UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 30 May, 2013 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Vectors and Matrices

(a) State de Moivre's theorem and use it to derive a formula for the roots of order n of a complex number $z = a+ib$. Using this formula compute the cube roots of $z = -8$.

(b) Consider the equation $|z+3i| = 3|z|$ for $z \in \mathbb{C}$. Give a geometric description of the set S of solutions and sketch S as a subset of the complex plane.

2A Vectors and Matrices

Let A be a real 3×3 matrix.

(i) For $B = R_1 A$ with

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

find an angle θ_1 so that the element $b_{31} = 0$, where b_{ij} denotes the ij^{th} entry of the matrix B.

(ii) For $C = R_2 B$ with $b_{31} = 0$ and

$$R_2 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

show that $c_{31} = 0$ and find an angle θ_2 so that $c_{21} = 0$.

(iii) For $D = R_3 C$ with $c_{31} = c_{21} = 0$ and

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{pmatrix}$$

show that $d_{31} = d_{21} = 0$ and find an angle θ_3 so that $d_{32} = 0$.

(iv) Deduce that any real 3×3 matrix can be written as a product of an orthogonal matrix and an upper triangular matrix.

3D Analysis I

Show that $\exp(x) \geqslant 1 + x$ for $x \geqslant 0$.

Let (a_j) be a sequence of positive real numbers. Show that for every n,

$$\sum_1^n a_j \leqslant \prod_1^n (1 + a_j) \leqslant \exp\left(\sum_1^n a_j\right).$$

Deduce that $\prod_1^n (1 + a_j)$ tends to a limit as $n \to \infty$ if and only if $\sum_1^n a_j$ does.

4F Analysis I

(a) Suppose $b_n \geqslant b_{n+1} \geqslant 0$ for $n \geqslant 1$ and $b_n \to 0$. Show that $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

(b) Does the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converge or diverge? Explain your answer.

SECTION II

5C Vectors and Matrices

Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in $\mathbb{R}^n$. What is meant by saying that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent? What is the dimension of the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$ and $\mathbf{y}$ if they are (1) linearly independent, (2) linearly dependent?

Define the scalar product $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Define the corresponding norm $\|\mathbf{x}\|$ of $\mathbf{x} \in \mathbb{R}^n$. State and prove the Cauchy-Schwarz inequality, and deduce the triangle inequality. Under what condition does equality hold in the Cauchy-Schwarz inequality?

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be unit vectors in $\mathbb{R}^3$. Let

$$S = \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{z} + \mathbf{z} \cdot \mathbf{x}\,.$$

Show that for any fixed, linearly independent vectors $\mathbf{x}$ and $\mathbf{y}$, the minimum of S over $\mathbf{z}$ is attained when $\mathbf{z} = \lambda(\mathbf{x} + \mathbf{y})$ for some $\lambda \in \mathbb{R}$, and that for this value of λ we have

(i) $\lambda \leqslant -\frac{1}{2}$ (for any choice of $\mathbf{x}$ and $\mathbf{y}$);

(ii) $\lambda = -1$ and $S = -\frac{3}{2}$ in the case where $\mathbf{x} \cdot \mathbf{y} = \cos\frac{2\pi}{3}$.

6A Vectors and Matrices

Define the kernel and the image of a linear map α from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ be a basis of $\mathbb{R}^m$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ a basis of $\mathbb{R}^n$. Explain how to represent α by a matrix A relative to the given bases.

A second set of bases $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_m\}$ and $\{\mathbf{f}'_1, \mathbf{f}'_2, \ldots, \mathbf{f}'_n\}$ is now used to represent α by a matrix A'. Relate the elements of A' to the elements of A.

Let β be a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by

$$\beta\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \qquad \beta\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}6\\4\\2\end{pmatrix}.$$

Either find one or more $\mathbf{x}$ in $\mathbb{R}^2$ such that

$$\beta\mathbf{x} = \begin{pmatrix}1\\-2\\1\end{pmatrix},$$

or explain why one cannot be found.

Let γ be a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by

$$\gamma\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}, \qquad \gamma\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}, \qquad \gamma\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

Find the kernel of γ.

7B Vectors and Matrices

(a) Let $\lambda_1, \ldots, \lambda_d$ be distinct eigenvalues of an $n \times n$ matrix A, with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$. Prove that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ is linearly independent.

(b) Consider the quadric surface Q in $\mathbb{R}^3$ defined by

$$2x^2 - 4xy + 5y^2 - z^2 + 6\sqrt{5}y = 0.$$

Find the position of the origin $\tilde{O}$ and orthonormal coordinate basis vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2$ and $\tilde{\mathbf{e}}_3$, for a coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ in which Q takes the form

$$\alpha\tilde{x}^2 + \beta\tilde{y}^2 + \gamma\tilde{z}^2 = 1.$$

Also determine the values of α, β and γ, and describe the surface geometrically.

8B Vectors and Matrices

(a) Let A and A' be the matrices of a linear map L on $\mathbb{C}^2$ relative to bases $\mathcal{B}$ and $\mathcal{B}'$ respectively. In this question you may assume without proof that A and A' are similar.

(i) State how the matrix A of L relative to the basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ is constructed from L and $\mathcal{B}$. Also state how A may be used to compute $L\mathbf{v}$ for any $\mathbf{v} \in \mathbb{C}^2$.

(ii) Show that A and A' have the same characteristic equation.

(iii) Show that for any $k \neq 0$ the matrices

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & c/k \\ bk & d \end{pmatrix}$$

are similar. [*Hint: if* $\{\mathbf{e}_1, \mathbf{e}_2\}$ *is a basis then so is* $\{k\mathbf{e}_1, \mathbf{e}_2\}$.]

(b) Using the results of (a), or otherwise, prove that any 2×2 complex matrix M with equal eigenvalues is similar to one of

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \quad \text{with } a \in \mathbb{C}.$$

(c) Consider the matrix

$$B(r) = \frac{1}{2}\begin{pmatrix} 1+r & 1-r & 1 \\ 1-r & 1+r & -1 \\ -1 & 1 & 2r \end{pmatrix}.$$

Show that there is a real value $r_0 > 0$ such that $B(r_0)$ is an orthogonal matrix. Show that $B(r_0)$ is a rotation and find the axis and angle of the rotation.

9D Analysis I

(a) Determine the radius of convergence of each of the following power series:

$$\sum_{n\geqslant 1} \frac{x^n}{n!}, \qquad \sum_{n\geqslant 1} n!x^n, \qquad \sum_{n\geqslant 1} (n!)^2 x^{n^2}.$$

(b) State Taylor's theorem.

Show that

$$(1+x)^{1/2} = 1 + \sum_{n\geqslant 1} c_n x^n,$$

for all $x \in (0,1)$, where

$$c_n = \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}.$$

10E Analysis I

(a) Let $f\colon [a,b] \to \mathbb{R}$. Suppose that for every sequence (x_n) in $[a,b]$ with limit $y \in [a,b]$, the sequence $(f(x_n))$ converges to $f(y)$. Show that f is continuous at y.

(b) State the Intermediate Value Theorem.

Let $f\colon [a,b] \to \mathbb{R}$ be a function with $f(a) = c < f(b) = d$. We say f is *injective* if for all x, $y \in [a,b]$ with $x \neq y$, we have $f(x) \neq f(y)$. We say f is *strictly increasing* if for all x, y with $x < y$, we have $f(x) < f(y)$.

(i) Suppose f is strictly increasing. Show that it is injective, and that if $f(x) < f(y)$ then $x < y$.

(ii) Suppose f is continuous and injective. Show that if $a < x < b$ then $c < f(x) < d$. Deduce that f is strictly increasing.

(iii) Suppose f is strictly increasing, and that for every $y \in [c,d]$ there exists $x \in [a,b]$ with $f(x) = y$. Show that f is continuous at b. Deduce that f is continuous on $[a,b]$.

11E Analysis I

(i) State (without proof) Rolle's Theorem.

(ii) State and prove the Mean Value Theorem.

(iii) Let $f,\, g\colon [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$. Show that there exists $\xi \in (a,b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Deduce that if moreover $f(a) = g(a) = 0$, and the limit

$$\ell = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

exists, then

$$\frac{f(x)}{g(x)} \to \ell \quad \text{as } x \to a.$$

(iv) Deduce that if $f\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable then for any $a \in \mathbb{R}$

$$f''(a) = \lim_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2}.$$

12F Analysis I

Fix a closed interval $[a,b]$. For a bounded function f on $[a,b]$ and a dissection $\mathcal{D}$ of $[a,b]$, how are the lower sum $s(f,\mathcal{D})$ and upper sum $S(f,\mathcal{D})$ defined? Show that $s(f,\mathcal{D}) \leqslant S(f,\mathcal{D})$.

Suppose $\mathcal{D}'$ is a dissection of $[a,b]$ such that $\mathcal{D} \subseteq \mathcal{D}'$. Show that

$$s(f,\mathcal{D}) \leqslant s(f,\mathcal{D}') \text{ and } S(f,\mathcal{D}') \leqslant S(f,\mathcal{D})\,.$$

By using the above inequalities or otherwise, show that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two dissections of $[a,b]$ then

$$s(f,\mathcal{D}_1) \leqslant S(f,\mathcal{D}_2)\,.$$

For a function f and dissection $\mathcal{D} = \{x_0,\ldots,x_n\}$ let

$$p(f,\mathcal{D}) = \prod_{k=1}^{n}\left[1 + (x_k - x_{k-1}) \inf_{x\in[x_{k-1},x_k]} f(x)\right]\,.$$

If f is non-negative and Riemann integrable, show that

$$p(f,\mathcal{D}) \leqslant e^{\int_a^b f(x)dx}\,.$$

[You may use without proof the inequality $e^t \geqslant t+1$ for all t.]

END OF PAPER