{x}{1+x}\tfrac{2}{3}\Big)F(2x)\Big]\Big\}.$$

Prove that $G(x) = (1+x)F(x)$ is a convex function of x.

By sketching a graph of G, describe the form of the optimal policy.

30J Stochastic Financial Models

What does it mean to say that $(Y_n, \mathcal{F}_n)_{n\geqslant 0}$ is a *supermartingale*?

State and prove Doob's Upcrossing Inequality for a supermartingale.

Let $(M_n, \mathcal{F}_n)_{n\leqslant 0}$ be a martingale indexed by negative time, that is, for each $n \leqslant 0$, $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n$, $M_n \in L^1(\mathcal{F}_n)$ and $E[M_n|\mathcal{F}_{n-1}] = M_{n-1}$. Using Doob's Upcrossing Inequality, prove that the limit $\lim_{n\to-\infty} M_n$ exists almost surely.

31C Partial Differential Equations

State the Lax–Milgram lemma.

Let $\mathbf{V} = \mathbf{V}(x_1, x_2, x_3)$ be a smooth vector field which is 2π-periodic in each coordinate x_j for $j = 1, 2, 3$. Write down the definition of a weak H^1_{per} solution for the equation

$$-\Delta u + \sum_j V_j \partial_j u + u = f \tag{$*$}$$

to be solved for $u = u(x_1, x_2, x_3)$ given $f = f(x_1, x_2, x_3)$ in H^0, with both u and f also 2π-periodic in each co-ordinate. [In this question use the definition

$$H^s_{per} = \left\{ u = \sum_{m \in \mathbb{Z}^3} \hat{u}(m) e^{im \cdot x} \in L^2 : \|u\|^2_{H^s} = \sum_{m \in \mathbb{Z}^3} (1 + \|m\|^2)^s |\hat{u}(m)|^2 < \infty \right\}$$

for the Sobolev spaces of functions 2π-periodic in each coordinate x_j and for $s = 0, 1, 2, \ldots$.]

If the vector field is divergence-free, prove that there exists a unique weak H^1_{per} solution for all such f.

Supposing that $\mathbf{V}$ is the constant vector field with components $(1, 0, 0)$, write down the solution of $(*)$ in terms of Fourier series and show that there exists $C > 0$ such that

$$\|u\|_{H^2} \leqslant C \|f\|_{H^0} .$$

32C Integrable Systems

Consider the Hamiltonian system

$$\mathbf{p}' = -\frac{\partial H}{\partial \mathbf{q}}, \qquad \mathbf{q}' = \frac{\partial H}{\partial \mathbf{p}},$$

where $H = H(\mathbf{p}, \mathbf{q})$.

When is the transformation $\mathbf{P} = \mathbf{P}(\mathbf{p}, \mathbf{q})$, $\mathbf{Q} = \mathbf{Q}(\mathbf{p}, \mathbf{q})$ canonical?

Prove that, if the transformation is canonical, then the equations in the new variables $(\mathbf{P}, \mathbf{Q})$ are also Hamiltonian, with the same Hamiltonian function H.

Let $\mathbf{P} = C^{-1}\mathbf{p} + Bq$, $\mathbf{Q} = C\mathbf{q}$, where C is a symmetric nonsingular matrix. Determine necessary and sufficient conditions on C for the transformation to be canonical.

33E Principles of Quantum Mechanics

(i) In units where $\hbar = 1$, angular momentum states $|j\ m\rangle$ obey

$$J^2|j\ m\rangle = j(j+1)|j\ m\rangle, \qquad J_3|j\ m\rangle = m|j\ m\rangle.$$

Use the algebra of angular momentum $[J_i,\ J_j] = i\epsilon_{ijk}J_k$ to derive the following in terms of J^2, $J_\pm = J_1 \pm iJ_2$ and J_3:

(a) $[J^2,\ J_i]$;

(b) $[J_3,\ J_\pm]$;

(c) $[J^2,\ J_\pm]$.

(ii) Find J_+J_- in terms of J^2 and J_3. Thus calculate the quantum numbers of the state $J_\pm|j\ m\rangle$ in terms of j and m. Derive the normalisation of the state $J_-|j\ m\rangle$. Therefore, show that

$$\langle j\ j-1|J_+^{j-1}J_-^{j}|j\ j\rangle = \sqrt{A}\,(2j-1)!,$$

finding A in terms of j.

(iii) Consider the combination of a spinless particle with an electron of spin $1/2$ and orbital angular momentum 1. Calculate the probability that the electron has a spin of $+1/2$ in the $z-$direction if the combined system has an angular momentum of $+1/2$ in the $z-$direction and a total angular momentum of $+3/2$. Repeat the calculation for a total angular momentum of $+1/2$.

34D Applications of Quantum Mechanics

(i) A particle of momentum $\hbar k$ and energy $E = \hbar^2 k^2/2m$ scatters off a spherically-symmetric target in three dimensions. Define the corresponding *scattering amplitude* f as a function of the scattering angle θ. Expand the scattering amplitude in partial waves of definite angular momentum l, and determine the coefficients of this expansion in terms of the phase shifts $\delta_l(k)$ appearing in the following asymptotic form of the wavefunction, valid at large distance from the target,

$$\psi(\mathbf{r}) \quad \sim \quad \sum_{l=0}^{\infty} \frac{2l+1}{2ik} \left[e^{2i\delta_l} \frac{e^{ikr}}{r} \; - \; (-1)^l \, \frac{e^{-ikr}}{r} \right] P_l(\cos\theta)\,.$$

Here, $r = |\mathbf{r}|$ is the distance from the target and P_l are the Legendre polynomials.

[You may use without derivation the following approximate relation between plane and spherical waves (valid asymptotically for large r):

$$\exp(ikz) \quad \sim \quad \sum_{l=0}^{\infty} (2l+1)\, i^l \, \frac{\sin\left(kr - \frac{1}{2}l\pi\right)}{kr} \, P_l(\cos\theta)\,.$$

]

(ii) Suppose that the potential energy takes the form $V(r) = \lambda\, U(r)$ where $\lambda \ll 1$ is a dimensionless coupling. By expanding the wavefunction in a power series in λ, derive the *Born Approximation* to the scattering amplitude in the form

$$f(\theta) \quad = \quad -\frac{2m\lambda}{\hbar^2} \int_0^{\infty} U(r)\, \frac{\sin qr}{q}\, r dr\,,$$

up to corrections of order λ^2, where $q = 2k \sin(\theta/2)$. [You may quote any results you need for the Green's function for the differential operator $\nabla^2 + k^2$ provided they are stated clearly.]

(iii) Derive the corresponding order λ contribution to the phase shift $\delta_l(k)$ of angular momentum l.

[You may use the orthogonality relations

$$\int_{-1}^{+1} P_l(w) P_m(w)\, dw \quad = \quad \frac{2}{(2l+1)}\, \delta_{lm}$$

and the integral formula

$$\int_0^1 P_l\left(1 - 2x^2\right) \sin(ax)\, dx \quad = \quad \frac{a}{2} \left[j_l\left(\frac{a}{2}\right) \right]^2\,,$$

where $j_l(z)$ is a spherical Bessel function.]

35A Statistical Physics

(i) The first law of thermodynamics is $dE = TdS - pdV + \mu dN$, where μ is the chemical potential. Briefly describe its meaning.

(ii) What is equipartition of energy? Under which conditions is it valid? Write down the heat capacity C_V at constant volume for a monatomic ideal gas.

(iii) Starting from the first law of thermodynamics, and using the fact that for an ideal gas $(\partial E/\partial V)_T = 0$, show that the entropy of an ideal gas containing N particles can be written as

$$S(T,V) = N\left(\int \frac{c_V(T)}{T}dT + k_{\mathrm{B}} \ln \frac{V}{N} + \text{const}\right) ,$$

where T and V are temperature and volume of the gas, k_{B} is the Boltzmann constant, and we define the heat capacity per particle as $c_V = C_V/N$.

(iv) The Gibbs free energy G is defined as $G = E + pV - TS$. Verify that it is a function of temperature T, pressure p and particle number N. Explain why G depends on the particle number N through $G = \mu(T,p)N$.

(v) Calculate the chemical potential μ for an ideal gas with heat capacity per particle $c_V(T)$. Calculate μ for the special case of a monatomic gas.

36D General Relativity

A spacetime contains a one-parameter family of geodesics $x^a = x^a(\lambda, \mu)$, where λ is a parameter along each geodesic, and μ labels the geodesics. The tangent to the geodesics is $T^a = \partial x^a/\partial\lambda$, and $N^a = \partial x^a/\partial\mu$ is a connecting vector. Prove that

$$\nabla_\mu T^a = \nabla_\lambda N^a \,,$$

and hence derive the equation of geodesic deviation:

$$\nabla_\lambda^2 N^a + R^a{}_{bcd}\, T^b N^c T^d = 0\,.$$

[You may assume $R^a{}_{bcd} = -R^a{}_{bdc}$ and the Ricci identity in the form

$$(\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda)\, T^a = R^a{}_{bcd}\, T^b T^c N^d\,. \quad]$$

Consider the two-dimensional space consisting of the sphere of radius r with line element

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

Show that one may choose $T^a = (1, 0)$, $N^a = (0, 1)$, and that

$$\nabla_\theta N^a = \cot\theta\ N^a\,.$$

Hence show that $R = 2/r^2$, using the geodesic deviation equation and the identity in any two-dimensional space

$$R^a{}_{bcd} = \frac{1}{2} R(\delta^a_c\, g_{bd} - \delta^a_d\, g_{bc})\,,$$

where R is the Ricci scalar.

Verify your answer by direct computation of R.

[You may assume that the only non-zero connection components are

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta$$

and

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\,.$$

You may also use the definition

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}\,. \quad]$$

37A Fluid Dynamics II

Write down the boundary-layer equations for steady two-dimensional flow of a viscous incompressible fluid with velocity $U(x)$ outside the boundary layer. Find the boundary layer thickness $\delta(x)$ when $U(x) = U_0$, a constant. Show that the boundary-layer equations can be satisfied in this case by a streamfunction $\psi(x, y) = g(x)f(\eta)$ with suitable scaling function $g(x)$ and similarity variable η. Find the equation satisfied by f and the associated boundary conditions.

Find the drag on a thin two-dimensional flat plate of finite length L placed parallel to a uniform flow. Why does the drag not increase in proportion to the length of the plate? [You may assume that the boundary-layer solution is applicable except in negligibly small regions near the leading and trailing edges. You may also assume that $f''(0) = 0.33$.]

38C Waves

Show that the equations governing linear elasticity have plane-wave solutions, distinguishing between P, SV and SH waves.

A semi-infinite elastic medium in $y < 0$ (where y is the vertical coordinate) with density ρ and Lamé moduli λ and μ is overlaid by a layer of thickness h (in $0 < y < h$) of a second elastic medium with density ρ' and Lamé moduli λ' and μ'. The top surface at $y = h$ is free, that is, the surface tractions vanish there. The speed of the S-waves is lower in the layer, that is, ${c'_S}^2 = \mu'/\rho' < \mu/\rho = {c_S}^2$. For a time-harmonic SH-wave with horizontal wavenumber k and frequency ω, which oscillates in the slow top layer and decays exponentially into the fast semi-infinite medium, derive the dispersion relation for the apparent horizontal wave speed $c(k) = \omega/k$:

$$\tan\left(kh\sqrt{(c^2/{c'_S}^2) - 1}\right) = \frac{\mu\sqrt{1 - (c^2/c_S^2)}}{\mu'\sqrt{(c^2/{c'_S}^2) - 1}}\,. \qquad (*)$$

Show graphically that for a given value of k there is always at least one real value of c which satisfies equation $(*)$. Show further that there are one or more higher modes if $\sqrt{c_S^2/{c'_S}^2 - 1} > \pi/kh$.

39C Numerical Analysis

Consider the advection equation $u_t = u_x$ on the unit interval $x \in [0,1]$ and $t \geqslant 0$, where $u = u(x,t)$, subject to the initial condition $u(x,0) = \varphi(x)$ and the boundary condition $u(1,t) = 0$, where φ is a given smooth function on $[0,1]$.

(i) We commence by discretising the advection equation above with finite differences on the equidistant space-time grid $\{(m\Delta x, n\Delta t),\ m = 0,\ldots,M+1,\ n = 0,\ldots,T\}$ with $\Delta x = 1/(M+1)$ and $\Delta t > 0$. We obtain an equation for $u_m^n \approx u(m\Delta x, n\Delta t)$ that reads

$$u_m^{n+1} = u_m^n + \frac{1}{2}\mu(u_{m+1}^n - u_{m-1}^n), \quad m = 1,\ldots,M,\ n \in \mathbb{Z}^+,$$

with the condition $u_0^n = 0$ for all $n \in \mathbb{Z}^+$ and $\mu = \Delta t/\Delta x$.

What is the order of approximation (that is, the order of the local error) in space and time of the above discrete solution to the exact solution of the advection equation? Write the scheme in matrix form and deduce for which choices of μ this approximation converges to the exact solution. State (without proof) any theorems you use. [You may use the fact that for a tridiagonal $M \times M$ matrix

$$\begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix}$$

the eigenvalues are given by $\lambda_\ell = \alpha + 2i\beta\cos\frac{\ell\pi}{M+1}$.]

(ii) How does the order change when we replace the central difference approximation of the first derivative in space by forward differences, that is $u_{m+1}^n - u_m^n$ instead of $(u_{m+1}^n - u_{m-1}^n)/2$? For which choices of μ is this new scheme convergent?

(iii) Instead of the approximation in (i) we consider the following method for numerically solving the advection equation,

$$u_m^{n+1} = \mu(u_{m+1}^n - u_{m-1}^n) + u_m^{n-1},$$

where we additionally assume that u_m^1 is given. What is the order of this method for a fixed μ?

END OF PAPER