UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday, 4 June, 2013 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

State Lagrange's Theorem.

Let G be a finite group, and H and K two subgroups of G such that

(i) the orders of H and K are coprime;

(ii) every element of G may be written as a product hk, with $h \in H$ and $k \in K$;

(iii) both H and K are normal subgroups of G.

Prove that G is isomorphic to $H \times K$.

2D Groups

Define what it means for a group to be *cyclic*, and for a group to be *abelian*. Show that every cyclic group is abelian, and give an example to show that the converse is false.

Show that a group homomorphism from the cyclic group C_n of order n to a group G determines, and is determined by, an element g of G such that $g^n = 1$.

Hence list all group homomorphisms from C_4 to the symmetric group S_4.

3C Vector Calculus

The curve C is given by

$$\mathbf{r}(t) = \left(\sqrt{2}e^t,\ -e^t \sin t,\ e^t \cos t\right), \quad -\infty < t < \infty\,.$$

(i) Compute the arc length of C between the points with $t = 0$ and $t = 1$.

(ii) Derive an expression for the curvature of C as a function of arc length s measured from the point with $t = 0$.

4C Vector Calculus

State a necessary and sufficient condition for a vector field $\mathbf{F}$ on $\mathbb{R}^3$ to be conservative.

Check that the field

$$\mathbf{F} = (2x \cos y - 2z^3,\ 3 + 2ye^z - x^2 \sin y,\ y^2 e^z - 6xz^2)$$

is conservative and find a scalar potential for $\mathbf{F}$.

SECTION II

5D Groups

(a) Let G be a finite group. Show that there exists an injective homomorphism $G \to Sym(X)$ to a symmetric group, for some set X.

(b) Let H be the full group of symmetries of the cube, and X the set of edges of the cube.

Show that H acts transitively on X, and determine the stabiliser of an element of X. Hence determine the order of H.

Show that the action of H on X defines an injective homomorphism $H \to Sym(X)$ to the group of permutations of X, and determine the number of cosets of H in $Sym(X)$.

Is H a normal subgroup of $Sym(X)$? Prove your answer.

6D Groups

(a) Let p be a prime, and let $G = SL_2(p)$ be the group of 2×2 matrices of determinant 1 with entries in the field $\mathbb{F}_p$ of integers mod p.

(i) Define the action of G on $X = \mathbb{F}_p \cup \{\infty\}$ by Möbius transformations. [You need not show that it is a group action.]

State the orbit-stabiliser theorem.

Determine the orbit of ∞ and the stabiliser of ∞. Hence compute the order of $SL_2(p)$.

(ii) Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Show that A is conjugate to B in G if $p = 11$, but not if $p = 5$.

(b) Let G be the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, x \in \mathbb{R}$. Show that G is a subgroup of the group of all invertible real matrices.

Let H be the subset of G given by matrices with $a = 0$. Show that H is a normal subgroup, and that the quotient group G/H is isomorphic to $\mathbb{R}$.

Determine the centre $Z(G)$ of G, and identify the quotient group $G/Z(G)$.

7D Groups

(a) Let G be the dihedral group of order $4n$, the symmetry group of a regular polygon with $2n$ sides.

Determine all elements of order 2 in G. For each element of order 2, determine its conjugacy class and the smallest normal subgroup containing it.

(b) Let G be a finite group.

(i) Prove that if H and K are subgroups of G, then $K \cup H$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

(ii) Let H be a proper subgroup of G, and write $G \setminus H$ for the elements of G not in H. Let K be the subgroup of G generated by $G \setminus H$.

Show that $K = G$.

8D Groups

Let p be a prime number.

Prove that every group whose order is a power of p has a non-trivial centre.

Show that every group of order p^2 is abelian, and that there are precisely two of them, up to isomorphism.

9C Vector Calculus

Give an explicit formula for $\mathcal{J}$ which makes the following result hold:

$$\int_D f \; dx \; dy \; dz = \int_{D'} \phi \; |\mathcal{J}| \; du \; dv \; dw \,,$$

where the region D, with coordinates x, y, z, and the region D', with coordinates u, v, w, are in one-to-one correspondence, and

$$\phi(u, v, w) = f(x(u, v, w), y(u, v, w), z(u, v, w))\,.$$

Explain, in outline, why this result holds.

Let D be the region in $\mathbb{R}^3$ defined by $4 \leqslant x^2 + y^2 + z^2 \leqslant 9$ and $z \geqslant 0$. Sketch the region and employ a suitable transformation to evaluate the integral

$$\int_D (x^2 + y^2) \; dx \; dy \; dz\,.$$

10C Vector Calculus

Consider the bounded surface S that is the union of $x^2 + y^2 = 4$ for $-2 \leqslant z \leqslant 2$ and $(4-z)^2 = x^2 + y^2$ for $2 \leqslant z \leqslant 4$. Sketch the surface.

Using suitable parametrisations for the two parts of S, calculate the integral

$$\int_S (\nabla \times \mathbf{F}\,) \cdot d\mathbf{S}$$

for $\mathbf{F} = yz^2\mathbf{i}$.

Check your result using Stokes's Theorem.

11C Vector Calculus

If $\mathbf{E}$ and $\mathbf{B}$ are vectors in $\mathbb{R}^3$, show that

$$T_{ij} = E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}\left(E_k E_k + B_k B_k\right)$$

is a second rank tensor.

Now assume that $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ obey Maxwell's equations, which in suitable units read

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\,,
\end{aligned}$$

where ρ is the charge density and $\mathbf{J}$ the current density. Show that

$$\frac{\partial}{\partial t}\left(\mathbf{E} \times \mathbf{B}\right) = \mathbf{M} - \rho\mathbf{E} - \mathbf{J} \times \mathbf{B} \quad \text{where} \quad M_i = \frac{\partial T_{ij}}{\partial x_j}.$$

12C Vector Calculus

(a) Prove that

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}\,.$$

(b) State the divergence theorem for a vector field $\mathbf{F}$ in a closed region $\Omega \subset \mathbb{R}^3$ bounded by $\partial\Omega$.

For a smooth vector field $\mathbf{F}$ and a smooth scalar function g prove that

$$\int_{\Omega} \mathbf{F} \cdot \nabla g + g \nabla \cdot \mathbf{F} \; dV = \int_{\partial\Omega} g\mathbf{F} \cdot \mathbf{n} \; dS\,,$$

where $\mathbf{n}$ is the outward unit normal on the surface $\partial\Omega$.

Use this identity to prove that the solution u to the Laplace equation $\nabla^2 u = 0$ in Ω with $u = f$ on $\partial\Omega$ is unique, provided it exists.

END OF PAPER