UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 3 June, 2013 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A****,* ***B****,* ***C****,* ***D****,* ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

Let m and n be positive integers. State what is meant by the *greatest common divisor* $\gcd(m,n)$ of m and n, and show that there exist integers a and b such that $\gcd(m,n) = am + bn$. Deduce that an integer k divides both m and n only if k divides $\gcd(m,n)$.

Prove (without using the Fundamental Theorem of Arithmetic) that for any positive integer k, $\gcd(km,kn) = k\gcd(m,n)$.

2E Numbers and Sets

Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. What does it mean to say that the sequence (x_n) is convergent? What does it mean to say the series $\sum x_n$ is convergent? Show that if $\sum x_n$ is convergent, then the sequence (x_n) converges to zero. Show that the converse is not necessarily true.

3B Dynamics and Relativity

A hot air balloon of mass M is equipped with a bag of sand of mass $m = m(t)$ which decreases in time as the sand is gradually released. In addition to gravity the balloon experiences a constant upwards buoyancy force T and we neglect air resistance effects. Show that if $v(t)$ is the upward speed of the balloon then

$$(M+m)\frac{dv}{dt} = T - (M+m)g.$$

Initially at $t = 0$ the mass of sand is $m(0) = m_0$ and the balloon is at rest in equilibrium. Subsequently the sand is released at a constant rate and is depleted in a time t_0. Show that the speed of the balloon at time t_0 is

$$gt_0\left(\left(1+\frac{M}{m_0}\right)\ln\left(1+\frac{m_0}{M}\right)-1\right).$$

[You may use without proof the indefinite integral $\int t/(A-t)\,dt = -t - A\ln|A-t| + C$.]

4B Dynamics and Relativity

A frame S' moves with constant velocity v along the x axis of an inertial frame S of Minkowski space. A particle P moves with constant velocity u' along the x' axis of S'. Find the velocity u of P in S.

The rapidity φ of any velocity w is defined by $\tanh\varphi = w/c$. Find a relation between the rapidities of u, u' and v.

Suppose now that P is initially at rest in S and is subsequently given n successive velocity increments of $c/2$ (each delivered in the instantaneous rest frame of the particle). Show that the resulting velocity of P in S is

$$c\left(\frac{e^{2n\alpha}-1}{e^{2n\alpha}+1}\right)$$

where $\tanh\alpha = 1/2$.

[You may use without proof the addition formulae $\sinh(a+b) = \sinh a\cosh b + \cosh a\sinh b$ and $\cosh(a+b) = \cosh a\cosh b + \sinh a\sinh b$.]

SECTION II

5E Numbers and Sets

(i) What does it mean to say that a function $f\colon X \to Y$ is *injective*? What does it mean to say that f is *surjective*? Let $g\colon Y \to Z$ be a function. Show that if $g \circ f$ is injective, then so is f, and that if $g \circ f$ is surjective, then so is g.

(ii) Let X_1, X_2 be two sets. Their *product* $X_1 \times X_2$ is the set of ordered pairs (x_1, x_2) with $x_i \in X_i$ $(i = 1,\, 2)$. Let p_i (for $i = 1,\, 2$) be the function

$$p_i\colon X_1 \times X_2 \to X_i, \quad p_i(x_1, x_2) = x_i.$$

When is p_i surjective? When is p_i injective?

(iii) Now let Y be any set, and let $f_1\colon Y \to X_1$, $f_2\colon Y \to X_2$ be functions. Show that there exists a unique $g\colon Y \to X_1 \times X_2$ such that $f_1 = p_1 \circ g$ and $f_2 = p_2 \circ g$.

Show that if f_1 or f_2 is injective, then g is injective. Is the converse true? Justify your answer.

Show that if g is surjective then both f_1 and f_2 are surjective. Is the converse true? Justify your answer.

6E Numbers and Sets

(i) Let N and r be integers with $N \geqslant 0$, $r \geqslant 1$. Let S be the set of $(r+1)$-tuples $(n_0, n_1, \ldots, n_r)$ of non-negative integers satisfying the equation $n_0 + \cdots + n_r = N$. By mapping elements of S to suitable subsets of $\{1, \ldots, N+r\}$ of size r, or otherwise, show that the number of elements of S equals

$$\binom{N+r}{r}.$$

(ii) State the Inclusion–Exclusion principle.

(iii) Let $a_0, \ldots, a_r$ be positive integers. Show that the number of $(r+1)$-tuples (n_i) of integers satisfying

$$n_0 + \cdots + n_r = N, \quad 0 \leqslant n_i < a_i \text{ for all } i$$

is

$$\binom{N+r}{r} - \sum_{0 \leqslant i \leqslant r} \binom{N+r-a_i}{r} + \sum_{0 \leqslant i < j \leqslant r} \binom{N+r-a_i-a_j}{r} - \sum_{0 \leqslant i < j < k \leqslant r} \binom{N+r-a_i-a_j-a_k}{r} + \cdots$$

where the binomial coefficient $\binom{m}{r}$ is defined to be zero if $m < r$.

7E Numbers and Sets

(i) What does it mean to say that a set is *countable*? Show directly from your definition that any subset of a countable set is countable, and that a countable union of countable sets is countable.

(ii) Let X be either $\mathbb{Z}$ or $\mathbb{Q}$. A function $f\colon X \to \mathbb{Z}$ is said to be *periodic* if there exists a positive integer n such that for every $x \in X$, $f(x+n) = f(x)$. Show that the set of periodic functions from $\mathbb{Z}$ to itself is countable. Is the set of periodic functions $f\colon \mathbb{Q} \to \mathbb{Z}$ countable? Justify your answer.

(iii) Show that $\mathbb{R}^2$ is not the union of a countable collection of lines.

[You may assume that $\mathbb{R}$ and the power set of $\mathbb{N}$ are uncountable.]

8E Numbers and Sets

Let p be a prime number, and x, n integers with $n \geqslant 1$.

(i) Prove Fermat's Little Theorem: for any integer x, $x^p \equiv x \pmod{p}$.

(ii) Show that if y is an integer such that $x \equiv y \pmod{p^n}$, then for every integer $r \geqslant 0$,

$$x^{p^r} \equiv y^{p^r} \pmod{p^{n+r}}.$$

Deduce that $x^{p^n} \equiv x^{p^{n-1}} \pmod{p^n}$.

(iii) Show that there exists a unique integer $y \in \{0, 1, \ldots, p^n - 1\}$ such that

$$y \equiv x \pmod{p} \quad \text{and} \quad y^p \equiv y \pmod{p^n}.$$

9B Dynamics and Relativity

(a) A particle P of unit mass moves in a plane with polar coordinates (r, θ). You may assume that the radial and angular components of the acceleration are given by $(\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\dot{\theta})$, where the dot denotes d/dt. The particle experiences a central force corresponding to a potential $V = V(r)$.

(i) Prove that $l = r^2\dot{\theta}$ is constant in time and show that the time dependence of the radial coordinate $r(t)$ is equivalent to the motion of a particle in one dimension x in a potential V_{eff} given by

$$V_{\text{eff}} = V(x) + \frac{l^2}{2x^2}.$$

(ii) Now suppose that $V(r) = -e^{-r}$. Show that if $l^2 < 27/e^3$ then two circular orbits are possible with radii $r_1 < 3$ and $r_2 > 3$. Determine whether each orbit is stable or unstable.

(b) Kepler's first and second laws for planetary motion are the following statements:
K1: the planet moves on an ellipse with a focus at the Sun;
K2: the line between the planet and the Sun sweeps out equal areas in equal times.
Show that **K2** implies that the force acting on the planet is a central force.
Show that **K2** together with **K1** implies that the force is given by the inverse square law.
[You may assume that an ellipse with a focus at the origin has polar equation $\frac{A}{r} = 1 + \varepsilon\cos\theta$ with $A > 0$ and $0 < \varepsilon < 1$.]

10B Dynamics and Relativity

(a) A rigid body Q is made up of N particles of masses m_i at positions $\mathbf{r}_i(t)$. Let $\mathbf{R}(t)$ denote the position of its centre of mass. Show that the total kinetic energy of Q may be decomposed into T_1, the kinetic energy of the centre of mass, plus a term T_2 representing the kinetic energy about the centre of mass.
Suppose now that Q is rotating with angular velocity $\boldsymbol{\omega}$ about its centre of mass. Define the moment of inertia I of Q (about the axis defined by $\boldsymbol{\omega}$) and derive an expression for T_2 in terms of I and $\omega = |\boldsymbol{\omega}|$.

(b) Consider a uniform rod AB of length $2l$ and mass M. Two such rods AB and BC are freely hinged together at B. The end A is attached to a fixed point O on a perfectly smooth horizontal floor and AB is able to rotate freely about O. The rods are initially at rest, lying in a vertical plane with C resting on the floor and each rod making angle α with the horizontal. The rods subsequently move under gravity in their vertical plane.
Find an expression for the angular velocity of rod AB when it makes angle θ with the floor. Determine the speed at which the hinge strikes the floor.

11B Dynamics and Relativity

(i) An inertial frame S has orthonormal coordinate basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. A second frame S' rotates with angular velocity $\boldsymbol{\omega}$ relative to S and has coordinate basis vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$. The motion of S' is characterised by the equations $d\mathbf{e}'_i/dt = \boldsymbol{\omega}\times\mathbf{e}'_i$ and at $t = 0$ the two coordinate frames coincide.
If a particle P has position vector $\mathbf{r}$ show that $\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}$ where $\mathbf{v}$ and $\mathbf{v}'$ are the velocity vectors of P as seen by observers fixed respectively in S and S'.

(ii) For the remainder of this question you may assume that $\mathbf{a} = \mathbf{a}' + 2\,\boldsymbol{\omega}\times\mathbf{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ where $\mathbf{a}$ and $\mathbf{a}'$ are the acceleration vectors of P as seen by observers fixed respectively in S and S', and that $\boldsymbol{\omega}$ is constant.

Consider again the frames S and S' in (i). Suppose that $\boldsymbol{\omega} = \omega\,\mathbf{e}_3$ with ω constant. A particle of mass m moves under a force $\mathbf{F} = -4m\omega^2\mathbf{r}$. When viewed in S' its position and velocity at time $t = 0$ are $(x', y', z') = (1, 0, 0)$ and $(\dot{x}', \dot{y}', \dot{z}') = (0, 0, 0)$. Find the motion of the particle in the coordinates of S'. Show that for an observer fixed in S', the particle achieves its maximum speed at time $t = \pi/(4\omega)$ and determine that speed. [*Hint: you may find it useful to consider the combination* $\zeta = x' + iy'$.]

12B Dynamics and Relativity

(a) Let S with coordinates (ct, x, y) and S' with coordinates (ct', x', y') be inertial frames in Minkowski space with two spatial dimensions. S' moves with velocity v along the x-axis of S and they are related by the standard Lorentz transformation:

$$\begin{pmatrix} ct \\ x \\ y \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v/c & 0 \\ \gamma v/c & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix}, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

A photon is emitted at the spacetime origin. In S' it has frequency ν' and propagates at angle θ' to the x'-axis.
Write down the 4-momentum of the photon in the frame S'.
Hence or otherwise find the frequency of the photon as seen in S. Show that it propagates at angle θ to the x-axis in S, where

$$\tan\theta = \frac{\tan\theta'}{\gamma\left(1 + \dfrac{v}{c}\sec\theta'\right)}.$$

A light source in S' emits photons uniformly in all directions in the $x'y'$-plane. Show that for large v, in S half of the light is concentrated into a narrow cone whose semi-angle α is given by $\cos\alpha = v/c$.

(b) The centre-of-mass frame for a system of relativistic particles in Minkowski space is the frame in which the total relativistic 3-momentum is zero.
Two particles A_1 and A_2 of rest masses m_1 and m_2 move collinearly with uniform velocities u_1 and u_2 respectively, along the x-axis of a frame S. They collide, coalescing to form a single particle A_3.
Determine the velocity of the centre-of-mass frame of the system comprising A_1 and A_2.
Find the speed of A_3 in S and show that its rest mass m_3 is given by

$$m_3^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma_1 \gamma_2 \left(1 - \frac{u_1 u_2}{c^2}\right),$$

where $\gamma_i = (1 - u_i^2/c^2)^{-1/2}$.

END OF PAPER