UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2013

List of Courses

Analysis I

Differential Equations

Dynamics and Relativity

Groups

Numbers and Sets

Probability

Vector Calculus

Vectors and Matrices

Paper 1, Section I
3D Analysis I

Show that $\exp(x) \geqslant 1 + x$ for $x \geqslant 0$.

Let (a_j) be a sequence of positive real numbers. Show that for every n,

$$\sum_1^n a_j \leqslant \prod_1^n (1 + a_j) \leqslant \exp\left(\sum_1^n a_j\right).$$

Deduce that $\prod_1^n (1 + a_j)$ tends to a limit as $n \to \infty$ if and only if $\sum_1^n a_j$ does.

Paper 1, Section I
4F Analysis I

(a) Suppose $b_n \geqslant b_{n+1} \geqslant 0$ for $n \geqslant 1$ and $b_n \to 0$. Show that $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

(b) Does the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converge or diverge? Explain your answer.

Paper 1, Section II
9D Analysis I

(a) Determine the radius of convergence of each of the following power series:

$$\sum_{n \geqslant 1} \frac{x^n}{n!}, \qquad \sum_{n \geqslant 1} n! x^n, \qquad \sum_{n \geqslant 1} (n!)^2 x^{n^2}.$$

(b) State Taylor's theorem.

Show that

$$(1 + x)^{1/2} = 1 + \sum_{n \geqslant 1} c_n x^n,$$

for all $x \in (0, 1)$, where

$$c_n = \frac{\frac{1}{2}(\frac{1}{2} - 1) \ldots (\frac{1}{2} - n + 1)}{n!}.$$

Paper 1, Section II

10E Analysis I

(a) Let $f\colon [a,b] \to \mathbb{R}$. Suppose that for every sequence (x_n) in $[a,b]$ with limit $y \in [a,b]$, the sequence $(f(x_n))$ converges to $f(y)$. Show that f is continuous at y.

(b) State the Intermediate Value Theorem.

Let $f\colon [a,b] \to \mathbb{R}$ be a function with $f(a) = c < f(b) = d$. We say f is *injective* if for all x, $y \in [a,b]$ with $x \neq y$, we have $f(x) \neq f(y)$. We say f is *strictly increasing* if for all x, y with $x < y$, we have $f(x) < f(y)$.

(i) Suppose f is strictly increasing. Show that it is injective, and that if $f(x) < f(y)$ then $x < y$.

(ii) Suppose f is continuous and injective. Show that if $a < x < b$ then $c < f(x) < d$. Deduce that f is strictly increasing.

(iii) Suppose f is strictly increasing, and that for every $y \in [c,d]$ there exists $x \in [a,b]$ with $f(x) = y$. Show that f is continuous at b. Deduce that f is continuous on $[a,b]$.

Paper 1, Section II
11E Analysis I

(i) State (without proof) Rolle's Theorem.

(ii) State and prove the Mean Value Theorem.

(iii) Let $f,\, g\colon [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$. Show that there exists $\xi \in (a,b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Deduce that if moreover $f(a) = g(a) = 0$, and the limit

$$\ell = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists, then

$$\frac{f(x)}{g(x)} \to \ell \quad \text{as } x \to a.$$

(iv) Deduce that if $f\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable then for any $a \in \mathbb{R}$

$$f''(a) = \lim_{h \to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2}.$$

Paper 1, Section II

12F Analysis I

Fix a closed interval $[a,b]$. For a bounded function f on $[a,b]$ and a dissection $\mathcal{D}$ of $[a,b]$, how are the lower sum $s(f,\mathcal{D})$ and upper sum $S(f,\mathcal{D})$ defined? Show that $s(f,\mathcal{D}) \leqslant S(f,\mathcal{D})$.

Suppose $\mathcal{D}'$ is a dissection of $[a,b]$ such that $\mathcal{D} \subseteq \mathcal{D}'$. Show that

$$s(f,\mathcal{D}) \leqslant s(f,\mathcal{D}') \text{ and } S(f,\mathcal{D}') \leqslant S(f,\mathcal{D})\,.$$

By using the above inequalities or otherwise, show that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two dissections of $[a,b]$ then

$$s(f,\mathcal{D}_1) \leqslant S(f,\mathcal{D}_2)\,.$$

For a function f and dissection $\mathcal{D} = \{x_0, \ldots, x_n\}$ let

$$p(f,\mathcal{D}) = \prod_{k=1}^{n} \left[1 + (x_k - x_{k-1}) \inf_{x \in [x_{k-1}, x_k]} f(x) \right] .$$

If f is non-negative and Riemann integrable, show that

$$p(f,\mathcal{D}) \leqslant e^{\int_a^b f(x)dx}\,.$$

[You may use without proof the inequality $e^t \geqslant t+1$ for all t.]

Paper 2, Section I

1A Differential Equations

Solve the equation

$$\ddot{y} - \dot{y} - 2y = 3e^{2t} + 3e^{-t} + 3 + 6t$$

subject to the conditions $y = \dot{y} = 0$ at $t = 0$.

Paper 2, Section I

2A Differential Equations

Use the transformation $z = \ln x$ to solve

$$\ddot{z} = -\dot{z}^2 - 1 - e^{-z}$$

subject to the conditions $z = 0$ and $\dot{z} = V$ at $t = 0$, where V is a positive constant.

Show that when $\dot{z}(t) = 0$

$$z = \ln\left(\sqrt{V^2+4} - 1\right) .$$

Paper 2, Section II

5A Differential Equations

The function $y(x)$ satisfies the equation

$$y'' + p(x)y' + q(x)y = 0 .$$

Give the definitions of the terms *ordinary point*, *singular point*, and *regular singular point* for this equation.

For the equation

$$xy'' + y = 0$$

classify the point $x = 0$ according to your definitions. Find the series solution about $x = 0$ which satisfies

$$y = 0 \quad \text{and} \quad y' = 1 \quad \text{at } x = 0.$$

For a second solution with $y = 1$ at $x = 0$, consider an expansion

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \dots ,$$

where $y_0 = 1$ and $xy''_{n+1} = -y_n$. Find y_1 and y_2 which have $y_n(0) = 0$ and $y'_n(1) = 0$. Comment on y' near $x = 0$ for this second solution.

Paper 2, Section II
6A Differential Equations
Consider the function

$$f(x,y) = (x^2 - y^4)(1 - x^2 - y^4).$$

Determine the type of each of the nine critical points.

Sketch contours of constant $f(x,y)$.

Paper 2, Section II
7A Differential Equations
Find $x(t)$ and $y(t)$ which satisfy

$$\begin{aligned} 3\dot{x} + \dot{y} \quad + \quad 5x - y &= 2e^{-t} + 4e^{-3t}, \\ \dot{x} + 4\dot{y} \quad - \quad 2x + 7y &= -3e^{-t} + 5e^{-3t}, \end{aligned}$$

subject to $x = y = 0$ at $t = 0$.

Paper 2, Section II
8A Differential Equations
Medical equipment is sterilised by placing it in a hot oven for a time T and then removing it and letting it cool for the same time. The equipment at temperature $\theta(t)$ warms and cools at a rate equal to the product of a constant α and the difference between its temperature and its surroundings, θ_1 when warming in the oven and θ_0 when cooling outside. The equipment starts the sterilisation process at temperature θ_0.

Bacteria are killed by the heat treatment. Their number $N(t)$ decreases at a rate equal to the product of the current number and a destruction factor β. This destruction factor varies linearly with temperature, vanishing at θ_0 and having a maximum $\beta_{\max}$ at θ_1.

Find an implicit equation for T such that the number of bacteria is reduced by a factor of 10^{-20} by the sterilisation process.

A second hardier species of bacteria requires the oven temperature to be increased to achieve the same destruction factor $\beta_{\max}$. How is the sterilisation time T affected?

Paper 4, Section I

3B Dynamics and Relativity

A hot air balloon of mass M is equipped with a bag of sand of mass $m = m(t)$ which decreases in time as the sand is gradually released. In addition to gravity the balloon experiences a constant upwards buoyancy force T and we neglect air resistance effects. Show that if $v(t)$ is the upward speed of the balloon then

$$(M+m)\frac{dv}{dt} = T - (M+m)g.$$

Initially at $t = 0$ the mass of sand is $m(0) = m_0$ and the balloon is at rest in equilibrium. Subsequently the sand is released at a constant rate and is depleted in a time t_0. Show that the speed of the balloon at time t_0 is

$$gt_0\left(\left(1+\frac{M}{m_0}\right)\ln\left(1+\frac{m_0}{M}\right)-1\right).$$

[You may use without proof the indefinite integral $\int t/(A-t)\,dt = -t - A\ln|A-t| + C$.]

Paper 4, Section I

4B Dynamics and Relativity

A frame S' moves with constant velocity v along the x axis of an inertial frame S of Minkowski space. A particle P moves with constant velocity u' along the x' axis of S'. Find the velocity u of P in S.

The rapidity φ of any velocity w is defined by $\tanh\varphi = w/c$. Find a relation between the rapidities of u, u' and v.

Suppose now that P is initially at rest in S and is subsequently given n successive velocity increments of $c/2$ (each delivered in the instantaneous rest frame of the particle). Show that the resulting velocity of P in S is

$$c\left(\frac{e^{2n\alpha}-1}{e^{2n\alpha}+1}\right)$$

where $\tanh\alpha = 1/2$.

[You may use without proof the addition formulae $\sinh(a+b) = \sinh a\cosh b + \cosh a\sinh b$ and $\cosh(a+b) = \cosh a\cosh b + \sinh a\sinh b$.]

Paper 4, Section II

9B Dynamics and Relativity

(a) A particle P of unit mass moves in a plane with polar coordinates (r, θ). You may assume that the radial and angular components of the acceleration are given by $(\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\dot{\theta})$, where the dot denotes d/dt. The particle experiences a central force corresponding to a potential $V = V(r)$.

(i) Prove that $l = r^2\dot{\theta}$ is constant in time and show that the time dependence of the radial coordinate $r(t)$ is equivalent to the motion of a particle in one dimension x in a potential V_{eff} given by

$$V_{\text{eff}} = V(x) + \frac{l^2}{2x^2}.$$

(ii) Now suppose that $V(r) = -e^{-r}$. Show that if $l^2 < 27/e^3$ then two circular orbits are possible with radii $r_1 < 3$ and $r_2 > 3$. Determine whether each orbit is stable or unstable.

(b) Kepler's first and second laws for planetary motion are the following statements:
K1: the planet moves on an ellipse with a focus at the Sun;
K2: the line between the planet and the Sun sweeps out equal areas in equal times.
Show that **K2** implies that the force acting on the planet is a central force.
Show that **K2** together with **K1** implies that the force is given by the inverse square law.
[You may assume that an ellipse with a focus at the origin has polar equation $\frac{A}{r} = 1 + \varepsilon \cos\theta$ with $A > 0$ and $0 < \varepsilon < 1$.]

Paper 4, Section II
10B Dynamics and Relativity

(a) A rigid body Q is made up of N particles of masses m_i at positions $\mathbf{r}_i(t)$. Let $\mathbf{R}(t)$ denote the position of its centre of mass. Show that the total kinetic energy of Q may be decomposed into T_1, the kinetic energy of the centre of mass, plus a term T_2 representing the kinetic energy about the centre of mass.
Suppose now that Q is rotating with angular velocity $\boldsymbol{\omega}$ about its centre of mass. Define the moment of inertia I of Q (about the axis defined by $\boldsymbol{\omega}$) and derive an expression for T_2 in terms of I and $\omega = |\boldsymbol{\omega}|$.

(b) Consider a uniform rod AB of length $2l$ and mass M. Two such rods AB and BC are freely hinged together at B. The end A is attached to a fixed point O on a perfectly smooth horizontal floor and AB is able to rotate freely about O. The rods are initially at rest, lying in a vertical plane with C resting on the floor and each rod making angle α with the horizontal. The rods subsequently move under gravity in their vertical plane.
Find an expression for the angular velocity of rod AB when it makes angle θ with the floor. Determine the speed at which the hinge strikes the floor.

Paper 4, Section II
11B Dynamics and Relativity

(i) An inertial frame S has orthonormal coordinate basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. A second frame S' rotates with angular velocity $\boldsymbol{\omega}$ relative to S and has coordinate basis vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$. The motion of S' is characterised by the equations $d\mathbf{e}'_i/dt = \boldsymbol{\omega}\times\mathbf{e}'_i$ and at $t = 0$ the two coordinate frames coincide.
If a particle P has position vector $\mathbf{r}$ show that $\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega}\times\mathbf{r}$ where $\mathbf{v}$ and $\mathbf{v}'$ are the velocity vectors of P as seen by observers fixed respectively in S and S'.

(ii) For the remainder of this question you may assume that $\mathbf{a} = \mathbf{a}' + 2\,\boldsymbol{\omega}\times\mathbf{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ where $\mathbf{a}$ and $\mathbf{a}'$ are the acceleration vectors of P as seen by observers fixed respectively in S and S', and that $\boldsymbol{\omega}$ is constant.

Consider again the frames S and S' in (i). Suppose that $\boldsymbol{\omega} = \omega\,\mathbf{e}_3$ with ω constant. A particle of mass m moves under a force $\mathbf{F} = -4m\omega^2\mathbf{r}$. When viewed in S' its position and velocity at time $t = 0$ are $(x', y', z') = (1, 0, 0)$ and $(\dot{x}', \dot{y}', \dot{z}') = (0, 0, 0)$. Find the motion of the particle in the coordinates of S'. Show that for an observer fixed in S', the particle achieves its maximum speed at time $t = \pi/(4\omega)$ and determine that speed. [*Hint: you may find it useful to consider the combination* $\zeta = x' + iy'$.]

Paper 4, Section II

12B Dynamics and Relativity

(a) Let S with coordinates (ct, x, y) and S' with coordinates (ct', x', y') be inertial frames in Minkowski space with two spatial dimensions. S' moves with velocity v along the x-axis of S and they are related by the standard Lorentz transformation:

$$\begin{pmatrix} ct \\ x \\ y \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v/c & 0 \\ \gamma v/c & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix}, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

A photon is emitted at the spacetime origin. In S' it has frequency ν' and propagates at angle θ' to the x'-axis.
Write down the 4-momentum of the photon in the frame S'.
Hence or otherwise find the frequency of the photon as seen in S. Show that it propagates at angle θ to the x-axis in S, where

$$\tan\theta = \frac{\tan\theta'}{\gamma\left(1 + \dfrac{v}{c}\sec\theta'\right)}.$$

A light source in S' emits photons uniformly in all directions in the $x'y'$-plane. Show that for large v, in S half of the light is concentrated into a narrow cone whose semi-angle α is given by $\cos\alpha = v/c$.

(b) The centre-of-mass frame for a system of relativistic particles in Minkowski space is the frame in which the total relativistic 3-momentum is zero.
Two particles A_1 and A_2 of rest masses m_1 and m_2 move collinearly with uniform velocities u_1 and u_2 respectively, along the x-axis of a frame S. They collide, coalescing to form a single particle A_3.
Determine the velocity of the centre-of-mass frame of the system comprising A_1 and A_2.
Find the speed of A_3 in S and show that its rest mass m_3 is given by

$$m_3^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2\left(1 - \frac{u_1u_2}{c^2}\right),$$

where $\gamma_i = (1 - u_i^2/c^2)^{-1/2}$.

Paper 3, Section I

1D Groups

State Lagrange's Theorem.

Let G be a finite group, and H and K two subgroups of G such that

(i) the orders of H and K are coprime;

(ii) every element of G may be written as a product hk, with $h \in H$ and $k \in K$;

(iii) both H and K are normal subgroups of G.

Prove that G is isomorphic to $H \times K$.

Paper 3, Section I

2D Groups

Define what it means for a group to be *cyclic*, and for a group to be *abelian.* Show that every cyclic group is abelian, and give an example to show that the converse is false.

Show that a group homomorphism from the cyclic group C_n of order n to a group G determines, and is determined by, an element g of G such that $g^n = 1$.

Hence list all group homomorphisms from C_4 to the symmetric group S_4.

Paper 3, Section II

5D Groups

(a) Let G be a finite group. Show that there exists an injective homomorphism $G \to Sym(X)$ to a symmetric group, for some set X.

(b) Let H be the full group of symmetries of the cube, and X the set of edges of the cube.

Show that H acts transitively on X, and determine the stabiliser of an element of X. Hence determine the order of H.

Show that the action of H on X defines an injective homomorphism $H \to Sym(X)$ to the group of permutations of X, and determine the number of cosets of H in $Sym(X)$.

Is H a normal subgroup of $Sym(X)$? Prove your answer.

Paper 3, Section II

6D Groups

(a) Let p be a prime, and let $G = SL_2(p)$ be the group of 2×2 matrices of determinant 1 with entries in the field $\mathbb{F}_p$ of integers mod p.

(i) Define the action of G on $X = \mathbb{F}_p \cup \{\infty\}$ by Möbius transformations. [You need not show that it is a group action.]

State the orbit-stabiliser theorem.

Determine the orbit of ∞ and the stabiliser of ∞. Hence compute the order of $SL_2(p)$.

(ii) Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Show that A is conjugate to B in G if $p = 11$, but not if $p = 5$.

(b) Let G be the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, x \in \mathbb{R}$. Show that G is a subgroup of the group of all invertible real matrices.

Let H be the subset of G given by matrices with $a = 0$. Show that H is a normal subgroup, and that the quotient group G/H is isomorphic to $\mathbb{R}$.

Determine the centre $Z(G)$ of G, and identify the quotient group $G/Z(G)$.

Paper 3, Section II
7D Groups

(a) Let G be the dihedral group of order $4n$, the symmetry group of a regular polygon with $2n$ sides.

Determine all elements of order 2 in G. For each element of order 2, determine its conjugacy class and the smallest normal subgroup containing it.

(b) Let G be a finite group.

(i) Prove that if H and K are subgroups of G, then $K \cup H$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

(ii) Let H be a proper subgroup of G, and write $G \setminus H$ for the elements of G not in H. Let K be the subgroup of G generated by $G \setminus H$.

Show that $K = G$.

Paper 3, Section II
8D Groups

Let p be a prime number.

Prove that every group whose order is a power of p has a non-trivial centre.

Show that every group of order p^2 is abelian, and that there are precisely two of them, up to isomorphism.

Paper 4, Section I

1E Numbers and Sets

Let m and n be positive integers. State what is meant by the *greatest common divisor* $\gcd(m,n)$ of m and n, and show that there exist integers a and b such that $\gcd(m,n) = am + bn$. Deduce that an integer k divides both m and n only if k divides $\gcd(m,n)$.

Prove (without using the Fundamental Theorem of Arithmetic) that for any positive integer k, $\gcd(km,kn) = k\gcd(m,n)$.

Paper 4, Section I

2E Numbers and Sets

Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. What does it mean to say that the sequence (x_n) is convergent? What does it mean to say the series $\sum x_n$ is convergent? Show that if $\sum x_n$ is convergent, then the sequence (x_n) converges to zero. Show that the converse is not necessarily true.

Paper 4, Section II

5E Numbers and Sets

(i) What does it mean to say that a function $f\colon X \to Y$ is *injective*? What does it mean to say that f is *surjective*? Let $g\colon Y \to Z$ be a function. Show that if $g \circ f$ is injective, then so is f, and that if $g \circ f$ is surjective, then so is g.

(ii) Let X_1, X_2 be two sets. Their *product* $X_1 \times X_2$ is the set of ordered pairs (x_1, x_2) with $x_i \in X_i$ $(i = 1, 2)$. Let p_i (for $i = 1, 2$) be the function

$$p_i\colon X_1 \times X_2 \to X_i, \quad p_i(x_1,x_2) = x_i.$$

When is p_i surjective? When is p_i injective?

(iii) Now let Y be any set, and let $f_1\colon Y \to X_1$, $f_2\colon Y \to X_2$ be functions. Show that there exists a unique $g\colon Y \to X_1 \times X_2$ such that $f_1 = p_1 \circ g$ and $f_2 = p_2 \circ g$.

Show that if f_1 or f_2 is injective, then g is injective. Is the converse true? Justify your answer.

Show that if g is surjective then both f_1 and f_2 are surjective. Is the converse true? Justify your answer.

Paper 4, Section II
6E Numbers and Sets

(i) Let N and r be integers with $N \geqslant 0$, $r \geqslant 1$. Let S be the set of $(r+1)$-tuples $(n_0, n_1, \ldots, n_r)$ of non-negative integers satisfying the equation $n_0 + \cdots + n_r = N$. By mapping elements of S to suitable subsets of $\{1, \ldots, N+r\}$ of size r, or otherwise, show that the number of elements of S equals

$$\binom{N+r}{r}.$$

(ii) State the Inclusion–Exclusion principle.

(iii) Let $a_0, \ldots, a_r$ be positive integers. Show that the number of $(r+1)$-tuples (n_i) of integers satisfying

$$n_0 + \cdots + n_r = N, \quad 0 \leqslant n_i < a_i \text{ for all } i$$

is

$$\binom{N+r}{r} - \sum_{0 \leqslant i \leqslant r} \binom{N+r-a_i}{r} + \sum_{0 \leqslant i < j \leqslant r} \binom{N+r-a_i-a_j}{r}$$
$$- \sum_{0 \leqslant i < j < k \leqslant r} \binom{N+r-a_i-a_j-a_k}{r} + \cdots$$

where the binomial coefficient $\binom{m}{r}$ is defined to be zero if $m < r$.

Paper 4, Section II
7E Numbers and Sets

(i) What does it mean to say that a set is *countable*? Show directly from your definition that any subset of a countable set is countable, and that a countable union of countable sets is countable.

(ii) Let X be either $\mathbb{Z}$ or $\mathbb{Q}$. A function $f\colon X \to \mathbb{Z}$ is said to be *periodic* if there exists a positive integer n such that for every $x \in X$, $f(x+n) = f(x)$. Show that the set of periodic functions from $\mathbb{Z}$ to itself is countable. Is the set of periodic functions $f\colon \mathbb{Q} \to \mathbb{Z}$ countable? Justify your answer.

(iii) Show that $\mathbb{R}^2$ is not the union of a countable collection of lines.

[You may assume that $\mathbb{R}$ and the power set of $\mathbb{N}$ are uncountable.]

Paper 4, Section II

8E Numbers and Sets

Let p be a prime number, and x, n integers with $n \geqslant 1$.

(i) Prove Fermat's Little Theorem: for any integer x, $x^p \equiv x \pmod{p}$.

(ii) Show that if y is an integer such that $x \equiv y \pmod{p^n}$, then for every integer $r \geqslant 0$,

$$x^{p^r} \equiv y^{p^r} \pmod{p^{n+r}}.$$

Deduce that $x^{p^n} \equiv x^{p^{n-1}} \pmod{p^n}$.

(iii) Show that there exists a unique integer $y \in \{0, 1, \ldots, p^n - 1\}$ such that

$$y \equiv x \pmod{p} \quad \text{and} \quad y^p \equiv y \pmod{p^n}.$$

Paper 2, Section I
3F Probability

Let X be a random variable with mean μ and variance σ^2. Let

$$G(a) = \mathbb{E}\big[(X-a)^2\big].$$

Show that $G(a) \geqslant \sigma^2$ for all a. For what value of a is there equality?

Let

$$H(a) = \mathbb{E}\big[|X-a|\big].$$

Supposing that X has probability density function f, express $H(a)$ in terms of f. Show that H is minimised when a is such that $\int_{-\infty}^{a} f(x)dx = 1/2$.

Paper 2, Section I
4F Probability

(i) Let X be a random variable. Use Markov's inequality to show that

$$\mathbb{P}(X \geqslant k) \leqslant \mathbb{E}(e^{tX})e^{-kt}$$

for all $t \geqslant 0$ and real k.

(ii) Calculate $\mathbb{E}(e^{tX})$ in the case where X is a Poisson random variable with parameter $\lambda = 1$. Using the inequality from part (i) with a suitable choice of t, prove that

$$\frac{1}{k!} + \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \ldots \leqslant \left(\frac{e}{k}\right)^k$$

for all $k \geqslant 1$.

Paper 2, Section II
9F Probability

Let Z be an exponential random variable with parameter $\lambda = 1$. Show that

$$\mathbb{P}(Z > s + t \mid Z > s) = \mathbb{P}(Z > t)$$

for any $s, t \geqslant 0$.

Let $Z_{\text{int}} = \lfloor Z \rfloor$ be the greatest integer less than or equal to Z. What is the probability mass function of Z_{int}? Show that $\mathbb{E}(Z_{\text{int}}) = \frac{1}{e-1}$.

Let $Z_{\text{frac}} = Z - Z_{\text{int}}$ be the fractional part of Z. What is the density of Z_{frac}?

Show that Z_{int} and Z_{frac} are independent.

Paper 2, Section II
10F Probability

Let X be a random variable taking values in the non-negative integers, and let G be the probability generating function of X. Assuming G is everywhere finite, show that

$$G'(1) = \mu \text{ and } G''(1) = \sigma^2 + \mu^2 - \mu$$

where μ is the mean of X and σ^2 is its variance. [You may interchange differentiation and expectation without justification.]

Consider a branching process where individuals produce independent random numbers of offspring with the same distribution as X. Let X_n be the number of individuals in the n-th generation, and let G_n be the probability generating function of X_n. Explain carefully why

$$G_{n+1}(t) = G_n(G(t))$$

Assuming $X_0 = 1$, compute the mean of X_n. Show that

$$\mathrm{Var}(X_n) = \sigma^2 \frac{\mu^{n-1}(\mu^n - 1)}{\mu - 1}.$$

Suppose $\mathbb{P}(X = 0) = 3/7$ and $\mathbb{P}(X = 3) = 4/7$. Compute the probability that the population will eventually become extinct. You may use standard results on branching processes as long as they are clearly stated.

Paper 2, Section II
11F Probability

Let X be a geometric random variable with $\mathbb{P}(X = 1) = p$. Derive formulae for $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ in terms of p.

A jar contains n balls. Initially, all of the balls are red. Every minute, a ball is drawn at random from the jar, and then replaced with a green ball. Let T be the number of minutes until the jar contains only green balls. Show that the expected value of T is $n\sum_{i=1}^{n} 1/i$. What is the variance of T?

Paper 2, Section II
12F Probability

Let Ω be the sample space of a probabilistic experiment, and suppose that the sets $B_1, B_2, \ldots, B_k$ are a partition of Ω into events of positive probability. Show that

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\sum_{j=1}^{k} \mathbb{P}(A|B_j)\mathbb{P}(B_j)}$$

for any event A of positive probability.

A drawer contains two coins. One is an unbiased coin, which when tossed, is equally likely to turn up heads or tails. The other is a biased coin, which will turn up heads with probability p and tails with probability $1-p$. One coin is selected (uniformly) at random from the drawer. Two experiments are performed:

(a) The selected coin is tossed n times. Given that the coin turns up heads k times and tails $n-k$ times, what is the probability that the coin is biased?

(b) The selected coin is tossed repeatedly until it turns up heads k times. Given that the coin is tossed n times in total, what is the probability that the coin is biased?

Paper 3, Section I

3C Vector Calculus

The curve C is given by

$$\mathbf{r}(t) = \left(\sqrt{2}e^t,\, -e^t \sin t,\, e^t \cos t\right), \quad -\infty < t < \infty\,.$$

(i) Compute the arc length of C between the points with $t = 0$ and $t = 1$.

(ii) Derive an expression for the curvature of C as a function of arc length s measured from the point with $t = 0$.

Paper 3, Section I

4C Vector Calculus

State a necessary and sufficient condition for a vector field $\mathbf{F}$ on $\mathbb{R}^3$ to be conservative.

Check that the field

$$\mathbf{F} = (2x\cos y - 2z^3,\, 3 + 2ye^z - x^2 \sin y,\, y^2 e^z - 6xz^2)$$

is conservative and find a scalar potential for $\mathbf{F}$.

Paper 3, Section II

9C Vector Calculus

Give an explicit formula for $\mathcal{J}$ which makes the following result hold:

$$\int_D f \; dx\; dy\; dz = \int_{D'} \phi \; |\mathcal{J}| \; du\; dv\; dw\,,$$

where the region D, with coordinates x, y, z, and the region D', with coordinates u, v, w, are in one-to-one correspondence, and

$$\phi(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w))\,.$$

Explain, in outline, why this result holds.

Let D be the region in $\mathbb{R}^3$ defined by $4 \leqslant x^2 + y^2 + z^2 \leqslant 9$ and $z \geqslant 0$. Sketch the region and employ a suitable transformation to evaluate the integral

$$\int_D (x^2 + y^2)\; dx\; dy\; dz\,.$$

Paper 3, Section II

10C Vector Calculus

Consider the bounded surface S that is the union of $x^2 + y^2 = 4$ for $-2 \leqslant z \leqslant 2$ and $(4-z)^2 = x^2 + y^2$ for $2 \leqslant z \leqslant 4$. Sketch the surface.

Using suitable parametrisations for the two parts of S, calculate the integral

$$\int_S (\nabla \times \mathbf{F}\ ) \cdot d\mathbf{S}$$

for $\mathbf{F} = yz^2\mathbf{i}$.

Check your result using Stokes's Theorem.

Paper 3, Section II

11C Vector Calculus

If $\mathbf{E}$ and $\mathbf{B}$ are vectors in $\mathbb{R}^3$, show that

$$T_{ij} = E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(E_kE_k + B_kB_k\right)$$

is a second rank tensor.

Now assume that $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ obey Maxwell's equations, which in suitable units read

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\,,
\end{aligned}$$

where ρ is the charge density and $\mathbf{J}$ the current density. Show that

$$\frac{\partial}{\partial t}\left(\mathbf{E} \times \mathbf{B}\right) = \mathbf{M} - \rho\mathbf{E} - \mathbf{J} \times \mathbf{B} \quad \text{where} \quad M_i = \frac{\partial T_{ij}}{\partial x_j}.$$

Paper 3, Section II

12C Vector Calculus

(a) Prove that

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} \,.$$

(b) State the divergence theorem for a vector field $\mathbf{F}$ in a closed region $\Omega \subset \mathbb{R}^3$ bounded by $\partial\Omega$.

For a smooth vector field $\mathbf{F}$ and a smooth scalar function g prove that

$$\int_{\Omega} \mathbf{F} \cdot \nabla g + g \nabla \cdot \mathbf{F} \; dV = \int_{\partial\Omega} g\mathbf{F} \cdot \mathbf{n} \; dS \,,$$

where $\mathbf{n}$ is the outward unit normal on the surface $\partial\Omega$.

Use this identity to prove that the solution u to the Laplace equation $\nabla^2 u = 0$ in Ω with $u = f$ on $\partial\Omega$ is unique, provided it exists.

Paper 1, Section I

1C Vectors and Matrices

(a) State de Moivre's theorem and use it to derive a formula for the roots of order n of a complex number $z = a + ib$. Using this formula compute the cube roots of $z = -8$.

(b) Consider the equation $|z + 3i| = 3|z|$ for $z \in \mathbb{C}$. Give a geometric description of the set S of solutions and sketch S as a subset of the complex plane.

Paper 1, Section I

2A Vectors and Matrices

Let A be a real 3×3 matrix.

(i) For $B = R_1 A$ with

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

find an angle θ_1 so that the element $b_{31} = 0$, where b_{ij} denotes the ij^{th} entry of the matrix B.

(ii) For $C = R_2 B$ with $b_{31} = 0$ and

$$R_2 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

show that $c_{31} = 0$ and find an angle θ_2 so that $c_{21} = 0$.

(iii) For $D = R_3 C$ with $c_{31} = c_{21} = 0$ and

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{pmatrix}$$

show that $d_{31} = d_{21} = 0$ and find an angle θ_3 so that $d_{32} = 0$.

(iv) Deduce that any real 3×3 matrix can be written as a product of an orthogonal matrix and an upper triangular matrix.

Paper 1, Section II

5C Vectors and Matrices

Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in $\mathbb{R}^n$. What is meant by saying that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent? What is the dimension of the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$ and $\mathbf{y}$ if they are (1) linearly independent, (2) linearly dependent?

Define the scalar product $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Define the corresponding norm $\|\mathbf{x}\|$ of $\mathbf{x} \in \mathbb{R}^n$. State and prove the Cauchy-Schwarz inequality, and deduce the triangle inequality. Under what condition does equality hold in the Cauchy-Schwarz inequality?

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be unit vectors in $\mathbb{R}^3$. Let

$$S = \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{z} + \mathbf{z} \cdot \mathbf{x}\,.$$

Show that for any fixed, linearly independent vectors $\mathbf{x}$ and $\mathbf{y}$, the minimum of S over $\mathbf{z}$ is attained when $\mathbf{z} = \lambda(\mathbf{x} + \mathbf{y})$ for some $\lambda \in \mathbb{R}$, and that for this value of λ we have

(i) $\lambda \leqslant -\frac{1}{2}$ (for any choice of $\mathbf{x}$ and $\mathbf{y}$);

(ii) $\lambda = -1$ and $S = -\frac{3}{2}$ in the case where $\mathbf{x} \cdot \mathbf{y} = \cos \frac{2\pi}{3}$.

Paper 1, Section II

6A Vectors and Matrices

Define the kernel and the image of a linear map α from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ be a basis of $\mathbb{R}^m$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ a basis of $\mathbb{R}^n$. Explain how to represent α by a matrix A relative to the given bases.

A second set of bases $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_m\}$ and $\{\mathbf{f}'_1, \mathbf{f}'_2, \ldots, \mathbf{f}'_n\}$ is now used to represent α by a matrix A'. Relate the elements of A' to the elements of A.

Let β be a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by

$$\beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}.$$

Either find one or more $\mathbf{x}$ in $\mathbb{R}^2$ such that

$$\beta\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

or explain why one cannot be found.

Let γ be a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by

$$\gamma \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \qquad \gamma \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \qquad \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the kernel of γ.

Paper 1, Section II

7B Vectors and Matrices

(a) Let $\lambda_1, \ldots, \lambda_d$ be distinct eigenvalues of an $n \times n$ matrix A, with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$. Prove that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ is linearly independent.

(b) Consider the quadric surface Q in $\mathbb{R}^3$ defined by

$$2x^2 - 4xy + 5y^2 - z^2 + 6\sqrt{5}y = 0.$$

Find the position of the origin $\tilde{O}$ and orthonormal coordinate basis vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2$ and $\tilde{\mathbf{e}}_3$, for a coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ in which Q takes the form

$$\alpha\tilde{x}^2 + \beta\tilde{y}^2 + \gamma\tilde{z}^2 = 1.$$

Also determine the values of α, β and γ, and describe the surface geometrically.

Paper 1, Section II
8B Vectors and Matrices

(a) Let A and A' be the matrices of a linear map L on $\mathbb{C}^2$ relative to bases $\mathcal{B}$ and $\mathcal{B}'$ respectively. In this question you may assume without proof that A and A' are similar.

(i) State how the matrix A of L relative to the basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ is constructed from L and $\mathcal{B}$. Also state how A may be used to compute $L\mathbf{v}$ for any $\mathbf{v} \in \mathbb{C}^2$.

(ii) Show that A and A' have the same characteristic equation.

(iii) Show that for any $k \neq 0$ the matrices

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \text{ and } \begin{pmatrix} a & c/k \\ bk & d \end{pmatrix}$$

are similar. [*Hint: if* $\{\mathbf{e}_1, \mathbf{e}_2\}$ *is a basis then so is* $\{k\mathbf{e}_1, \mathbf{e}_2\}$.]

(b) Using the results of (a), or otherwise, prove that any 2×2 complex matrix M with equal eigenvalues is similar to one of

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \text{ and } \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \quad \text{with } a \in \mathbb{C}.$$

(c) Consider the matrix

$$B(r) = \frac{1}{2}\begin{pmatrix} 1+r & 1-r & 1 \\ 1-r & 1+r & -1 \\ -1 & 1 & 2r \end{pmatrix}.$$

Show that there is a real value $r_0 > 0$ such that $B(r_0)$ is an orthogonal matrix. Show that $B(r_0)$ is a rotation and find the axis and angle of the rotation.