UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday, 31 May, 2013 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Differential Equations

Solve the equation

$$\ddot{y} - \dot{y} - 2y = 3e^{2t} + 3e^{-t} + 3 + 6t$$

subject to the conditions $y = \dot{y} = 0$ at $t = 0$.

2A Differential Equations

Use the transformation $z = \ln x$ to solve

$$\ddot{z} = -\dot{z}^2 - 1 - e^{-z}$$

subject to the conditions $z = 0$ and $\dot{z} = V$ at $t = 0$, where V is a positive constant.

Show that when $\dot{z}(t) = 0$

$$z = \ln\left(\sqrt{V^2 + 4} - 1\right).$$

3F Probability

Let X be a random variable with mean μ and variance σ^2. Let

$$G(a) = \mathbb{E}\left[(X - a)^2\right].$$

Show that $G(a) \geqslant \sigma^2$ for all a. For what value of a is there equality?

Let

$$H(a) = \mathbb{E}\big[|X - a|\big].$$

Supposing that X has probability density function f, express $H(a)$ in terms of f. Show that H is minimised when a is such that $\int_{-\infty}^{a} f(x)dx = 1/2$.

4F Probability

(i) Let X be a random variable. Use Markov's inequality to show that

$$\mathbb{P}(X \geqslant k) \leqslant \mathbb{E}(e^{tX})e^{-kt}$$

for all $t \geqslant 0$ and real k.

(ii) Calculate $\mathbb{E}(e^{tX})$ in the case where X is a Poisson random variable with parameter $\lambda = 1$. Using the inequality from part (i) with a suitable choice of t, prove that

$$\frac{1}{k!} + \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \ldots \leqslant \left(\frac{e}{k}\right)^k$$

for all $k \geqslant 1$.

SECTION II

5A Differential Equations

The function $y(x)$ satisfies the equation

$$y'' + p(x)y' + q(x)y = 0\,.$$

Give the definitions of the terms *ordinary point*, *singular point*, and *regular singular point* for this equation.

For the equation

$$xy'' + y = 0$$

classify the point $x = 0$ according to your definitions. Find the series solution about $x = 0$ which satisfies

$$y = 0 \quad \text{and} \quad y' = 1 \quad \text{at } x = 0.$$

For a second solution with $y = 1$ at $x = 0$, consider an expansion

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \ldots\,,$$

where $y_0 = 1$ and $xy''_{n+1} = -y_n$. Find y_1 and y_2 which have $y_n(0) = 0$ and $y'_n(1) = 0$. Comment on y' near $x = 0$ for this second solution.

6A Differential Equations

Consider the function

$$f(x, y) = (x^2 - y^4)(1 - x^2 - y^4).$$

Determine the type of each of the nine critical points.

Sketch contours of constant $f(x, y)$.

7A Differential Equations

Find $x(t)$ and $y(t)$ which satisfy

$$\begin{aligned} 3\dot{x} + \dot{y} \quad + \quad 5x - y &= 2e^{-t} + 4e^{-3t}, \\ \dot{x} + 4\dot{y} \quad - \quad 2x + 7y &= -3e^{-t} + 5e^{-3t}, \end{aligned}$$

subject to $x = y = 0$ at $t = 0$.

8A Differential Equations

Medical equipment is sterilised by placing it in a hot oven for a time T and then removing it and letting it cool for the same time. The equipment at temperature $\theta(t)$ warms and cools at a rate equal to the product of a constant α and the difference between its temperature and its surroundings, θ_1 when warming in the oven and θ_0 when cooling outside. The equipment starts the sterilisation process at temperature θ_0.

Bacteria are killed by the heat treatment. Their number $N(t)$ decreases at a rate equal to the product of the current number and a destruction factor β. This destruction factor varies linearly with temperature, vanishing at θ_0 and having a maximum $\beta_{\max}$ at θ_1.

Find an implicit equation for T such that the number of bacteria is reduced by a factor of 10^{-20} by the sterilisation process.

A second hardier species of bacteria requires the oven temperature to be increased to achieve the same destruction factor $\beta_{\max}$. How is the sterilisation time T affected?

9F Probability

Let Z be an exponential random variable with parameter $\lambda = 1$. Show that

$$\mathbb{P}(Z > s + t \mid Z > s) = \mathbb{P}(Z > t)$$

for any $s, t \geqslant 0$.

Let $Z_{\text{int}} = \lfloor Z \rfloor$ be the greatest integer less than or equal to Z. What is the probability mass function of Z_{int}? Show that $\mathbb{E}(Z_{\text{int}}) = \frac{1}{e-1}$.

Let $Z_{\text{frac}} = Z - Z_{\text{int}}$ be the fractional part of Z. What is the density of Z_{frac}?

Show that Z_{int} and Z_{frac} are independent.

10F Probability

Let X be a random variable taking values in the non-negative integers, and let G be the probability generating function of X. Assuming G is everywhere finite, show that

$$G'(1) = \mu \text{ and } G''(1) = \sigma^2 + \mu^2 - \mu$$

where μ is the mean of X and σ^2 is its variance. [You may interchange differentiation and expectation without justification.]

Consider a branching process where individuals produce independent random numbers of offspring with the same distribution as X. Let X_n be the number of individuals in the n-th generation, and let G_n be the probability generating function of X_n. Explain carefully why

$$G_{n+1}(t) = G_n(G(t))$$

Assuming $X_0 = 1$, compute the mean of X_n. Show that

$$\mathrm{Var}(X_n) = \sigma^2 \frac{\mu^{n-1}(\mu^n - 1)}{\mu - 1}.$$

Suppose $\mathbb{P}(X = 0) = 3/7$ and $\mathbb{P}(X = 3) = 4/7$. Compute the probability that the population will eventually become extinct. You may use standard results on branching processes as long as they are clearly stated.

11F Probability

Let X be a geometric random variable with $\mathbb{P}(X = 1) = p$. Derive formulae for $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ in terms of p.

A jar contains n balls. Initially, all of the balls are red. Every minute, a ball is drawn at random from the jar, and then replaced with a green ball. Let T be the number of minutes until the jar contains only green balls. Show that the expected value of T is $n \sum_{i=1}^{n} 1/i$. What is the variance of T?

12F Probability

Let Ω be the sample space of a probabilistic experiment, and suppose that the sets $B_1, B_2, \ldots, B_k$ are a partition of Ω into events of positive probability. Show that

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\sum_{j=1}^{k} \mathbb{P}(A|B_j)\mathbb{P}(B_j)}$$

for any event A of positive probability.

A drawer contains two coins. One is an unbiased coin, which when tossed, is equally likely to turn up heads or tails. The other is a biased coin, which will turn up heads with probability p and tails with probability $1-p$. One coin is selected (uniformly) at random from the drawer. Two experiments are performed:

(a) The selected coin is tossed n times. Given that the coin turns up heads k times and tails $n-k$ times, what is the probability that the coin is biased?

(b) The selected coin is tossed repeatedly until it turns up heads k times. Given that the coin is tossed n times in total, what is the probability that the coin is biased?

END OF PAPER