UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 7 June, 2012 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, ..., ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Define the *discriminant* of the binary quadratic form $f(x,y) = ax^2 + bxy + cy^2$.

Assuming that this form is positive definite, define what it means for f to be *reduced*.

Show that there are precisely two reduced positive definite binary quadratic forms of discriminant -35.

2F Topics in Analysis

State and prove Liouville's theorem concerning approximation of algebraic numbers by rationals.

3G Geometry and Groups

Let A be a Möbius transformation acting on the Riemann sphere. Show that, if A is not loxodromic, then there is a disc Δ in the Riemann sphere with $A(\Delta) = \Delta$. Describe all such discs for each Möbius transformation A.

Hence, or otherwise, show that the group G of Möbius transformations generated by

$$A : z \mapsto iz \qquad \text{and} \qquad B : z \mapsto 2z$$

does not map any disc onto itself.

Describe the set of points of the Riemann sphere at which G acts discontinuously. What is the quotient of this set by the action of G?

4G Coding and Cryptography

Describe the RSA system with public key (N, e) and private key d. Give a simple example of how the system is vulnerable to a homomorphism attack. Explain how a signature system prevents such an attack.

5K Statistical Modelling

Consider the linear model

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i,$$

for $i = 1, 2, \ldots, n$, where the ε_i are independent and identically distributed with $N(0, \sigma^2)$ distribution. What does it mean for the pair β_1 and β_2 to be *orthogonal*? What does it mean for all the three parameters β_0, β_1 and β_2 to be *mutually orthogonal*? Give necessary and sufficient conditions on $(x_{i1})_{i=1}^n, (x_{i2})_{i=1}^n$ so that β_0, β_1 and β_2 are mutually orthogonal. If $\beta_0, \beta_1, \beta_2$ are mutually orthogonal, find the joint distribution of the corresponding maximum likelihood estimators $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\beta}_2$.

6C Mathematical Biology

Consider a model of insect dispersal in two dimensions given by

$$\frac{\partial C}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rDC\frac{\partial C}{\partial r}\right),$$

where r is a radial coordinate, t is time, $C(r,t)$ is the density of insects and D is a constant coefficient such that DC is a diffusivity.

Show that under suitable assumptions

$$2\pi \int_0^\infty rC\,dr = N,$$

where N is constant, and interpret this condition.

Suppose that after a long time the form of C depends only on r, t, D and N (and is thus independent of any detailed form of the initial condition). Show that there is a solution of the form

$$C(r,t) = \left(\frac{N}{Dt}\right)^{1/2} g\left(\frac{r}{(NDt)^{1/4}}\right),$$

and deduce that the function $g(\xi)$ satisfies

$$\frac{d}{d\xi}\left(\xi g\frac{dg}{d\xi} + \frac{1}{4}\xi^2 g\right) = 0\,.$$

Show that this equation has a continuous solution with $g > 0$ for $\xi < \xi_0$ and $g = 0$ for $\xi \geqslant \xi_0$, and determine ξ_0. Hence determine the area within which $C(r,t) > 0$ as a function of t.

7D Dynamical Systems

State without proof Lyapunov's first theorem, carefully defining all the terms that you use.

Consider the dynamical system

$$\begin{aligned}\dot{x} &= -2x + y - xy + 3y^2 - xy^2 + x^3\,,\\ \dot{y} &= -2y - x - y^2 - 3xy + 2x^2y\,.\end{aligned}$$

By choosing a Lyapunov function $V(x, y) = x^2 + y^2$, prove that the origin is asymptotically stable.

By factorising the expression for $\dot{V}$, or otherwise, show that the basin of attraction of the origin includes the set $V < 7/4$.

8E Further Complex Methods

The Beta function, denoted by $B(z_1, z_2)$, is defined by

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}\,, \quad z_1, z_2 \in \mathbb{C}\,,$$

where $\Gamma(z)$ denotes the Gamma function. It can be shown that

$$B(z_1, z_2) = \int_0^\infty \frac{v^{z_2-1}\,dv}{(1+v)^{z_1+z_2}}\,, \quad \text{Re } z_1 > 0\,,\ \text{Re } z_2 > 0\,.$$

By computing this integral for the particular case of $z_1 + z_2 = 1$, and by employing analytic continuation, deduce that $\Gamma(z)$ satisfies the functional equation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}\,, \quad z \in \mathbb{C}.$$

9A Classical Dynamics

The motion of a particle of charge q and mass m in an electromagnetic field with scalar potential $\phi(\mathbf{r},t)$ and vector potential $\mathbf{A}(\mathbf{r},t)$ is characterized by the Lagrangian

$$L = \frac{m\dot{\mathbf{r}}^2}{2} - q(\phi - \dot{\mathbf{r}} \cdot \mathbf{A}).$$

(a) Show that the Euler–Lagrange equation is invariant under the gauge transformation

$$\phi \to \phi - \frac{\partial \Lambda}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla\Lambda,$$

for an arbitrary function $\Lambda(\mathbf{r},t)$.

(b) Derive the equations of motion in terms of the electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$.

[Recall that $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$.]

10E Cosmology

For an ideal Fermi gas in equilibrium at temperature T and chemical potential μ, the average occupation number of the kth energy state, with energy E_k, is

$$\bar{n}_k = \frac{1}{e^{(E_k-\mu)/k_BT} + 1}\,.$$

Discuss the limit $T \to 0$. What is the Fermi energy ϵ_F? How is it related to the Fermi momentum p_F? Explain why the density of states with momentum between p and $p+dp$ is proportional to p^2dp and use this fact to deduce that the fermion number density at zero temperature takes the form

$$n \propto p_F^3\,.$$

Consider an ideal Fermi gas that, at zero temperature, is either (i) non-relativistic or (ii) ultra-relativistic. In each case show that the fermion energy density ϵ takes the form

$$\epsilon \propto n^\gamma\,,$$

for some constant γ which you should compute.

SECTION II

11I Number Theory

Let p be an odd prime. Prove that the multiplicative groups $(\mathbb{Z}/p^n\mathbb{Z})^\times$ are cyclic for $n \geqslant 2$. [You may assume that the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic.]

Find an integer which generates $(\mathbb{Z}/7^n\mathbb{Z})^\times$ for all $n \geqslant 1$, justifying your answer.

12F Topics in Analysis

State Brouwer's fixed point theorem on the plane, and also an equivalent version of it concerning continuous retractions. Prove the equivalence of the two statements.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map with the property that $|f(x)| \leqslant 1$ whenever $|x| = 1$. Show that f has a fixed point. [*Hint. Compose f with the map that sends x to the nearest point to x inside the closed unit disc.*]

13C Mathematical Biology

Consider the two-variable reaction-diffusion system

$$\frac{\partial u}{\partial t} = a - u + u^2 v + \nabla^2 u\,,$$
$$\frac{\partial v}{\partial t} = b - u^2 v + d\nabla^2 v\,,$$

where a, b and d are positive constants.

Show that there is one possible spatially homogeneous steady state with $u > 0$ and $v > 0$ and show that it is stable to small-amplitude spatially homogeneous disturbances provided that $\gamma < \beta$, where

$$\gamma = \frac{b-a}{b+a} \qquad \text{and} \qquad \beta = (a+b)^2.$$

Now assuming that the condition $\gamma < \beta$ is satisfied, investigate the stability of the homogeneous steady state to spatially varying perturbations by considering the time-dependence of disturbances whose spatial form is such that $\nabla^2 u = -k^2 u$ and $\nabla^2 v = -k^2 v$, with k constant. Show that such disturbances vary as e^{pt}, where p is one of the roots of

$$p^2 + (\beta - \gamma + dk^2 + k^2)p + dk^4 + (\beta - d\gamma)k^2 + \beta.$$

By comparison with the stability condition for the homogeneous case above, give a simple argument as to why the system must be stable if $d = 1$.

Show that the boundary between stability and instability (as some combination of β, γ and d is varied) must correspond to $p = 0$.

Deduce that $d\gamma > \beta$ is a necessary condition for instability and, furthermore, that instability will occur for some k if

$$d > \frac{\beta}{\gamma}\left\{1 + \frac{2}{\gamma} + 2\sqrt{\frac{1}{\gamma} + \frac{1}{\gamma^2}}\right\}.$$

Deduce that the value of k^2 at which instability occurs as the stability boundary is crossed is given by

$$k^2 = \sqrt{\frac{\beta}{d}}\,.$$

14D Dynamical Systems

Consider the dynamical system

$$\ddot{x} - (a - bx)\dot{x} + x - x^2 = 0, \quad a, b > 0\,. \tag{1}$$

(a) Show that the fixed point at the origin is an unstable node or focus, and that the fixed point at $x = 1$ is a saddle point.

(b) By considering the phase plane $(x, \dot{x})$, or otherwise, show graphically that the maximum value of x for any periodic orbit is less than one.

(c) By writing the system in terms of the variables x and $z = \dot{x} - (ax - bx^2/2)$, or otherwise, show that for any periodic orbit $\mathcal{C}$

$$\oint_{\mathcal{C}} (x - x^2)(2ax - bx^2)\, dt = 0\,. \tag{2}$$

Deduce that if $a/b > 1/2$ there are no periodic orbits.

(d) If $a = b = 0$ the system (1) is Hamiltonian and has homoclinic orbit

$$X(t) = \frac{1}{2}\left(3\tanh^2\left(\frac{t}{2}\right) - 1\right)\,, \tag{3}$$

which approaches $X = 1$ as $t \to \pm\infty$. Now suppose that a, b are very small and that we seek the value of a/b corresponding to a periodic orbit very close to $X(t)$. By using equation (3) in equation (2), find an approximation to the largest value of a/b for a periodic orbit when a, b are very small.

[*Hint. You may use the fact that* $(1 - X) = \frac{3}{2}\mathrm{sech}^2(\frac{t}{2}) = 3\frac{d}{dt}(\tanh(\frac{t}{2}))$]

15E Cosmology

In a flat expanding universe with scale factor $a(t)$, average mass density $\bar{\rho}$ and average pressure $\bar{P} \ll \bar{\rho}c^2$, the fractional density perturbations $\delta_k(t)$ at co-moving wavenumber k satisfy the equation

$$\ddot{\delta}_k = -2\left(\frac{\dot{a}}{a}\right)\dot{\delta}_k + 4\pi G\bar{\rho}\delta_k - \frac{c_s^2 k^2}{a^2}\delta_k\,. \tag{$*$}$$

Discuss briefly the meaning of each term on the right hand side of this equation. What is the Jeans length λ_J, and what is its significance? How is it related to the Jeans mass?

How does the equation $(*)$ simplify at $\lambda \gg \lambda_J$ in a flat universe? Use your result to show that density perturbations can grow. For a growing density perturbation, how does $\dot{\delta}/\delta$ compare to the inverse Hubble time?

Explain qualitatively why structure only forms after decoupling, and why cold dark matter is needed for structure formation.

16H Logic and Set Theory

Write down **either** the synthetic **or** the recursive definitions of ordinal addition and multiplication. Using your definitions, give proofs or counterexamples for the following statements:

(i) For all α, β and γ, we have $\alpha.(\beta+\gamma) = \alpha.\beta + \alpha.\gamma$.

(ii) For all α, β and γ, we have $(\alpha+\beta).\gamma = \alpha.\gamma + \beta.\gamma$.

(iii) For all α and β with $\beta > 0$, there exist γ and δ with $\delta < \beta$ and $\alpha = \beta.\gamma + \delta$.

(iv) For all α and β with $\beta > 0$, there exist γ and δ with $\delta < \beta$ and $\alpha = \gamma.\beta + \delta$.

(v) For every α, either there exists a cofinal map $f\colon \omega \to \alpha$ (that is, one such that $\alpha = \bigcup\{f(n)^+ \mid n \in \omega\}$), or there exists β such that $\alpha = \omega_1.\beta$.

17F Graph Theory

Let H be a graph with at least one edge. Define $\mathrm{ex}\,(n;H)$, where n is an integer with $n \geqslant |H|$. Without assuming the Erdős–Stone theorem, show that the sequence $\mathrm{ex}\,(n;H)/\binom{n}{2}$ converges as $n \to \infty$.

State precisely the Erdős–Stone theorem. Hence determine, with justification, $\lim_{n\to\infty} \mathrm{ex}\,(n;H)/\binom{n}{2}$.

Let K be another graph with at least one edge. For each integer n such that $n \geqslant \max\{|H|,|K|\}$, let

$$f(n) = \max\{e(G) : |G| = n; H \not\subset G \text{ and } K \not\subset G\}$$

and let

$$g(n) = \max\{e(G) : |G| = n; H \not\subset G \text{ or } K \not\subset G\}.$$

Find, with justification, $\lim_{n\to\infty} f(n)/\binom{n}{2}$ and $\lim_{n\to\infty} g(n)/\binom{n}{2}$.

18H Galois Theory

Let $q = p^f$ $(f \geqslant 1)$ be a power of the prime p, and $\mathbb{F}_q$ be a finite field consisting of q elements.

Let N be a positive integer prime to p, and $\mathbb{F}_q(\boldsymbol{\mu}_N)$ be the cyclotomic extension obtained by adjoining all Nth roots of unity to $\mathbb{F}_q$. Prove that $\mathbb{F}_q(\boldsymbol{\mu}_N)$ is a finite field with q^n elements, where n is the order of the element $q \bmod N$ in the multiplicative group $(\mathbb{Z}/N\mathbb{Z})^\times$ of the ring $\mathbb{Z}/N\mathbb{Z}$.

Explain why what is proven above specialises to the following fact: the finite field $\mathbb{F}_p$ for an odd prime p contains a square root of -1 if and only if $p \equiv 1 \pmod 4$.

[Standard facts on finite fields and their extensions can be quoted without proof, as long as they are clearly stated.]

19H Representation Theory

Show that every complex representation of a finite group G is equivalent to a unitary representation. Let χ be a character of some finite group G and let $g \in G$. Explain why there are roots of unity $\omega_1, \ldots, \omega_d$ such that

$$\chi(g^i) = \omega_1^i + \cdots + \omega_d^i$$

for all integers i.

For the rest of the question let G be the symmetric group on some finite set. Explain why $\chi(g) = \chi(g^i)$ whenever i is coprime to the order of g.

Prove that $\chi(g) \in \mathbb{Z}$.

State without proof a formula for $\sum_{g \in G} \chi(g)^2$ when χ is irreducible. Is there an irreducible character χ of degree at least 2 with $\chi(g) \neq 0$ for all $g \in G$? Explain your answer.

[You may assume basic facts about the symmetric group, and about algebraic integers, without proof. You may also use without proof the fact that $\displaystyle\sum_{\substack{1 \leqslant i \leqslant n \\ \gcd(i,n)=1}} \omega^i \in \mathbb{Z}$ for any nth root of unity ω.]

20G Algebraic Topology

State the Mayer–Vietoris Theorem for a simplicial complex K expressed as the union of two subcomplexes L and M. Explain briefly how the connecting homomorphism $\delta_* \colon H_n(K) \to H_{n-1}(L \cap M)$, which appears in the theorem, is defined. [You should include a proof that δ_* is well-defined, but need not verify that it is a homomorphism.]

Now suppose that $|K| \cong S^3$, that $|L|$ is a solid torus $S^1 \times B^2$, and that $|L \cap M|$ is the boundary torus of $|L|$. Show that $\delta_* \colon H_3(K) \to H_2(L \cap M)$ is an isomorphism, and hence calculate the homology groups of M. [You may assume that a generator of $H_3(K)$ may be represented by a 3-cycle which is the sum of all the 3-simplices of K, with 'matching' orientations.]

21G Linear Analysis

State the closed graph theorem.

(i) Let X be a Banach space and Y a vector space. Suppose that Y is endowed with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and that there is a constant $c > 0$ such that $\|y\|_2 \geqslant c\|y\|_1$ for all $y \in Y$. Suppose that Y is a Banach space with respect to both norms. Suppose that $T : X \to Y$ is a linear operator, and that it is bounded when Y is endowed with the $\|\cdot\|_1$ norm. Show that it is also bounded when Y is endowed with the $\|\cdot\|_2$ norm.

(ii) Suppose that X is a normed space and that $(x_n)_{n=1}^{\infty} \subseteq X$ is a sequence with $\sum_{n=1}^{\infty} |f(x_n)| < \infty$ for all f in the dual space X^*. Show that there is an M such that

$$\sum_{n=1}^{\infty} |f(x_n)| \leqslant M\|f\|$$

for all $f \in X^*$.

(iii) Suppose that X is the space of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the sup norm, and that $Y \subseteq X$ is the subspace of continuously differentiable functions with bounded derivative. Let $T : Y \to X$ be defined by $Tf = f'$. Show that the graph of T is closed, but that T is not bounded.

22I Riemann Surfaces

Let Λ be the lattice $\mathbb{Z} + \mathbb{Z}i$, X the torus $\mathbb{C}/\Lambda$, and $\wp$ the Weierstrass elliptic function with respect to Λ.

(i) Let $x \in X$ be the point given by $0 \in \Lambda$. Determine the group

$$G = \{f \in \mathrm{Aut}(X) \mid f(x) = x\}.$$

(ii) Show that $\wp^2$ defines a degree 4 holomorphic map $h\colon X \to \mathbb{C} \cup \{\infty\}$, which is invariant under the action of G, that is, $h(f(y)) = h(y)$ for any $y \in X$ and any $f \in G$. Identify a ramification point of h distinct from x which is fixed by every element of G.

[If you use the Monodromy theorem, then you should state it correctly. You may use the fact that $\mathrm{Aut}(\mathbb{C}) = \{az + b \mid a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C}\}$, and may assume without proof standard facts about $\wp$.]

23I Algebraic Geometry

Let $X \subset \mathbf{P}^2(\mathbf{C})$ be the projective closure of the affine curve $y^3 = x^4 + 1$. Let ω denote the differential dx/y^2. Show that X is smooth, and compute $v_p(\omega)$ for all $p \in X$.

Calculate the genus of X.

24I Differential Geometry

For a surface $S \subset \mathbb{R}^3$, define what is meant by the *exponential mapping* $\exp_p$ at $p \in S$, *geodesic polar coordinates* (r, θ) and *geodesic circles*.

Let E, F, G be the coefficients of the first fundamental form in geodesic polar coordinates (r, θ). Prove that $\lim_{r\to 0} \sqrt{G}(r,\theta) = 0$ and $\lim_{r\to 0}(\sqrt{G})_r(r,\theta) = 1$. Give an expression for the Gaussian curvature K in terms of G.

Prove that the Gaussian curvature at a point $p \in S$ satisfies

$$K(p) = \lim_{r\to 0} \frac{12(\pi r^2 - A_p(r))}{\pi r^4},$$

where $A_p(r)$ is the area of the region bounded by the geodesic circle of radius r centred at p.

[You may assume that $E = 1$, $F = 0$ and $d(\exp_p)_0$ is an isometry. Taylor's theorem with any form of the remainder may be assumed if accurately stated.]

25J Probability and Measure

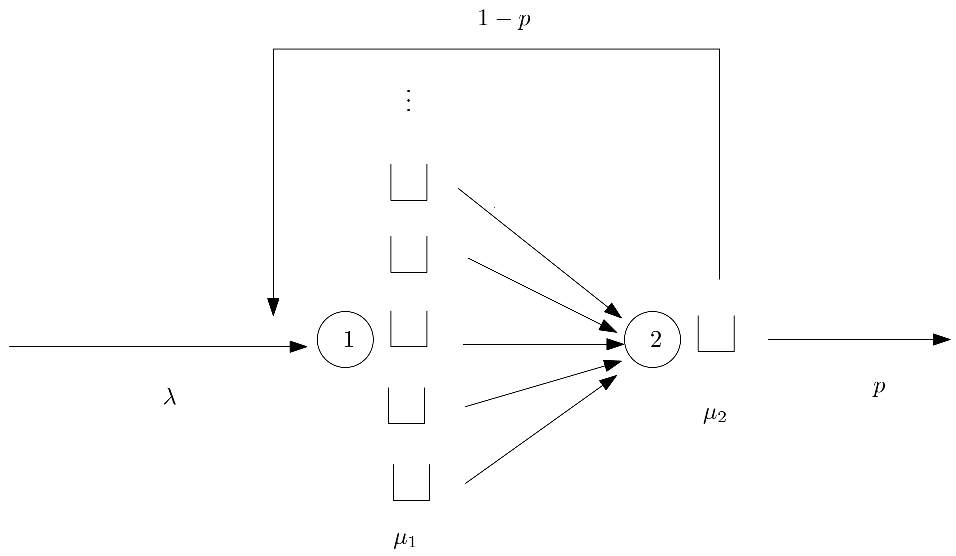
Carefully state and prove the first and second Borel–Cantelli lemmas.

Now let $(A_n : n \in \mathbb{N})$ be a sequence of events that are *pairwise independent*; that is, $\mathbb{P}(A_n \cap A_m) = \mathbb{P}(A_n)\mathbb{P}(A_m)$ whenever $m \neq n$. For $N \geqslant 1$, let $S_N = \sum_{n=1}^{N} 1_{A_n}$. Show that $\mathrm{Var}(S_N) \leqslant \mathbb{E}(S_N)$.

Using Chebyshev's inequality or otherwise, deduce that if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then $\lim_{N\to\infty} S_N = \infty$ almost surely. Conclude that $\mathbb{P}(A_n \text{ infinitely often}) = 1$.

26K Applied Probability

We consider a system of two queues in tandem, as follows. Customers arrive in the first queue at rate λ. Each arriving customer is immediately served by one of infinitely many servers at rate μ_1. Immediately after service, customers join a single-server second queue which operates on a first-come, first-served basis, and has a service rate μ_2. After service in this second queue, each customer returns to the first queue with probability $0 < 1-p < 1$, and otherwise leaves the system forever. A schematic representation is given below:



(a) Let M_t and N_t denote the number of customers at time t in queues number 1 and 2 respectively, including those currently in service at time t. Give the transition rates of the Markov chain $(M_t, N_t)_{t\geqslant 0}$.

(b) Write down an equation satisfied by any invariant measure π for this Markov chain. Let $\alpha > 0$ and $\beta \in (0,1)$. Define a measure π by

$$\pi(m,n) := e^{-\alpha}\frac{\alpha^m}{m!}\beta^n(1-\beta), \qquad m,n \in \{0,1,\ldots\}.$$

Show that it is possible to find $\alpha > 0, \beta \in (0,1)$ so that π is an invariant measure of $(M_t, N_t)_{t\geqslant 0}$, if and only if $\lambda < \mu_2 p$. Give the values of α and β in this case.

(c) Assume now that $\lambda p > \mu_2$. Show that the number of customers is not positive recurrent.

[*Hint. One way to solve the problem is as follows. Assume it is positive recurrent. Observe that* M_t *is greater than a* $M/M/\infty$ *queue with arrival rate* λ*. Deduce that* N_t *is greater than a* $M/M/1$ *queue with arrival rate* λp *and service rate* μ_2*. You may use without proof the fact that the departure process from the first queue is then, at equilibrium, a Poisson process with rate* λ*, and you may use without proof properties of thinned Poisson processes.*]

27K Principles of Statistics

The parameter vector is $\mathbf{\Theta} \equiv (\Theta_1, \Theta_2, \Theta_3)$, with $\Theta_i > 0$, $\Theta_1 + \Theta_2 + \Theta_3 = 1$. Given $\mathbf{\Theta} = \boldsymbol{\theta} \equiv (\theta_1, \theta_2, \theta_3)$, the integer random vector $\boldsymbol{X} = (X_1, X_2, X_3)$ has a trinomial distribution, with probability mass function

$$p(\boldsymbol{x} \mid \boldsymbol{\theta}) = \frac{n!}{x_1!\, x_2!\, x_3!}\, \theta_1^{x_1}\, \theta_2^{x_2}\, \theta_3^{x_3}\,, \qquad \left(x_i \geqslant 0,\ \sum_{i=1}^{3} x_i = n\right). \tag{1}$$

Compute the score vector for the parameter $\mathbf{\Theta}^* := (\Theta_1, \Theta_2)$, and, quoting any relevant general result, use this to determine $\mathbb{E}(X_i)$ $(i = 1, 2, 3)$.

Considering (1) as an exponential family with mean-value parameter $\mathbf{\Theta}^*$, what is the corresponding natural parameter $\mathbf{\Phi} \equiv (\Phi_1, \Phi_2)$?

Compute the information matrix I for $\mathbf{\Theta}^*$, which has (i,j)-entry

$$I_{ij} = -\mathbb{E}\left(\frac{\partial^2 l}{\partial\theta_i \partial\theta_j}\right) \quad (i, j = 1, 2)\,,$$

where l denotes the log-likelihood function, based on $\boldsymbol{X}$, expressed in terms of (θ_1, θ_2).

Show that the variance of $\log(X_1/X_3)$ is asymptotic to $n^{-1}(\theta_1^{-1} + \theta_3^{-1})$ as $n \to \infty$. [*Hint. The information matrix* I_Φ *for* $\mathbf{\Phi}$ *is* I^{-1} *and the dispersion matrix of the maximum likelihood estimator* $\widehat{\mathbf{\Phi}}$ *behaves, asymptotically (for* $n \to \infty$*) as* I_Φ^{-1}.]

28J Optimization and Control

A state variable $x = (x_1, x_2) \in \mathbb{R}^2$ is subject to dynamics

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t), \end{aligned}$$

where $u = u(t)$ is a scalar control variable constrained to the interval $[-1, 1]$. Given an initial value $x(0) = (x_1, x_2)$, let $F(x_1, x_2)$ denote the minimal time required to bring the state to $(0, 0)$. Prove that

$$\max_{u \in [-1,1]} \left\{ -x_2 \frac{\partial F}{\partial x_1} - u \frac{\partial F}{\partial x_2} - 1 \right\} = 0\,.$$

Explain how this equation figures in Pontryagin's maximum principle.

Use Pontryagin's maximum principle to show that, on an optimal trajectory, $u(t)$ only takes the values 1 and -1, and that it makes at most one switch between them.

Show that $u(t) = 1$, $0 \leqslant t \leqslant 2$ is optimal when $x(0) = (2, -2)$.

Find the optimal control when $x(0) = (7, -2)$.

29J Stochastic Financial Models

(i) Let $\mathcal{F} = \{\mathcal{F}_n\}_{n=0}^{\infty}$ be a filtration. Give the definition of a martingale and a stopping time with respect to the filtration $\mathcal{F}$.

(ii) State Doob's optional stopping theorem. Give an example of a martingale M and a stopping time T such that $\mathbb{E}(M_T) \neq \mathbb{E}(M_0)$.

(iii) Let S_n be a standard random walk on $\mathbb{Z}$, that is, $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, where X_i are i.i.d. and $X_i = 1$ or -1 with probability $1/2$.

Let $T_a = \inf\{n \geqslant 0 : S_n = a\}$ where a is a positive integer. Show that for all $\theta > 0$,

$$\mathbb{E}\left(e^{-\theta T_a}\right) = \left(e^{\theta} - \sqrt{e^{2\theta} - 1}\right)^a .$$

Carefully justify all steps in your derivation.

[*Hint. For all* $\lambda > 0$ *find* θ *such that* $M_n = \exp(-\theta n + \lambda S_n)$ *is a martingale. You may assume that* T_a *is almost surely finite.*]

Let $T = T_a \wedge T_{-a} = \inf\{n \geqslant 0 : |S_n| = a\}$. By introducing a suitable martingale, compute $\mathbb{E}(e^{-\theta T})$.

30B Partial Differential Equations

Consider the nonlinear partial differential equation for a function $u(x,t)$, $x \in \mathbb{R}^n$, $t > 0$,

$$u_t = \Delta u - \alpha |\nabla u|^2, \tag{1}$$

$$\text{subject to} \quad u(x,0) = u_0(x), \tag{2}$$

where $u_0 \in L^{\infty}(\mathbb{R}^n)$.

(i) Find a transformation $w := F(u)$ such that w satisfies the heat equation

$$w_t = \Delta w, \quad x \in \mathbb{R}^n,$$

if (1) holds for u.

(ii) Use the transformation obtained in (i) (and its inverse) to find a solution to the initial value problem (1), (2).
[*Hint. Use the fundamental solution of the heat equation.*]

(iii) The equation (1) is posed on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary, subject to the initial condition (2) on Ω and inhomogeneous Dirichlet boundary conditions

$$u = u_D \quad \text{on } \partial\Omega,$$

where u_D is a bounded function. Use the maximum-minimum principle to prove that there exists at most one classical solution of this boundary value problem.

31B Asymptotic Methods

Find the two leading terms in the asymptotic expansion of the Laplace integral

$$I(x) = \int_0^1 f(t) e^{xt^4} dt$$

as $x \to \infty$, where $f(t)$ is smooth and positive on $[0, 1]$.

32D Integrable Systems

Consider a one-parameter group of transformations acting on $\mathbb{R}^4$

$$(x, y, t, u) \longrightarrow (\exp(\epsilon\alpha)x, \exp(\epsilon\beta)y, \exp(\epsilon\gamma)t, \exp(\epsilon\delta)u) \,, \qquad (1)$$

where ϵ is a group parameter and $(\alpha, \beta, \gamma, \delta)$ are constants.

(a) Find a vector field W which generates this group.

(b) Find two independent Lie point symmetries S_1 and S_2 of the PDE

$$(u_t - uu_x)_x = u_{yy}, \quad u = u(x, y, t) \,, \qquad (2)$$

which are of the form (1).

(c) Find three functionally-independent invariants of S_1, and do the same for S_2. Find a non-constant function $G = G(x, y, t, u)$ which is invariant under both S_1 and S_2.

(d) Explain why all the solutions of (2) that are invariant under a two-parameter group of transformations generated by vector fields

$$W = u\frac{\partial}{\partial u} + x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y}, \quad V = \frac{\partial}{\partial y} \,,$$

are of the form $u = xF(t)$, where F is a function of one variable. Find an ODE for F characterising these group-invariant solutions.

33A Principles of Quantum Mechanics

Discuss the consequences of indistinguishability for a quantum mechanical state consisting of two identical, non-interacting particles when the particles have (a) spin zero, (b) spin 1/2.

The stationary Schrödinger equation for one particle in the potential

$$\frac{-2e^2}{4\pi\epsilon_0 r}$$

has normalised, spherically-symmetric real wavefunctions $\psi_n(\mathbf{r})$ and energy eigenvalues E_n with $E_0 < E_1 < E_2 < \cdots$. The helium atom can be modelled by considering two non-interacting spin 1/2 particles in the above potential. What are the consequences of the Pauli exclusion principle for the ground state? Write down the two-electron state for this model in the form of a spatial wavefunction times a spin state. Assuming that wavefunctions are spherically-symmetric, find the states of the first excited energy level of the helium atom. What combined angular momentum quantum numbers J, M does each state have?

Assuming standard perturbation theory results, arrive at a multi-dimensional integral in terms of the one-particle wavefunctions for the first-order correction to the helium ground state energy, arising from the electron-electron interaction.

34E Applications of Quantum Mechanics

A simple model of a crystal consists of a 1D linear array of sites at positions $x = na$, for all integer n and separation a, each occupied by a similar atom. The potential due to the atom at the origin is $U(x)$, which is symmetric: $U(-x) = U(x)$. The Hamiltonian, H_0, for the atom at the n-th site in isolation has electron eigenfunction $\psi_n(x)$ with energy E_0. Write down H_0 and state the relationship between $\psi_n(x)$ and $\psi_0(x)$.

The Hamiltonian H for an electron moving in the crystal is $H = H_0 + V(x)$. Give an expression for $V(x)$.

In the tight-binding approximation for this model the ψ_n are assumed to be orthonormal, $(\psi_n, \psi_m) = \delta_{nm}$, and the only non-zero matrix elements of H_0 and V are

$$(\psi_n, H_0\psi_n) = E_0, \quad (\psi_n, V\psi_n) = \alpha, \quad (\psi_n, V\psi_{n\pm1}) = -A\,,$$

where $A > 0$. By considering the trial wavefunction $\Psi(x,t) = \sum_n c_n(t)\psi_n(x)$, show that the time-dependent Schrödinger equation governing the amplitudes $c_n(t)$ is

$$i\hbar\dot{c}_n = (E_0 + \alpha)c_n - A(c_{n+1} + c_{n-1})\,.$$

By examining a solution of the form

$$c_n = e^{i(kna - Et/\hbar)}\,,$$

show that E, the energy of the electron in the crystal, lies in a band given by

$$E = E_0 + \alpha - 2A\cos ka\,.$$

Using the fact that $\psi_0(x)$ is a parity eigenstate show that

$$(\psi_n, x\psi_n) = na\,.$$

The electron in this model is now subject to an electric field $\mathcal{E}$ in the direction of increasing x, so that $V(x)$ is replaced by $V(x) - e\mathcal{E}x$, where $-e$ is the charge on the electron. Assuming that $(\psi_n, x\psi_m) = 0,\ n \neq m$, write down the new form of the time-dependent Schrödinger equation for the probability amplitudes c_n. Verify that it has solutions of the form

$$c_n = \exp\left[-\frac{i}{\hbar}\int_0^t \epsilon(t')dt' + i\left(k + \frac{e\mathcal{E}t}{\hbar}\right)na\right]\,,$$

where

$$\epsilon(t) = E_0 + \alpha - 2A\cos\left[\left(k + \frac{e\mathcal{E}t}{\hbar}\right)a\right]\,.$$

Use this result to show that the dynamical behaviour of an electron near the bottom of an energy band is the same as that for a free particle in the presence of an electric field with an effective mass $m^* = \hbar^2/(2Aa^2)$.

35C Statistical Physics

A ferromagnet has magnetization order parameter m and is at temperature T. The free energy is given by

$$F(T;m) = F_0(T) + \frac{a}{2}(T - T_c)\, m^2 + \frac{b}{4}\, m^4 \ ,$$

where a, b and T_c are positive constants. Find the equilibrium value of the magnetization at both high and low temperatures.

Evaluate the free energy of the ground state as a function of temperature. Hence compute the entropy and heat capacity. Determine the jump in the heat capacity and identify the order of the phase transition.

After imposing a background magnetic field B, the free energy becomes

$$F(T;m) = F_0(T) + Bm + \frac{a}{2}(T - T_c)\, m^2 + \frac{b}{4}\, m^4 \ .$$

Explain graphically why the system undergoes a first-order phase transition at low temperatures as B changes sign.

The *spinodal* point occurs when the meta-stable vacuum ceases to exist. Determine the temperature T of the spinodal point as a function of T_c, a, b and B.

36B Electrodynamics

The non-relativistic Larmor formula for the power, P, radiated by a particle of charge q and mass m that is being accelerated with an acceleration $\mathbf{a}$ is

$$P = \frac{\mu_0}{6\pi} q^2 |\mathbf{a}|^{\mathbf{2}} \ .$$

Starting from the Liénard–Wiechert potentials, sketch a derivation of this result. Explain briefly why the relativistic generalization of this formula is

$$P = \frac{\mu_0}{6\pi}\frac{q^2}{m^2}\left(\frac{dp^\mu}{d\tau}\frac{dp^\nu}{d\tau}\eta_{\mu\nu}\right) \ ,$$

where p^μ is the relativistic momentum of the particle and τ is the proper time along the worldline of the particle.

A particle of mass m and charge q moves in a plane perpendicular to a constant magnetic field B. At time $t = 0$ as seen by an observer $\mathbf{O}$ at rest, the particle has energy $E = \gamma m$. At what rate is electromagnetic energy radiated by this particle?

At time t according to the observer $\mathbf{O}$, the particle has energy $E' = \gamma' m$. Find an expression for γ' in terms of γ and t.

37B General Relativity

(i) The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

Consider a time-like geodesic $x^a(\tau)$, where τ is the proper time, lying in the plane $\theta = \pi/2$. Use the Lagrangian $L = g_{ab}\dot{x}^a\dot{x}^b$ to derive the equations governing the geodesic, showing that

$$r^2\dot{\phi} = h\,,$$

with h constant, and hence demonstrate that

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{h^2} + 3Mu^2\,,$$

where $u = 1/r$. State which term in this equation makes it different from an analogous equation in Newtonian theory.

(ii) Now consider Kruskal coordinates, in which the Schwarzschild t and r are replaced by U and V, defined for $r > 2M$ by

$$U \equiv \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/(4M)} \cosh\left(\frac{t}{4M}\right)$$
$$V \equiv \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/(4M)} \sinh\left(\frac{t}{4M}\right)$$

and for $r < 2M$ by

$$U \equiv \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/(4M)} \sinh\left(\frac{t}{4M}\right)$$
$$V \equiv \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/(4M)} \cosh\left(\frac{t}{4M}\right)\,.$$

Given that the metric in these coordinates is

$$ds^2 = \frac{32M^3}{r} e^{-r/(2M)}(-dV^2 + dU^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

where $r = r(U, V)$ is defined implicitly by

$$\left(\frac{r}{2M} - 1\right) e^{r/(2M)} = U^2 - V^2\,,$$

sketch the Kruskal diagram, indicating the positions of the singularity at $r = 0$, the event horizon at $r = 2M$, and general lines of constant r and of constant t.

38C Fluid Dynamics II

For two Stokes flows $\mathbf{u}^{(1)}(\mathbf{x})$ and $\mathbf{u}^{(2)}(\mathbf{x})$ inside the same volume V with different boundary conditions on its boundary S, prove the reciprocal theorem

$$\int_S \sigma_{ij}^{(1)} n_j u_i^{(2)}\, dS = \int_S \sigma_{ij}^{(2)} n_j u_i^{(1)}\, dS\,,$$

where $\sigma^{(1)}$ and $\sigma^{(2)}$ are the stress fields associated with the flows.

When a rigid sphere of radius a translates with velocity $\mathbf{U}$ through unbounded fluid at rest at infinity, it may be shown that the traction per unit area, $\boldsymbol{\sigma}\cdot\mathbf{n}$, exerted by the sphere on the fluid has the uniform value $3\mu\mathbf{U}/2a$ over the sphere surface. Find the drag on the sphere.

Suppose that the same sphere is now free of external forces and is placed with its centre at the origin in an unbounded Stokes flow given in the absence of the sphere as $\mathbf{u}^*(\mathbf{x})$. By applying the reciprocal theorem to the perturbation to the flow generated by the presence of the sphere, and assuming this tends to zero sufficiently rapidly at infinity, show that the instantaneous velocity of the centre of the sphere is

$$\frac{1}{4\pi a^2}\int \mathbf{u}^*(\mathbf{x})\, dS\,,$$

where the integral is taken over the sphere of radius a.

39D Waves

The function $\phi(x,t)$ satisfies the equation

$$\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x} + \frac{1}{5}\frac{\partial^5\phi}{\partial x^5} = 0\,,$$

where $U > 0$ is a constant. Find the dispersion relation for waves of frequency ω and wavenumber k. Sketch a graph showing both the phase velocity $c(k)$ and the group velocity $c_g(k)$, and state whether wave crests move faster or slower than a wave packet.

Suppose that $\phi(x,0)$ is real and given by a Fourier transform as

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk\,.$$

Use the method of stationary phase to obtain an approximation for $\phi(Vt,t)$ for fixed $V > U$ and large t. If, in addition, $\phi(x,0) = \phi(-x,0)$, deduce an approximation for the sequence of times at which $\phi(Vt,t) = 0$.

What can be said about $\phi(Vt,t)$ if $V < U$? [Detailed calculation is **not** required in this case.]

[You may assume that $\displaystyle\int_{-\infty}^{\infty} e^{-au^2} du = \sqrt{\frac{\pi}{a}}$ for $\mathrm{Re}(a) \geqslant 0$, $a \neq 0$.]

40D Numerical Analysis

The inverse discrete Fourier transform $\mathcal{F}_n^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is given by the formula

$$\mathbf{x} = \mathcal{F}_n^{-1}\mathbf{y}, \quad \text{where} \quad x_l = \sum_{j=0}^{n-1} \omega_n^{jl} y_j, \quad l = 0, \ldots, n-1\,.$$

Here, $\omega_n = \exp(2\pi i/n)$ is the primitive root of unity of degree n and $n = 2^p$, $p = 1, 2, \ldots$

(i) Show how to assemble $\mathbf{x} = \mathcal{F}_{2m}^{-1}\mathbf{y}$ in a small number of operations if the Fourier transforms of the even and odd parts of $\mathbf{y}$,

$$\mathbf{x}^{(E)} = \mathcal{F}_m^{-1}\mathbf{y}^{(E)}, \quad \mathbf{x}^{(O)} = \mathcal{F}_m^{-1}\mathbf{y}^{(O)}\,,$$

are already known.

(ii) Describe the Fast Fourier Transform (FFT) method for evaluating $\mathbf{x}$, and draw a relevant diagram for $n = 8$.

(iii) Find the costs of the FFT method for $n = 2^p$ (only multiplications count).

(iv) For $n = 4$ use the FFT method to find $\mathbf{x} = \mathcal{F}_4^{-1}\mathbf{y}$ when:

(a) $\mathbf{y} = (1, 1, -1, -1)$,
(b) $\mathbf{y} = (1, -1, 1, -1)$.

END OF PAPER