20H Markov Chains

A Markov chain $(X_n)_{n\geqslant 0}$ has as its state space the integers, with

$$p_{i,i+1} = p, \quad p_{i,i-1} = q = 1 - p,$$

and $p_{ij} = 0$ otherwise. Assume $p > q$.

Let $T_j = \inf\{n \geqslant 1 : X_n = j\}$ if this is finite, and $T_j = \infty$ otherwise. Let V_0 be the total number of hits on 0, and let $V_0(n)$ be the total number of hits on 0 within times $0, \ldots, n-1$. Let

$$\begin{aligned} h_i &= P(V_0 > 0 \mid X_0 = i) \\ r_i(n) &= E[V_0(n) \mid X_0 = i] \\ r_i &= E[V_0 \mid X_0 = i]. \end{aligned}$$

(i) Quoting an appropriate theorem, find, for every i, the value of h_i.

(ii) Show that if $(x_i,\ i \in \mathbb{Z})$ is any non-negative solution to the system of equations

$$\begin{aligned} x_0 &= 1 + px_1 + qx_{-1}, \\ x_i &= qx_{i-1} + px_{i+1}, \quad \text{for all } i \neq 0, \end{aligned}$$

then $x_i \geqslant r_i(n)$ for all i and n.

(iii) Show that $P(V_0(T_1) \geqslant k \mid X_0 = 1) = q^k$ and $E[V_0(T_1) \mid X_0 = 1] = q/p$.

(iv) Explain why $r_{i+1} = (q/p)r_i$ for $i > 0$.

(v) Find r_i for all i.