

MATHEMATICAL TRIPOS Part IB

Tuesday, 5 June, 2012 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

*Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most **four** questions from Section I and at most **six** questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in separate bundles labelled **A, B, ..., H** according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS

Gold cover sheets

Green master cover sheet

SPECIAL REQUIREMENTS

None

<p>You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.</p>

SECTION I**1F Linear Algebra**

Define the notions of basis and dimension of a vector space. Prove that two finite-dimensional real vector spaces with the same dimension are isomorphic.

In each case below, determine whether the set S is a basis of the real vector space V :

- (i) $V = \mathbb{C}$ is the complex numbers; $S = \{1, i\}$.
- (ii) $V = \mathbb{R}[x]$ is the vector space of all polynomials in x with real coefficients; $S = \{1, (x-1), (x-1)(x-2), (x-1)(x-2)(x-3), \dots\}$.
- (iii) $V = \{f : [0, 1] \rightarrow \mathbb{R}\}$; $S = \{\chi_p \mid p \in [0, 1]\}$, where

$$\chi_p(x) = \begin{cases} 1 & x = p \\ 0 & x \neq p. \end{cases}$$

2A Complex Analysis or Complex Methods

Find a conformal transformation $\zeta = \zeta(z)$ that maps the domain D , $0 < \arg z < \frac{3\pi}{2}$, on to the strip $0 < \text{Im}(\zeta) < 1$.

Hence find a bounded harmonic function ϕ on D subject to the boundary conditions $\phi = 0, A$ on $\arg z = 0, \frac{3\pi}{2}$, respectively, where A is a real constant.

3G Geometry

Describe a collection of charts which cover a circular cylinder of radius R . Compute the first fundamental form, and deduce that the cylinder is locally isometric to the plane.

4B Variational Principles

State how to find the stationary points of a C^1 function $f(x, y)$ restricted to the circle $x^2 + y^2 = 1$, using the method of Lagrange multipliers. Explain why, in general, the method of Lagrange multipliers works, in the case where there is just one constraint.

Find the stationary points of $x^4 + 2y^3$ restricted to the circle $x^2 + y^2 = 1$.

5A Fluid Dynamics

Viscous fluid, with viscosity μ and density ρ flows along a straight circular pipe of radius R . The average velocity of the flow is U . Define a Reynolds number for the flow.

The flow is driven by a constant pressure gradient $-G > 0$ along the pipe and the velocity is parallel to the axis of the pipe with magnitude $u(r)$ that satisfies

$$\frac{\mu}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = -G,$$

where r is the radial distance from the axis.

State the boundary conditions on u and find the velocity as a function of r assuming that it is finite on the axis $r = 0$. Hence, show that the shear stress τ at the pipe wall is independent of the viscosity. Why is this the case?

6D Numerical Analysis

Let

$$A = \begin{bmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ a^2 & a^3 & 1 & a \\ a & a^2 & a^3 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} \gamma \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma = 1 - a^4 \neq 0.$$

Find the LU factorization of the matrix A and use it to solve the system $Ax = b$ via forward and backward substitution. [Other methods of solution are not acceptable.]

7H Statistics

Describe the generalised likelihood ratio test and the type of statistical question for which it is useful.

Suppose that X_1, \dots, X_n are independent and identically distributed random variables with the Gamma(2, λ) distribution, having density function $\lambda^2 x \exp(-\lambda x)$, $x \geq 0$. Similarly, Y_1, \dots, Y_n are independent and identically distributed with the Gamma(2, μ) distribution. It is desired to test the hypothesis $H_0 : \lambda = \mu$ against $H_1 : \lambda \neq \mu$. Derive the generalised likelihood ratio test and express it in terms of $R = \sum_i X_i / \sum_i Y_i$.

Let $F_{\nu_1, \nu_2}^{(1-\alpha)}$ denote the value that a random variable having the F_{ν_1, ν_2} distribution exceeds with probability α . Explain how to decide the outcome of a size 0.05 test when $n = 5$ by knowing only the value of R and the value $F_{\nu_1, \nu_2}^{(1-\alpha)}$, for some ν_1, ν_2 and α , which you should specify.

[You may use the fact that the χ_k^2 distribution is equivalent to the Gamma($k/2, 1/2$) distribution.]

8H Optimization

State the Lagrangian sufficiency theorem.

Use Lagrange multipliers to find the optimal values of x_1 and x_2 in the problem:

$$\text{maximize } x_1^2 + x_2 \quad \text{subject to } x_1^2 + \frac{1}{2}x_2^2 \leq b_1, \quad x_1 \geq b_2 \quad \text{and} \quad x_1, x_2 \geq 0,$$

for all values of b_1, b_2 such that $b_1 - b_2^2 \geq 0$.

SECTION II

9F Linear Algebra

Define what it means for two $n \times n$ matrices to be similar to each other. Show that if two $n \times n$ matrices are similar, then the linear transformations they define have isomorphic kernels and images.

If A and B are $n \times n$ real matrices, we define $[A, B] = AB - BA$. Let

$$K_A = \{X \in M_{n \times n}(\mathbb{R}) \mid [A, X] = 0\}$$

$$L_A = \{[A, X] \mid X \in M_{n \times n}(\mathbb{R})\}.$$

Show that K_A and L_A are linear subspaces of $M_{n \times n}(\mathbb{R})$. If A and B are similar, show that $K_A \cong K_B$ and $L_A \cong L_B$.

Suppose that A is diagonalizable and has characteristic polynomial

$$(x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2},$$

where $\lambda_1 \neq \lambda_2$. What are $\dim K_A$ and $\dim L_A$?

10G Groups, Rings and Modules

Let G be a finite group. What is a *Sylow p -subgroup* of G ?

Assuming that a Sylow p -subgroup H exists, and that the number of conjugates of H is congruent to 1 mod p , prove that all Sylow p -subgroups are conjugate. If n_p denotes the number of Sylow p -subgroups, deduce that

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid |G|.$$

If furthermore G is simple prove that either $G = H$ or

$$|G| \mid n_p!$$

Deduce that a group of order 1,000,000 cannot be simple.

11E Analysis II

State the inverse function theorem for a function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Suppose F is a differentiable bijection with F^{-1} also differentiable. Show that the derivative of F at any point in \mathbb{R}^n is a linear isomorphism.

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function such that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist and are continuous. Assume there is a point $(a, b) \in \mathbb{R}^2$ for which $f(a, b) = 0$ and $\frac{\partial f}{\partial x}(a, b) \neq 0$. Prove that there exist open sets $U \subset \mathbb{R}^2$ and $W \subset \mathbb{R}$ containing (a, b) and b , respectively, such that for every $y \in W$ there exists a unique x such that $(x, y) \in U$ and $f(x, y) = 0$. Moreover, if we define $g : W \rightarrow \mathbb{R}$ by $g(y) = x$, prove that g is differentiable with continuous derivative. Find the derivative of g at b in terms of $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$.

12F Metric and Topological Spaces

A topological space X is said to be *normal* if each point of X is a closed subset of X and for each pair of closed sets $C_1, C_2 \subset X$ with $C_1 \cap C_2 = \emptyset$ there are open sets $U_1, U_2 \subset X$ so that $C_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$. In this case we say that the U_i *separate* the C_i .

Show that a compact Hausdorff space is normal. [*Hint: first consider the case where C_2 is a point.*]

For $C \subset X$ we define an equivalence relation \sim_C on X by $x \sim_C y$ for all $x, y \in C$, $x \sim_C x$ for $x \notin C$. If C, C_1 and C_2 are pairwise disjoint closed subsets of a normal space X , show that C_1 and C_2 may be separated by open subsets U_1 and U_2 such that $U_i \cap C = \emptyset$. Deduce that the quotient space X / \sim_C is also normal.

13A Complex Analysis or Complex Methods

Using Cauchy's integral theorem, write down the value of a holomorphic function $f(z)$ where $|z| < 1$ in terms of a contour integral around the unit circle, $\zeta = e^{i\theta}$.

By considering the point $1/\bar{z}$, or otherwise, show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} d\theta.$$

By setting $z = re^{i\alpha}$, show that for any harmonic function $u(r, \alpha)$,

$$u(r, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} u(1, \theta) \frac{1 - r^2}{1 - 2r \cos(\alpha - \theta) + r^2} d\theta$$

if $r < 1$.

Assuming that the function $v(r, \alpha)$, which is the conjugate harmonic function to $u(r, \alpha)$, can be written as

$$v(r, \alpha) = v(0) + \frac{1}{\pi} \int_0^{2\pi} u(1, \theta) \frac{r \sin(\alpha - \theta)}{1 - 2r \cos(\alpha - \theta) + r^2} d\theta,$$

deduce that

$$f(z) = iv(0) + \frac{1}{2\pi} \int_0^{2\pi} u(1, \theta) \frac{\zeta + z}{\zeta - z} d\theta.$$

[You may use the fact that on the unit circle, $\zeta = 1/\bar{\zeta}$, and hence

$$\frac{\zeta}{\zeta - 1/\bar{\zeta}} = -\frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}}. \quad]$$

14C Methods

Consider the regular Sturm-Liouville (S-L) system

$$(\mathcal{L}y)(x) - \lambda\omega(x)y(x) = 0, \quad a \leq x \leq b,$$

where

$$(\mathcal{L}y)(x) := -[p(x)y'(x)]' + q(x)y(x)$$

with $\omega(x) > 0$ and $p(x) > 0$ for all x in $[a, b]$, and the boundary conditions on y are

$$\begin{cases} A_1 y(a) + A_2 y'(a) = 0 \\ B_1 y(b) + B_2 y'(b) = 0. \end{cases}$$

Show that with these boundary conditions, \mathcal{L} is self-adjoint. By considering $y\mathcal{L}y$, or otherwise, show that the eigenvalue λ can be written as

$$\lambda = \frac{\int_a^b (py'^2 + qy^2) dx - [pyy']_a^b}{\int_a^b \omega y^2 dx}.$$

Now suppose that $a = 0$ and $b = \ell$, that $p(x) = 1$, $q(x) \geq 0$ and $\omega(x) = 1$ for all $x \in [0, \ell]$, and that $A_1 = 1$, $A_2 = 0$, $B_1 = k \in \mathbb{R}^+$ and $B_2 = 1$. Show that the eigenvalues of this regular S-L system are strictly positive. Assuming further that $q(x) = 0$, solve the system explicitly, and with the aid of a graph, show that there exist infinitely many eigenvalues $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$. Describe the behaviour of λ_n as $n \rightarrow \infty$.

15C Quantum Mechanics

Show that if the energy levels are discrete, the general solution of the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{x})\psi$$

is a linear superposition of stationary states

$$\psi(\mathbf{x}, t) = \sum_{n=1}^{\infty} a_n \chi_n(\mathbf{x}) \exp(-iE_n t/\hbar),$$

where $\chi_n(\mathbf{x})$ is a solution of the time-independent Schrödinger equation and a_n are complex coefficients. Can this general solution be considered to be a stationary state? Justify your answer.

A linear operator \hat{O} acts on the orthonormal energy eigenfunctions χ_n as follows:

$$\begin{aligned} \hat{O}\chi_1 &= \chi_1 + \chi_2 \\ \hat{O}\chi_2 &= \chi_1 + \chi_2 \\ \hat{O}\chi_n &= 0, \quad n \geq 3. \end{aligned}$$

Obtain the eigenvalues of \hat{O} . Hence, find the normalised eigenfunctions of \hat{O} . In an experiment a measurement is made of \hat{O} at $t = 0$ yielding an eigenvalue of 2. What is the probability that a measurement at some later time t will yield an eigenvalue of 2?

16B Electromagnetism

A sphere of radius a carries an electric charge Q uniformly distributed over its surface. Calculate the electric field outside and inside the sphere. Also calculate the electrostatic potential outside and inside the sphere, assuming it vanishes at infinity. State the integral formula for the energy U of the electric field and use it to evaluate U as a function of Q .

Relate $\frac{dU}{dQ}$ to the potential on the surface of the sphere and explain briefly the physical interpretation of the relation.

17A Fluid Dynamics

Consider inviscid, incompressible fluid flow confined to the (x, y) plane. The fluid has density ρ , and gravity can be neglected. Using the conservation of volume flux, determine the velocity potential $\phi(r)$ of a point source of strength m , in terms of the distance r from the source.

Two point sources each of strength m are located at $\mathbf{x}_+ = (0, a)$ and $\mathbf{x}_- = (0, -a)$. Find the velocity potential of the flow.

Show that the flow in the region $y \geq 0$ is equivalent to the flow due to a source at \mathbf{x}_+ and a fixed boundary at $y = 0$.

Find the pressure on the boundary $y = 0$ and hence determine the force on the boundary.

[Hint: you may find the substitution $x = a \tan \theta$ useful for the calculation of the pressure.]

18D Numerical Analysis

For a numerical method for solving $y' = f(t, y)$, define the linear stability domain, and state when such a method is A-stable.

Determine all values of the real parameter a for which the Runge-Kutta method

$$\begin{aligned} k_1 &= f\left(t_n + \left(\frac{1}{2} - a\right)h, y_n + \left(\frac{1}{4}hk_1 + \left(\frac{1}{4} - a\right)hk_2\right)\right), \\ k_2 &= f\left(t_n + \left(\frac{1}{2} + a\right)h, y_n + \left(\left(\frac{1}{4} + a\right)hk_1 + \frac{1}{4}hk_2\right)\right), \\ y_{n+1} &= y_n + \frac{1}{2}h(k_1 + k_2) \end{aligned}$$

is A-stable.

19H Statistics

State and prove the Neyman-Pearson lemma.

A sample of two independent observations, (x_1, x_2) , is taken from a distribution with density $f(x; \theta) = \theta x^{\theta-1}$, $0 \leq x \leq 1$. It is desired to test $H_0 : \theta = 1$ against $H_1 : \theta = 2$. Show that the best test of size α can be expressed using the number c such that

$$1 - c + c \log c = \alpha.$$

Is this the uniformly most powerful test of size α for testing H_0 against $H_1 : \theta > 1$?

Suppose that the prior distribution of θ is $P(\theta = 1) = 4\gamma/(1 + 4\gamma)$, $P(\theta = 2) = 1/(1+4\gamma)$, where $1 > \gamma > 0$. Find the test of H_0 against H_1 that minimizes the probability of error.

Let $w(\theta)$ denote the power function of this test at $\theta (\geq 1)$. Show that

$$w(\theta) = 1 - \gamma^\theta + \gamma^\theta \log \gamma^\theta.$$

20H Markov Chains

A Markov chain $(X_n)_{n \geq 0}$ has as its state space the integers, with

$$p_{i,i+1} = p, \quad p_{i,i-1} = q = 1 - p,$$

and $p_{ij} = 0$ otherwise. Assume $p > q$.

Let $T_j = \inf\{n \geq 1 : X_n = j\}$ if this is finite, and $T_j = \infty$ otherwise. Let V_0 be the total number of hits on 0, and let $V_0(n)$ be the total number of hits on 0 within times $0, \dots, n-1$. Let

$$\begin{aligned} h_i &= P(V_0 > 0 \mid X_0 = i) \\ r_i(n) &= E[V_0(n) \mid X_0 = i] \\ r_i &= E[V_0 \mid X_0 = i]. \end{aligned}$$

- (i) Quoting an appropriate theorem, find, for every i , the value of h_i .
- (ii) Show that if $(x_i, i \in \mathbb{Z})$ is any non-negative solution to the system of equations

$$\begin{aligned} x_0 &= 1 + qx_1 + px_{-1}, \\ x_i &= qx_{i-1} + px_{i+1}, \quad \text{for all } i \neq 0, \end{aligned}$$

then $x_i \geq r_i(n)$ for all i and n .

- (iii) Show that $P(V_0(T_1) \geq k \mid X_0 = 1) = q^k$ and $E[V_0(T_1) \mid X_0 = 1] = q/p$.
- (iv) Explain why $r_{i+1} = (q/p)r_i$ for $i > 0$.
- (v) Find r_i for all i .

END OF PAPER