UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday, 1 June, 2012 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Differential Equations

Find two linearly independent solutions of

$$y'' + 4y' + 4y = 0\,.$$

Find the solution in $x \geqslant 0$ of

$$y'' + 4y' + 4y = e^{-2x}\,,$$

subject to $y = y' = 0$ at $x = 0$.

2A Differential Equations

Find the constant solutions (those with $u_{n+1} = u_n$) of the discrete equation

$$u_{n+1} = \tfrac{1}{2}u_n\,(1 + u_n)\ ,$$

and determine their stability.

3F Probability

Given two events A and B with $P(A) > 0$ and $P(B) > 0$, define the conditional probability $P(A \mid B)$.

Show that

$$P(B \mid A) = P(A \mid B)\frac{P(B)}{P(A)}\,.$$

A random number N of fair coins are tossed, and the total number of heads is denoted by H. If $P(N = n) = 2^{-n}$ for $n = 1, 2, \ldots$, find $P(N = n \mid H = 1)$.

4F Probability

Define the *probability generating function* $G(s)$ of a random variable X taking values in the non-negative integers.

A coin shows heads with probability $p \in (0, 1)$ on each toss. Let N be the number of tosses up to and including the first appearance of heads, and let $k \geqslant 1$. Find the probability generating function of $X = \min\{N, k\}$.

Show that $E(X) = p^{-1}(1 - q^k)$ where $q = 1 - p$.

SECTION II

5A Differential Equations

Find the first three non-zero terms in the series solutions $y_1(x)$ and $y_2(x)$ for the differential equation

$$x^2y'' - 2xy' + (2 - x^2)y = 0\,,$$

that satisfy

$$y_1'(0) = a \quad \text{and} \quad y_1''(0) = 0\,,$$
$$y_2'(0) = 0 \quad \text{and} \quad y_2''(0) = 2b\,.$$

Identify these solutions in closed form.

6A Differential Equations

Consider the function

$$V(x, y) = x^4 - x^2 + 2xy + y^2.$$

Find the critical (stationary) points of $V(x, y)$. Determine the type of each critical point. Sketch the contours of $V(x, y) = \text{constant}$.

Now consider the coupled differential equations

$$\frac{dx}{dt} = -\frac{\partial V}{\partial x}, \qquad \frac{dy}{dt} = -\frac{\partial V}{\partial y}.$$

Show that $V(x(t), y(t))$ is a non-increasing function of t. If $x = 1$ and $y = -\frac{1}{2}$ at $t = 0$, where does the solution tend to as $t \to \infty$?

7A Differential Equations

Find the solution to the system of equations

$$\frac{dx}{dt} + \frac{-4x + 2y}{t} = -9,$$
$$\frac{dy}{dt} + \frac{x - 5y}{t} = 3$$

in $t \geqslant 1$ subject to

$$x = 0 \quad \text{and} \quad y = 0 \quad \text{at} \quad t = 1.$$

[*Hint: powers of t.*]

8A Differential Equations

Consider the second-order differential equation for $y(t)$ in $t \geqslant 0$

$$\ddot{y} + 2k\dot{y} + (k^2 + \omega^2)y = f(t)\,. \qquad (*)$$

(i) For $f(t) = 0$, find the general solution $y_1(t)$ of $(*)$.

(ii) For $f(t) = \delta(t - a)$ with $a > 0$, find the solution $y_2(t, a)$ of $(*)$ that satisfies $y = 0$ and $\dot{y} = 0$ at $t = 0$.

(iii) For $f(t) = H(t - b)$ with $b > 0$, find the solution $y_3(t, b)$ of $(*)$ that satisfies $y = 0$ and $\dot{y} = 0$ at $t = 0$.

(iv) Show that

$$y_2(t, b) = -\frac{\partial y_3}{\partial b}\,.$$

9F Probability

(i) Define the *moment generating function* $M_X(t)$ of a random variable X. If X, Y are independent and $a, b \in \mathbb{R}$, show that the moment generating function of $Z = aX + bY$ is $M_X(at)M_Y(bt)$.

(ii) Assume $T > 0$, and $M_X(t) < \infty$ for $|t| < T$. Explain the expansion

$$M_X(t) = 1 + \mu t + \tfrac{1}{2}s^2 t^2 + \mathrm{o}(t^2)$$

where $\mu = E(X)$ and $s^2 = E(X^2)$. [You may assume the validity of interchanging expectation and differentiation.]

(iii) Let X, Y be independent, identically distributed random variables with mean 0 and variance 1, and assume their moment generating function M satisfies the condition of part (ii) with $T = \infty$.

Suppose that $X + Y$ and $X - Y$ are independent. Show that $M(2t) = M(t)^3 M(-t)$, and deduce that $\psi(t) = M(t)/M(-t)$ satisfies $\psi(t) = \psi(t/2)^2$.

Show that $\psi(h) = 1 + \mathrm{o}(h^2)$ as $h \to 0$, and deduce that $\psi(t) = 1$ for all t.

Show that X and Y are normally distributed.

10F Probability

(i) Define the distribution function F of a random variable X, and also its density function f assuming F is differentiable. Show that

$$f(x) = -\frac{d}{dx} P(X > x)\,.$$

(ii) Let U, V be independent random variables each with the uniform distribution on $[0,1]$. Show that

$$P(V^2 > U > x) = \tfrac{1}{3} - x + \tfrac{2}{3}x^{3/2}, \qquad x \in (0,1)\,.$$

What is the probability that the random quadratic equation $x^2 + 2Vx + U = 0$ has real roots?

Given that the two roots R_1, R_2 of the above quadratic are real, what is the probability that both $|R_1| \leqslant 1$ and $|R_2| \leqslant 1$?

11F Probability

(i) Let X_n be the size of the n^{th} generation of a branching process with family-size probability generating function $G(s)$, and let $X_0 = 1$. Show that the probability generating function $G_n(s)$ of X_n satisfies $G_{n+1}(s) = G(G_n(s))$ for $n \geqslant 0$.

(ii) Suppose the family-size mass function is $P(X_1 = k) = 2^{-k-1}$, $k = 0, 1, 2, \ldots$. Find $G(s)$, and show that

$$G_n(s) = \frac{n - (n-1)s}{n + 1 - ns} \qquad \text{for } |s| < 1 + \frac{1}{n}\,.$$

Deduce the value of $P(X_n = 0)$.

(iii) Write down the moment generating function of X_n/n. Hence or otherwise show that, for $x \geqslant 0$,

$$P(X_n/n > x \mid X_n > 0) \to e^{-x} \qquad \text{as } n \to \infty\,.$$

[You may use the continuity theorem but, if so, should give a clear statement of it.]

12F Probability

Let X, Y be independent random variables with distribution functions F_X, F_Y. Show that $U = \min\{X, Y\}$, $V = \max\{X, Y\}$ have distribution functions

$$F_U(u) = 1 - (1 - F_X(u))(1 - F_Y(u)), \quad F_V(v) = F_X(v)F_Y(v).$$

Now let X, Y be independent random variables, each having the exponential distribution with parameter 1. Show that U has the exponential distribution with parameter 2, and that $V - U$ is independent of U.

Hence or otherwise show that V has the same distribution as $X + \frac{1}{2}Y$, and deduce the mean and variance of V.

[You may use without proof that X has mean 1 and variance 1.]

END OF PAPER