UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday, 5 June, 2012 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups

State Lagrange's Theorem. Deduce that if G is a finite group of order n, then the order of every element of G is a divisor of n.

Let G be a group such that, for every $g \in G$, $g^2 = e$. Show that G is abelian. Give an example of a non-abelian group in which every element g satisfies $g^4 = e$.

2E Groups

What is a *cycle* in the symmetric group S_n? Show that a cycle of length p and a cycle of length q in S_n are conjugate if and only if $p = q$.

Suppose that p is odd. Show that any two p-cycles in A_{p+2} are conjugate. Are any two 3-cycles in A_4 conjugate? Justify your answer.

3C Vector Calculus

Define what it means for a differential $P\,dx + Q\,dy$ to be exact, and derive a necessary condition on $P(x,y)$ and $Q(x,y)$ for this to hold. Show that one of the following two differentials is exact and the other is not:

$$\begin{gathered} y^2\,dx + 2xy\,dy\,, \\ y^2\,dx + xy^2\,dy\,. \end{gathered}$$

Show that the differential which is not exact can be written in the form $g\,df$ for functions $f(x,y)$ and $g(y)$, to be determined.

4C Vector Calculus

What does it mean for a second-rank tensor T_{ij} to be *isotropic*? Show that δ_{ij} is isotropic. By considering rotations through $\pi/2$ about the coordinate axes, or otherwise, show that the most general isotropic second-rank tensor in $\mathbb{R}^3$ has the form $T_{ij} = \lambda\delta_{ij}$, for some scalar λ.

SECTION II

5E Groups

(i) State and prove the Orbit-Stabilizer Theorem.

Show that if G is a finite group of order n, then G is isomorphic to a subgroup of the symmetric group S_n.

(ii) Let G be a group acting on a set X with a single orbit, and let H be the stabilizer of some element of X. Show that the homomorphism $G \to \mathrm{Sym}(X)$ given by the action is injective if and only if the intersection of all the conjugates of H equals $\{e\}$.

(iii) Let Q_8 denote the quaternion group of order 8. Show that for every $n < 8$, Q_8 is not isomorphic to a subgroup of S_n.

6E Groups

Let G be $SL_2(\mathbb{R})$, the groups of *real* 2×2 matrices of determinant 1, acting on $\mathbb{C} \cup \{\infty\}$ by Möbius transformations.

For each of the points 0, i, $-i$, compute its stabilizer and its orbit under the action of G. Show that G has exactly 3 orbits in all.

Compute the orbit of i under the subgroup

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a,\ b,\ d \in \mathbb{R},\ ad = 1 \right\} \subset G\,.$$

Deduce that every element g of G may be expressed in the form $g = hk$ where $h \in H$ and for some $\theta \in \mathbb{R}$,

$$k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\,.$$

How many ways are there of writing g in this form?

7E Groups

Let $\mathbb{F}_p$ be the set of (residue classes of) integers mod p, and let

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\, b,\, c,\, d \in \mathbb{F}_p,\ ad - bc \neq 0 \right\}$$

Show that G is a group under multiplication. [You may assume throughout this question that multiplication of matrices is associative.]

Let X be the set of 2-dimensional column vectors with entries in $\mathbb{F}_p$. Show that the mapping $G \times X \to X$ given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

is a group action.

Let $g \in G$ be an element of order p. Use the orbit-stabilizer theorem to show that there exist $x,\, y \in \mathbb{F}_p$, not both zero, with

$$g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} .$$

Deduce that g is conjugate in G to the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

8E Groups

Let p be a prime number, and a an integer with $1 \leqslant a \leqslant p-1$. Let G be the Cartesian product

$$G = \{\, (x,u) \mid x \in \{0,1,\ldots,p-2\},\ u \in \{0,1,\ldots,p-1\} \,\}$$

Show that the binary operation

$$(x,u) * (y,v) = (z,w)$$

where

$$\begin{aligned} z &\equiv x + y \pmod{p-1} \\ w &\equiv a^y u + v \pmod{p} \end{aligned}$$

makes G into a group. Show that G is abelian if and only if $a = 1$.

Let H and K be the subsets

$$H = \{\, (x,0) \mid x \in \{0,1,\ldots,p-2\} \,\}, \qquad K = \{\, (0,u) \mid u \in \{0,1,\ldots,p-1\} \,\}$$

of G. Show that K is a normal subgroup of G, and that H is a subgroup which is normal if and only if $a = 1$.

Find a homomorphism from G to another group whose kernel is K.

9C Vector Calculus

State Stokes' Theorem for a vector field $\mathbf{B}(\mathbf{x})$ on $\mathbb{R}^3$.

Consider the surface S defined by

$$z = x^2 + y^2, \qquad \frac{1}{9} \leqslant z \leqslant 1.$$

Sketch the surface and calculate the area element $d\mathbf{S}$ in terms of suitable coordinates or parameters. For the vector field

$$\mathbf{B} = (-y^3, x^3, z^3)$$

compute $\nabla \times \mathbf{B}$ and calculate $I = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S}$.

Use Stokes' Theorem to express I as an integral over ∂S and verify that this gives the same result.

10C Vector Calculus

Consider the transformation of variables

$$x = 1 - u, \quad y = \frac{1-v}{1-uv}.$$

Show that the interior of the unit square in the uv plane

$$\{(u,v) : \ 0 < u < 1, \ 0 < v < 1\}$$

is mapped to the interior of the unit square in the xy plane,

$$R = \{(x,y) : \ 0 < x < 1, \ 0 < y < 1\}.$$

[*Hint: Consider the relation between* v *and* y *when* $u = \alpha$, *for* $0 < \alpha < 1$ *constant.*]

Show that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{(1-(1-x)y)^2}{x}.$$

Now let

$$u = \frac{1-t}{1-wt}, \quad v = 1 - w.$$

By calculating

$$\frac{\partial(x,y)}{\partial(t,w)} = \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(t,w)}$$

as a function of x and y, or otherwise, show that

$$\int_R \frac{x(1-y)}{(1-(1-x)y)(1-(1-x^2)y)^2} \, dx \, dy = 1.$$

11C Vector Calculus

(a) Prove the identity

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}).$$

(b) If $\mathbf{E}$ is an irrotational vector field (i.e. $\nabla\times\mathbf{E} = \mathbf{0}$ everywhere), prove that there exists a scalar potential $\phi(\mathbf{x})$ such that $\mathbf{E} = -\nabla\phi$.

Show that the vector field

$$(xy^2ze^{-x^2z}, -ye^{-x^2z}, \tfrac{1}{2}x^2y^2e^{-x^2z})$$

is irrotational, and determine the corresponding potential ϕ.

12C Vector Calculus

(i) Let V be a bounded region in $\mathbb{R}^3$ with smooth boundary $S = \partial V$. Show that Poisson's equation in V

$$\nabla^2 u = \rho$$

has at most one solution satisfying $u = f$ on S, where ρ and f are given functions.

Consider the alternative boundary condition $\partial u/\partial n = g$ on S, for some given function g, where n is the outward pointing normal on S. Derive a necessary condition in terms of ρ and g for a solution u of Poisson's equation to exist. Is such a solution unique?

(ii) Find the most general spherically symmetric function $u(r)$ satisfying

$$\nabla^2 u = 1$$

in the region $r = |\mathbf{r}| \leqslant a$ for $a > 0$. Hence in each of the following cases find all possible solutions satisfying the given boundary condition at $r = a$:

(a) $u = 0\,,$

(b) $\frac{\partial u}{\partial n} = 0\,.$

Compare these with your results in part (i).

END OF PAPER