UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday, 5 June, 2013 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

If A is an $n \times n$ invertible Hermitian matrix, let

$$U_A = \{U \in M_{n\times n}(\mathbb{C}) \,|\, \overline{U}^T AU = A\}.$$

Show that U_A with the operation of matrix multiplication is a group, and that $\det U$ has norm 1 for any $U \in U_A$. What is the relation between U_A and the complex Hermitian form defined by A?

If $A = I_n$ is the $n \times n$ identity matrix, show that any element of U_A is diagonalizable.

2G Groups, Rings and Modules

Show that every Euclidean domain is a PID. Define the notion of a Noetherian ring, and show that $\mathbb{Z}[i]$ is Noetherian by using the fact that it is a Euclidean domain.

3F Analysis II

Let $\mathcal{C}[a,b]$ denote the vector space of continuous real-valued functions on the interval $[a,b]$, and let $\mathcal{C}'[a,b]$ denote the subspace of continuously differentiable functions.

Show that $\|f\|_1 = \max|f| + \max|f'|$ defines a norm on $\mathcal{C}'[a,b]$. Show furthermore that the map $\Phi : f \mapsto f'((a+b)/2)$ takes the closed unit ball $\{\|f\|_1 \leqslant 1\} \subset \mathcal{C}'[a,b]$ to a bounded subset of $\mathbb{R}$.

If instead we had used the norm $\|f\|_0 = \max|f|$ restricted from $\mathcal{C}[a,b]$ to $\mathcal{C}'[a,b]$, would Φ take the closed unit ball $\{\|f\|_0 \leqslant 1\} \subset \mathcal{C}'[a,b]$ to a bounded subset of $\mathbb{R}$? Justify your answer.

4G Metric and Topological Spaces

Let X be a topological space. Prove or disprove the following statements.

(i) If X is discrete, then X is compact if and only if it is a finite set.

(ii) If Y is a subspace of X and X, Y are both compact, then Y is closed in X.

5B Methods

Consider the equation

$$xu_x + (x+y)u_y = 1$$

subject to the Cauchy data $u(1,y) = y$. Using the method of characteristics, obtain a solution to this equation.

6D Electromagnetism

Use Maxwell's equations to obtain the equation of continuity

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0\,.$$

Show that, for a body made from material of uniform conductivity σ, the charge density at any fixed internal point decays exponentially in time. If the body is finite and isolated, explain how this result can be consistent with overall charge conservation.

7A Fluid Dynamics

An incompressible, inviscid fluid occupies the region beneath the free surface $y = \eta(x,t)$ and moves with a velocity field determined by the velocity potential $\phi(x,y,t)$. Gravity acts in the $-y$ direction. You may assume Bernoulli's integral of the equation of motion:

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gy = F(t)\,.$$

Give the kinematic and dynamic boundary conditions that must be satisfied by ϕ on $y = \eta(x,t)$.

In the absence of waves, the fluid has constant uniform velocity U in the x direction. Derive the linearised form of the boundary conditions for small amplitude waves.

Assume that the free surface and velocity potential are of the form:

$$\begin{aligned} \eta &= ae^{i(kx-\omega t)} \\ \phi &= Ux + ibe^{ky}e^{i(kx-\omega t)} \end{aligned}$$

(where implicitly the real parts are taken). Show that

$$(\omega - kU)^2 = gk\,.$$

8H Statistics

State and prove the Rao–Blackwell theorem.

Individuals in a population are independently of three types $\{0, 1, 2\}$, with unknown probabilities p_0, p_1, p_2 where $p_0 + p_1 + p_2 = 1$. In a random sample of n people the ith person is found to be of type $x_i \in \{0, 1, 2\}$.

Show that an unbiased estimator of $\theta = p_0 p_1 p_2$ is

$$\hat{\theta} = \begin{cases} 1, & \text{if } (x_1, x_2, x_3) = (0, 1, 2), \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that n_i of the individuals are of type i. Find an unbiased estimator of θ, say θ^*, such that $\operatorname{var}(\theta^*) < \theta(1 - \theta)$.

9H Optimization

Given a network with a source A, a sink B, and capacities on directed arcs, define what is meant by a minimum cut.

The m streets and n intersections of a town are represented by sets of edges E and vertices V of a connected graph. A city planner wishes to make all streets one-way while ensuring it possible to drive away from each intersection along at least k different streets.

Use a theorem about min-cut and max-flow to prove that the city planner can achieve his goal provided that the following is true:

$$d(U) \geqslant k|\,U| \text{ for all } U \subseteq V,$$

where $|\,U|$ is the size of U and $d(U)$ is the number edges with at least one end in U. How could the planner find street directions that achieve his goal?

[*Hint: Consider a network having nodes A, B, nodes $a_1, \dots, a_m$ for the streets and nodes $b_1, \dots, b_n$ for the intersections. There are directed arcs from A to each a_i, and from each b_i to B. From each a_i there are two further arcs, directed towards b_j and $b_{j'}$ that correspond to endpoints of street i.*]

SECTION II

10E Linear Algebra

Define what it means for a set of vectors in a vector space V to be linearly dependent. Prove from the definition that any set of $n+1$ vectors in $\mathbb{R}^n$ is linearly dependent.

Using this or otherwise, prove that if V has a finite basis consisting of n elements, then any basis of V has exactly n elements.

Let V be the vector space of bounded continuous functions on $\mathbb{R}$. Show that V is infinite dimensional.

11G Groups, Rings and Modules

(i) State the structure theorem for finitely generated modules over Euclidean domains.

(ii) Let $\mathbb{C}[X]$ be the polynomial ring over the complex numbers. Let M be a $\mathbb{C}[X]$-module which is 4-dimensional as a $\mathbb{C}$-vector space and such that $(X-2)^4 \cdot x = 0$ for all $x \in M$. Find all possible forms we obtain when we write $M \cong \bigoplus_{i=1}^{m} \mathbb{C}[X]/(P_i^{n_i})$ for irreducible $P_i \in \mathbb{C}[X]$ and $n_i \geqslant 1$.

(iii) Consider the quotient ring $M = \mathbb{C}[X]/(X^3 + X)$ as a $\mathbb{C}[X]$-module. Show that M is isomorphic as a $\mathbb{C}[X]$-module to the direct sum of three copies of $\mathbb{C}$. Give the isomorphism and its inverse explicitly.

12F Analysis II

Let $f : U \to \mathbb{R}$ be continuous on an open set $U \subset \mathbb{R}^2$. Suppose that on U the partial derivatives $D_1 f$, $D_2 f$, $D_1 D_2 f$ and $D_2 D_1 f$ exist and are continuous. Prove that $D_1 D_2 f \;=\; D_2 D_1 f$ on U.

If f is infinitely differentiable, and $m \in \mathbb{N}$, what is the maximum number of distinct m-th order partial derivatives that f may have on U ?

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y^2}{x^4 + y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x,y) = \begin{cases} \dfrac{xy(x^4 - y^4)}{x^4 + y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

For each of f and g, determine whether they are (i) differentiable, (ii) infinitely differentiable at the origin. Briefly justify your answers.

13D Complex Analysis or Complex Methods

Let

$$I = \oint_C \frac{e^{iz^2/\pi}}{1 + e^{-2z}} dz\,,$$

where C is the rectangle with vertices at $\pm R$ and $\pm\ R + i\pi$, traversed anti-clockwise.

(i) Show that $I = \dfrac{\pi(1+i)}{\sqrt{2}}$.

(ii) Assuming that the contribution to I from the vertical sides of the rectangle is negligible in the limit $R \to \infty$, show that

$$\int_{-\infty}^{\infty} e^{ix^2/\pi} dx = \frac{\pi(1+i)}{\sqrt{2}}\,.$$

(iii) Justify briefly the assumption that the contribution to I from the vertical sides of the rectangle is negligible in the limit $R \to \infty$.

14F Geometry

Let A and B be disjoint circles in $\mathbb{C}$. Prove that there is a Möbius transformation which takes A and B to two concentric circles.

A collection of circles $X_i \subset \mathbb{C}$, $0 \leqslant i \leqslant n-1$, for which

1. X_i is tangent to A, B and X_{i+1}, where indices are *mod* n;
2. the circles are disjoint away from tangency points;

is called a *constellation* on (A, B). Prove that for any $n \geqslant 2$ there is some pair (A, B) and a constellation on (A, B) made up of precisely n circles. Draw a picture illustrating your answer.

Given a constellation on (A, B), prove that the tangency points $X_i \cap X_{i+1}$ for $0 \leqslant i \leqslant n-1$ all lie on a circle. Moreover, prove that if we take any other circle Y_0 tangent to A and B, and then construct Y_i for $i \geqslant 1$ inductively so that Y_i is tangent to A, B and Y_{i-1}, then we will have $Y_n = Y_0$, i.e. the chain of circles will again close up to form a constellation.

15A Variational Principles

Starting from the Euler–Lagrange equation, show that a condition for

$$\int f(y, y')dx$$

to be stationary is

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}.$$

In the half-plane $y > 0$, light has speed $c(y) = y + c_0$ where $c_0 > 0$. Find the equation for a light ray between $(-a, 0)$ and $(a, 0)$. Sketch the solution.

16B Methods

The steady-state temperature distribution $u(x)$ in a uniform rod of finite length satisfies the boundary value problem

$$-D\frac{d^2}{dx^2}u(x) = f(x)\,, \qquad 0 < x < l$$

$$u(0) = 0\,, \qquad u(l) = 0$$

where $D > 0$ is the (constant) diffusion coefficient. Determine the Green's function $G(x,\xi)$ for this problem. Now replace the above homogeneous boundary conditions with the inhomogeneous boundary conditions $u(0) = \alpha, \quad u(l) = \beta$ and give a solution to the new boundary value problem. Hence, obtain the steady-state solution for the following problem with the specified boundary conditions:

$$-D\frac{\partial^2}{\partial x^2}u(x,t) + \frac{\partial}{\partial t}u(x,t) = x\,, \quad 0 < x < 1\,,$$

$$u(0,t) = 1/D\,, \qquad u(1,t) = 2/D\,, \qquad t > 0\,.$$

[You may assume that a steady-state solution exists.]

17B Quantum Mechanics

(i) Consider a particle of mass m confined to a one-dimensional potential well of depth $U > 0$ and potential

$$V(x) = \begin{cases} -U, & |x| < l \\ 0, & |x| > l. \end{cases}$$

If the particle has energy E where $-U \leqslant E < 0$, show that for even states

$$\alpha \tan \alpha l = \beta$$

where $\alpha = [\frac{2m}{\hbar^2}(U+E)]^{1/2}$ and $\beta = [-\frac{2m}{\hbar^2}E]^{1/2}$.

(ii) A particle of mass m that is incident from the left scatters off a one-dimensional potential given by

$$V(x) = k\delta(x)$$

where $\delta(x)$ is the Dirac delta. If the particle has energy $E > 0$ and $k > 0$, obtain the reflection and transmission coefficients R and T, respectively. Confirm that $R + T = 1$.

For the case $k < 0$ and $E < 0$ show that the energy of the only even parity bound state of the system is given by

$$E = -\frac{mk^2}{2\hbar^2}.$$

Use part (i) to verify this result by taking the limit $U \to \infty,\ l \to 0$ with Ul fixed.

18D Electromagnetism

Starting with the expression

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')\, dV'}{|\mathbf{r} - \mathbf{r}'|}$$

for the magnetic vector potential at the point $\mathbf{r}$ due to a current distribution of density $\mathbf{J}(\mathbf{r})$, obtain the Biot-Savart law for the magnetic field due to a current I flowing in a simple loop C:

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{r}' \times (\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} \qquad (\mathbf{r} \notin C).$$

Verify by direct differentiation that this satisfies $\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{0}$. You may use without proof the identity $\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{v}) = \mathbf{a}(\boldsymbol{\nabla} \cdot \mathbf{v}) - (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{v}$, where $\mathbf{a}$ is a constant vector and $\mathbf{v}$ is a vector field.

Given that C is planar, and is described in cylindrical polar coordinates by $z = 0$, $r = f(\theta)$, show that the magnetic field at the origin is

$$\widehat{\mathbf{z}}\, \frac{\mu_0 I}{4\pi} \oint \frac{d\theta}{f(\theta)}\,.$$

If C is the ellipse $r(1 - e\cos\theta) = \ell$, find the magnetic field at the focus due to a current I.

19C Numerical Analysis

Explain briefly what is meant by the convergence of a numerical method for solving the ordinary differential equation

$$y'(t) = f(t, y), \qquad t \in [0, T], \qquad y(0) = y_0\,.$$

Prove from first principles that if the function f is sufficiently smooth and satisfies the Lipschitz condition

$$|f(t, x) - f(t, y)| \leqslant L\,|x - y|, \qquad x, y \in \mathbb{R}, \qquad t \in [0, T],$$

for some $L > 0$, then the backward Euler method

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})\,,$$

converges and find the order of convergence.

Find the linear stability domain of the backward Euler method.

20H Markov Chains

(i) Suppose $(X_n)_{n\geqslant 0}$ is an irreducible Markov chain and $f_{ij} = P(X_n = j \text{ for some } n \geqslant 1 \mid X_0 = i)$. Prove that $f_{ii} \geqslant f_{ij} f_{ji}$ and that

$$\sum_{n=0}^{\infty} P_i(X_n = i) = \sum_{n=1}^{\infty} f_{ii}^{n-1}.$$

(ii) Let $(X_n)_{n\geqslant 0}$ be a symmetric random walk on the $\mathbb{Z}^2$ lattice. Prove that $(X_n)_{n\geqslant 0}$ is recurrent. You may assume, for $n \geqslant 1$,

$$1/2 < 2^{-2n}\sqrt{n}\binom{2n}{n} < 1.$$

(iii) A princess and monster perform independent random walks on the $\mathbb{Z}^2$ lattice. The trajectory of the princess is the symmetric random walk $(X_n)_{n\geqslant 0}$. The monster's trajectory, denoted $(Z_n)_{n\geqslant 0}$, is a sleepy version of an independent symmetric random walk $(Y_n)_{n\geqslant 0}$. Specifically, given an infinite sequence of integers $0 = n_0 < n_1 < \cdots$, the monster sleeps between these times, so $Z_{n_i+1} = \cdots = Z_{n_{i+1}} = Y_{i+1}$. Initially, $X_0 = (100, 0)$ and $Z_0 = Y_0 = (0, 100)$. The princess is captured if and only if at some future time she and the monster are simultaneously at $(0, 0)$.

Compare the capture probabilities for an active monster, who takes $n_{i+1} = n_i + 1$ for all i, and a sleepy monster, who takes n_i spaced sufficiently widely so that

$$P\Big(X_k = (0,0) \text{ for some } k \in \{n_i + 1, \ldots, n_{i+1}\}\Big) > 1/2.$$

END OF PAPER