UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 9 June, 2011 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *. . . ,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

(i) State Lagrange's Theorem, and prove that, if p is an odd prime,

$$(p-1)! \equiv -1 \mod p\,.$$

(ii) Still assuming p is an odd prime, prove that

$$3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{\frac{p+1}{2}} \mod p\,.$$

2F Topics in Analysis

Let $\Gamma = \{z \in \mathbb{C} \,:\, z \neq 1,\ |\mathrm{Re}(z)| + |\mathrm{Im}(z)| = 1\}$.

(i) Prove that, for any $\zeta \in \mathbb{C}$ with $|\mathrm{Re}(\zeta)| + |\mathrm{Im}(\zeta)| > 1$ and any $\epsilon > 0$, there exists a complex polynomial p such that

$$\sup_{z\in\Gamma} |p(z) - (z-\zeta)^{-1}| < \epsilon\,.$$

(ii) Does there exist a sequence of polynomials p_n such that $p_n(z) \to (z-1)^{-1}$ for every $z \in \Gamma$? Justify your answer.

3G Geometry and Groups

Define a *Kleinian group.*

Give an example of a Kleinian group that is a free group on two generators and explain why it has this property.

4G Coding and Cryptography

What is the rank of a binary linear code C? What is the weight enumeration polynomial W_C of C?

Show that $W_C(1,1) = 2^r$ where r is the rank of C. Show that $W_C(s,t) = W_C(t,s)$ for all s and t if and only if $W_C(1,0) = 1$.

Find, with reasons, the weight enumeration polynomial of the repetition code of length n, and of the simple parity check code of length n.

5J Statistical Modelling

Define a generalised linear model for a sample $Y_1, \ldots, Y_n$ of independent random variables. Define further the concept of the link function. Define the binomial regression model with logistic and probit link functions. Which of these is the canonical link function?

6B Mathematical Biology

The dynamics of a directly transmitted microparasite can be modelled by the system

$$\begin{aligned} \frac{dX}{dt} &= bN - \beta XY - bX\,, \\ \frac{dY}{dt} &= \beta XY - (b+r)Y\,, \\ \frac{dZ}{dt} &= rY - bZ\,, \end{aligned}$$

where b, β and r are positive constants and X, Y and Z are respectively the numbers of susceptible, infected and immune (i.e. infected by the parasite, but showing no further symptoms of infection) individuals in a population of size N, independent of t, where $N = X + Y + Z$.

Consider the possible steady states of these equations. Show that there is a threshold population size N_c such that if $N < N_c$ there is no steady state with the parasite maintained in the population. Show that in this case the number of infected and immune individuals decreases to zero for all possible initial conditions.

Show that for $N > N_c$ there is a possible steady state with $X = X_s < N$ and $Y = Y_s > 0$, and find expressions for X_s and Y_s.

By linearising the equations for dX/dt and dY/dt about the steady state $X = X_s$ and $Y = Y_s$, derive a quadratic equation for the possible growth or decay rate in terms of X_s and Y_s and hence show that the steady state is stable.

7C Dynamical Systems

For the map $x_{n+1} = \lambda x_n(1 - x_n^2)$, with $\lambda > 0$, show the following:

(i) If $\lambda < 1$, then the origin is the only fixed point and is stable.

(ii) If $\lambda > 1$, then the origin is unstable. There are two further fixed points which are stable for $1 < \lambda < 2$ and unstable for $\lambda > 2$.

(iii) If $\lambda < 3\sqrt{3}/2$, then x_n has the same sign as the starting value x_0 if $|x_0| < 1$.

(iv) If $\lambda < 3$, then $|x_{n+1}| < 2\sqrt{3}/3$ when $|x_n| < 2\sqrt{3}/3$. Deduce that iterates starting sufficiently close to the origin remain bounded, though they may change sign.

[*Hint: For (iii) and (iv) a graphical representation may be helpful.*]

8E Further Complex Methods

Explain the meaning of z_j in the Weierstrass canonical product formula

$$f(z) = f(0)\exp\left[\frac{f'(0)}{f(0)}z\right]\prod_{j=1}^{\infty}\left\{\left(1 - \frac{z}{z_j}\right)e^{\frac{z}{z_j}}\right\}.$$

Show that

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

Deduce that

$$\pi\cot(\pi z) = \frac{1}{z} + 2\sum_{n=1}^{\infty}\frac{z}{z^2 - n^2}.$$

9C Classical Dynamics

The Lagrangian for a heavy symmetric top is

$$L = \frac{1}{2}I_1\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot\psi + \dot\phi\cos\theta\right)^2 - Mgl\cos\theta\,.$$

State Noether's Theorem. Hence, or otherwise, find two conserved quantities linear in momenta, and a third conserved quantity quadratic in momenta.

Writing $\mu = \cos\theta$, deduce that μ obeys an equation of the form

$$\dot\mu^2 = F(\mu)\,,$$

where $F(\mu)$ is cubic in μ. [You need not determine the explicit form of $F(\mu)$.]

10E Cosmology

For an ideal gas of fermions of mass m in volume V, and at temperature T and chemical potential μ, the number density n and kinetic energy E are given by

$$n = \frac{4\pi g_s}{h^3}\int_0^\infty \bar{n}(p)\, p^2\, dp\,, \qquad E = \frac{4\pi g_s}{h^3}V\int_0^\infty \bar{n}(p)\epsilon(p)p^2\, dp\,,$$

where g_s is the spin-degeneracy factor, h is Planck's constant, $\epsilon(p) = c\sqrt{p^2+m^2c^2}$ is the single-particle energy as a function of the momentum p, and

$$\bar{n}(p) = \left[\exp\left(\frac{\epsilon(p)-\mu}{kT}\right)+1\right]^{-1},$$

where k is Boltzmann's constant.

(i) Sketch the function $\bar{n}(p)$ at zero temperature, explaining why $\bar{n}(p) = 0$ for $p > p_F$ (the Fermi momentum). Find an expression for n at zero temperature as a function of p_F.

Assuming that a typical fermion is ultra-relativistic ($pc \gg mc^2$) even at zero temperature, obtain an estimate of the energy density E/V as a function of p_F, and hence show that

$$E \sim hc\, n^{4/3} V \qquad (*)$$

in the ultra-relativistic limit at zero temperature.

(ii) A white dwarf star of radius R has total mass $M = \frac{4\pi}{3} m_p n_p R^3$, where m_p is the proton mass and n_p the average proton number density. On the assumption that the star's degenerate electrons are ultra-relativistic, so that $(*)$ applies with n replaced by the average electron number density n_e, deduce the following estimate for the star's internal kinetic energy:

$$E_{\rm kin} \sim hc\left(\frac{M}{m_p}\right)^{4/3}\frac{1}{R}\,.$$

By comparing this with the total gravitational potential energy, briefly discuss the consequences for white dwarf stability.

SECTION II

11I Number Theory

Let $\zeta(s)$ be the Riemann zeta function, and put $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$.

(i) If $\sigma > 1$, prove that

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

where the product is taken over all primes p.

(ii) Assuming that, for $\sigma > 1$, we have

$$\zeta(s) = \sum_{n=1}^{\infty} n(n^{-s} - (n+1)^{-s}),$$

prove that $\zeta(s) - \frac{1}{s-1}$ has an analytic continuation to the half plane $\sigma > 0$.

12F Topics in Analysis

Let $f : [0,1] \to \mathbb{R}$ be continuous and let n be a positive integer. For $g : [0,1] \to \mathbb{R}$ a continuous function, write $\|f-g\|_{L^\infty} = \sup_{x\in[0,1]} |f(x)-g(x)|$.

(i) Let p be a polynomial of degree at most n with the property that there are $(n+2)$ distinct points $x_1, x_2, \ldots, x_{n+2} \in [0,1]$ with $x_1 < x_2 < \ldots < x_{n+2}$ such that

$$f(x_j) - p(x_j) = (-1)^j \|f-p\|_{L^\infty}$$

for each $j = 1, 2, \ldots, n+2$. Prove that

$$\|f-p\|_{L^\infty} \leqslant \|f-q\|_{L^\infty}$$

for every polynomial q of degree at most n.

(ii) Prove that there exists a polynomial p of degree at most n such that

$$\|f-p\|_{L^\infty} \leqslant \|f-q\|_{L^\infty}$$

for every polynomial q of degree at most n.

[If you deduce this from a more general result about abstract normed spaces, you must prove that result.]

(iii) Let $Y = \{y_1, y_2, \ldots, y_{n+2}\}$ be any set of $(n+2)$ distinct points in $[0,1]$.

(a) For $j = 1, 2, \ldots, n+2$, let

$$r_j(x) = \prod_{k=1,\, k\neq j}^{n+2} \frac{x-y_k}{y_j-y_k},$$

$t(x) = \sum_{j=1}^{n+2} f(y_j) r_j(x)$ and $r(x) = \sum_{j=1}^{n+2} (-1)^j r_j(x)$. Explain why there is a unique number $\lambda \in \mathbb{R}$ such that the degree of the polynomial $t - \lambda r$ is at most n.

(b) Let $\|f-g\|_{L^\infty(Y)} = \sup_{x\in Y} |f(x)-g(x)|$. Deduce from part (a) that there exists a polynomial p of degree at most n such that

$$\|f-p\|_{L^\infty(Y)} \leqslant \|f-q\|_{L^\infty(Y)}$$

for every polynomial q of degree at most n.

13B Mathematical Biology

The number density of a population of amoebae is $n(\mathbf{x}, t)$. The amoebae exhibit chemotaxis and are attracted to high concentrations of a chemical which has concentration $a(\mathbf{x}, t)$. The equations governing n and a are

$$\frac{\partial n}{\partial t} = \alpha n(n_0^2 - n^2) + \nabla^2 n - \boldsymbol{\nabla} \cdot (\chi(n) n \boldsymbol{\nabla} a)\,,$$

$$\frac{\partial a}{\partial t} = \beta n - \gamma a + D\nabla^2 a\,,$$

where the constants n_0, α, β, γ and D are all positive.

(i) Give a biological interpretation of each term in these equations and discuss the sign of $\chi(n)$.

(ii) Show that there is a non-trivial (i.e. $a \neq 0$, $n \neq 0$) steady-state solution for n and a, independent of $\mathbf{x}$, and show further that it is stable to small disturbances that are also independent of $\mathbf{x}$.

(iii) Consider small spatially varying disturbances to the steady state, with spatial structure such that $\nabla^2 \psi = -k^2 \psi$, where ψ is any disturbance quantity. Show that if such disturbances also satisfy $\partial \psi / \partial t = p \psi$, where p is a constant, then p satisfies a quadratic equation, to be derived. By considering the conditions required for $p = 0$ to be a possible solution of this quadratic equation, or otherwise, deduce that instability is possible if

$$\beta \chi_0 n_0 > 2\alpha n_0^2 D + \gamma + 2(2D\alpha n_0^2 \gamma)^{1/2}\,,$$

where $\chi_0 = \chi(n_0)$.

(iv) Explain briefly how your conclusions might change if an additional geometric constraint implied that $k^2 > k_0^2$, where k_0 is a given constant.

14C Dynamical Systems

Explain what is meant by a *steady-state bifurcation* of a fixed point $\mathbf{x}_0(\mu)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$ in $\mathbb{R}^n$, where μ is a real parameter.

Consider the system in $x \geqslant 0$, $y \geqslant 0$, with $\mu > 0$,

$$\begin{aligned}\dot{x} &= x(1 - y^2 - x^2)\,, \\ \dot{y} &= y(\mu - y - x^2)\,.\end{aligned}$$

(i) Show that both the fixed point $(0, \mu)$ and the fixed point $(1, 0)$ have a steady-state bifurcation when $\mu = 1$.

(ii) By finding the first approximation to the extended centre manifold, construct the normal form near the bifurcation point $(1, 0)$ when μ is close to unity, and show that there is a transcritical bifurcation there. Explain why the symmetries of the equations mean that the bifurcation at $(0, 1)$ must be of pitchfork type.

(iii) Show that two fixed points with $x, y > 0$ exist in the range $1 < \mu < 5/4$. Show that the solution with $y < 1/2$ is stable. Identify the bifurcation that occurs at $\mu = 5/4$.

(iv) Draw a sketch of the values of y at the fixed points as functions of μ, indicating the bifurcation points and the regions where each branch is stable. [Detailed calculations are not required.]

15E Cosmology

An expanding universe with scale factor $a(t)$ is filled with (pressure-free) cold dark matter (CDM) of average mass density $\bar{\rho}(t)$. In the Zel'dovich approximation to gravitational clumping, the perturbed position $\mathbf{r}(\mathbf{q},t)$ of a CDM particle with unperturbed comoving position $\mathbf{q}$ is given by

$$\mathbf{r}(\mathbf{q},t) = a(t)[\mathbf{q} + \boldsymbol{\psi}(\mathbf{q},t)]\,, \qquad (1)$$

where $\boldsymbol{\psi}$ is the comoving displacement.

(i) Explain why the conservation of CDM particles implies that

$$\rho(\mathbf{r},t)\, d^3r = a^3\bar{\rho}(t)\, d^3q\,,$$

where $\rho(\mathbf{r},t)$ is the CDM mass density. Use (1) to verify that $d^3q = a^{-3}[1 - \boldsymbol{\nabla}_\mathbf{q}\cdot\boldsymbol{\psi}]d^3r$, and hence deduce that the fractional density perturbation is, to first order,

$$\delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} = -\boldsymbol{\nabla}_\mathbf{q}\cdot\boldsymbol{\psi}\,.$$

Use this result to integrate the Poisson equation $\nabla^2\Phi = 4\pi G\bar{\rho}$ for the gravitational potential Φ. Then use the particle equation of motion $\ddot{\mathbf{r}} = -\boldsymbol{\nabla}\Phi$ to deduce a second-order differential equation for $\boldsymbol{\psi}$, and hence that

$$\ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\dot{\delta} - 4\pi G\bar{\rho}\,\delta = 0\,. \qquad (2)$$

[You may assume that $\nabla^2\Phi = 4\pi G\bar{\rho}$ implies $\boldsymbol{\nabla}\Phi = (4\pi G/3)\bar{\rho}\,\mathbf{r}$ and that the pressure-free acceleration equation is $\ddot{a} = -(4\pi G/3)\bar{\rho}a$.]

(ii) A flat matter-dominated universe with background density $\bar{\rho} = (6\pi G t^2)^{-1}$ has scale factor $a(t) = (t/t_0)^{2/3}$. The universe is filled with a pressure-free homogeneous (non-clumping) fluid of mass density $\rho_H(t)$, as well as cold dark matter of mass density $\rho_C(\mathbf{r},t)$.

Assuming that the Zel'dovich perturbation equation in this case is as in (2) but with $\bar{\rho}$ replaced by $\bar{\rho}_C$, i.e. that

$$\ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\dot{\delta} - 4\pi G\bar{\rho}_C\delta = 0\,,$$

seek power-law solutions $\delta \propto t^\alpha$ to find growing and decaying modes with

$$\alpha = \frac{1}{6}\left(-1 \pm \sqrt{25 - 24\,\Omega_H}\right)\,,$$

where $\Omega_H = \rho_H/\bar{\rho}$.

Given that matter domination starts ($t = t_{\rm eq}$) at a redshift $z \approx 10^5$, and given an initial perturbation $\delta(t_{\rm eq}) \approx 10^{-5}$, show that $\Omega_H = 2/3$ yields a model that is not compatible with the large-scale structure observed today.

16H Logic and Set Theory

State and prove the Upward Löwenheim–Skolem Theorem.

[You may assume the Compactness Theorem, provided that you state it clearly.]

A total ordering $(X, <)$ is called *dense* if for any $x < y$ there exists z with $x < z < y$. Show that a dense total ordering (on more than one point) cannot be a well-ordering.

For each of the following theories, either give axioms, in the language of posets, for the theory or prove carefully that the theory is not axiomatisable in the language of posets.

(i) The theory of dense total orderings.

(ii) The theory of countable dense total orderings.

(iii) The theory of uncountable dense total orderings.

(iv) The theory of well-orderings.

17F Graph Theory

Define the *Turán graph* $T_r(n)$. State and prove Turán's theorem. Hence, or otherwise, find $\mathrm{ex}(K_3; n)$.

Let G be a bipartite graph with n vertices in each class. Let k be an integer, $1 \leqslant k \leqslant n$, and assume $e(G) > (k-1)n$. Show that G contains a set of k independent edges.

[*Hint: Suppose G contains a set D of a independent edges but no set of $a+1$ independent edges. Let U be the set of vertices of the edges in D and let F be the set of edges in G with precisely one vertex in U; consider $|F|$.*]

Hence, or otherwise, show that if H is a triangle-free tripartite graph with n vertices in each class then $e(H) \leqslant 2n^2$.

18H Galois Theory

Let $n \geqslant 1$ and $K = \mathbb{Q}(\boldsymbol{\mu}_n)$ be the cyclotomic field generated by the nth roots of unity. Let $a \in \mathbb{Q}$ with $a \neq 0$, and consider $F = K(\sqrt[n]{a})$.

(i) State, without proof, the theorem which determines $\mathrm{Gal}(K/\mathbb{Q})$.

(ii) Show that $F/\mathbb{Q}$ is a Galois extension and that $\mathrm{Gal}(F/\mathbb{Q})$ is soluble. [When using facts about general Galois extensions and their generators, you should state them clearly.]

(iii) When $n = p$ is prime, list all possible degrees $[F : \mathbb{Q}]$, with justification.

19I Representation Theory

Define the character $\mathrm{Ind}_H^G\psi$ of a finite group G which is induced by a character ψ of a subgroup H of G.

State and prove the Frobenius reciprocity formula for the characters ψ of H and χ of G.

Now suppose that H has index 2 in G. An irreducible character ψ of H and an irreducible character χ of G are said to be 'related' if

$$\langle \mathrm{Ind}_H^G\psi, \chi\rangle_G = \langle \psi, \mathrm{Res}_H^G\chi\rangle_H > 0.$$

Show that each ψ of degree d is either 'monogamous' in the sense that it is related to one χ (of degree $2d$), or 'bigamous' in the sense that it is related to precisely two distinct characters χ_1, χ_2 (of degree d). Show that each χ is related to one bigamous ψ, or to two monogamous characters ψ_1, ψ_2 (of the same degree).

Write down the degrees of the complex irreducible characters of the alternating group A_5. Find the degrees of the irreducible characters of a group G containing A_5 as a subgroup of index 2, distinguishing two possible cases.

20H Algebraic Topology

Let K and L be (finite) simplicial complexes. Explain carefully what is meant by a *simplicial approximation* to a continuous map $f\colon |K| \to |L|$. Indicate briefly how the cartesian product $|K| \times |L|$ may be triangulated.

Two simplicial maps $g, h\colon K \to L$ are said to be *contiguous* if, for each simplex σ of K, there exists a simplex $\sigma*$ of L such that both $g(\sigma)$ and $h(\sigma)$ are faces of $\sigma*$. Show that:

(i) any two simplicial approximations to a given map $f\colon |K| \to |L|$ are contiguous;

(ii) if g and h are contiguous, then they induce homotopic maps $|K| \to |L|$;

(iii) if f and g are homotopic maps $|K| \to |L|$, then for some subdivision $K^{(n)}$ of K there exists a sequence $(h_1, h_2, \ldots, h_m)$ of simplicial maps $K^{(n)} \to L$ such that h_1 is a simplicial approximation to f, h_m is a simplicial approximation to g and each pair (h_i, h_{i+1}) is contiguous.

21G Linear Analysis

Let H be a complex Hilbert space with orthonormal basis $(e_n)_{n=-\infty}^{\infty}$. Let $T : H \to H$ be a bounded linear operator. What is meant by the spectrum $\sigma(T)$ of T?

Define T by setting $T(e_n) = e_{n-1} + e_{n+1}$ for $n \in \mathbb{Z}$. Show that T has a unique extension to a bounded, self-adjoint linear operator on H. Determine the norm $\|T\|$. Exhibit, with proof, an element of $\sigma(T)$.

Show that T has no eigenvectors. Is T compact?

[General results from spectral theory may be used without proof. You may also use the fact that if a sequence (x_n) satisfies a linear recurrence $\lambda x_n = x_{n-1} + x_{n+1}$ with $\lambda \in \mathbb{R}$, $|\lambda| \leqslant 2$, $\lambda \neq 0$, then it has the form $x_n = A\alpha^n \sin(\theta_1 n + \theta_2)$ or $x_n = (A + nB)\alpha^n$, where $A, B, \alpha \in \mathbb{R}$ and $0 \leqslant \theta_1 < \pi$, $|\theta_2| \leqslant \pi/2$.]

22G Riemann Surfaces

State the Classical Monodromy Theorem for analytic continuations in subdomains of the plane.

Let n, r be positive integers with $r > 1$ and set $h(z) = z^n - 1$. By removing n semi-infinite rays from $\mathbb{C}$, find a subdomain $U \subset \mathbb{C}$ on which an analytic function $h^{1/r}$ may be defined, justifying this assertion. Describe *briefly* a gluing procedure which will produce the Riemann surface R for the complete analytic function $h^{1/r}$.

Let Z denote the set of nth roots of unity and assume that the natural analytic covering map $\pi : R \to \mathbb{C} \setminus Z$ extends to an analytic map of Riemann surfaces $\tilde{\pi} : \tilde{R} \to \mathbb{C}_\infty$, where $\tilde{R}$ is a compactification of R and $\mathbb{C}_\infty$ denotes the extended complex plane. Show that $\tilde{\pi}$ has precisely n branch points if and only if r divides n.

23H Algebraic Geometry

Let X be a smooth projective curve over an algebraically closed field k of characteristic 0.

(i) Let D be a divisor on X.

Define $\mathcal{L}(D)$, and show $\dim \mathcal{L}(D) \leqslant \deg D + 1$.

(ii) Define the space of *rational differentials* $\Omega^1_{k(X)/k}$.

If p is a point on X, and t a local parameter at p, show that $\Omega^1_{k(X)/k} = k(X)dt$.

Use that equality to give a definition of $v_p(\omega) \in \mathbb{Z}$, for $\omega \in \Omega^1_{k(X)/k}$, $p \in X$. [You need not show that your definition is independent of the choice of local parameter.]

24I Differential Geometry

For an oriented surface S in $\mathbb{R}^3$, define the *Gauss map*, the *second fundamental form* and the *normal curvature* in the direction $w \in T_pS$ at a point $p \in S$.

Let $\tilde{k}_1, \ldots, \tilde{k}_m$ be normal curvatures at p in the directions $v_1, \ldots, v_m$, such that the angle between v_i and v_{i+1} is π/m for each $i = 1, \ldots, m-1$ ($m \geqslant 2$). Show that

$$\tilde{k}_1 + \ldots + \tilde{k}_m = mH \,,$$

where H is the mean curvature of S at p.

What is a minimal surface? Show that if S is a minimal surface, then its Gauss map N at each point $p \in S$ satisfies

$$\langle dN_p(w_1), dN_p(w_2)\rangle = \mu(p)\langle w_1, w_2\rangle \,, \quad \text{for all } w_1, w_2 \in T_pS \,, \qquad (*)$$

where $\mu(p) \in \mathbb{R}$ depends only on p. Conversely, if the identity $(*)$ holds at each point in S, must S be minimal? Justify your answer.

25K Probability and Measure

(i) State and prove Kolmogorov's zero-one law.

(ii) Let $(E, \mathcal{E}, \mu)$ be a finite measure space and suppose that $(B_n)_{n \geqslant 1}$ is a sequence of events such that $B_{n+1} \subset B_n$ for all $n \geqslant 1$. Show carefully that $\mu(B_n) \to \mu(B)$, where $B = \cap_{n=1}^{\infty} B_n$.

(iii) Let $(X_i)_{i \geqslant 1}$ be a sequence of independent and identically distributed random variables such that $\mathbb{E}(X_1^2) = \sigma^2 < \infty$ and $\mathbb{E}(X_1) = 0$. Let $K > 0$ and consider the event A_n defined by

$$A_n = \left\{ \frac{S_n}{\sqrt{n}} \geqslant K \right\}, \qquad \text{where} \quad S_n = \sum_{i=1}^{n} X_i \,.$$

Prove that there exists $c > 0$ such that for all n large enough, $\mathbb{P}(A_n) \geqslant c$. Any result used in the proof must be stated clearly.

(iv) Prove using the results above that A_n occurs infinitely often, almost surely. Deduce that

$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{n}} = \infty \,,$$

almost surely.

26J Applied Probability

(i) Define an inhomogeneous Poisson process with rate function $\lambda(u)$.

(ii) Show that the number of arrivals in an inhomogeneous Poisson process during the interval $(0,t)$ has the Poisson distribution with mean

$$\int_0^t \lambda(u)\,\mathrm{d}u\,.$$

(iii) Suppose that $\Lambda = \{\Lambda(t),\ t \geqslant 0\}$ is a non-negative real-valued random process. Conditional on Λ, let $N = \{N(t),\ t \geqslant 0\}$ be an inhomogeneous Poisson process with rate function $\Lambda(u)$. Such a process N is called a doubly-stochastic Poisson process. Show that the variance of $N(t)$ cannot be less than its mean.

(iv) Now consider the process $M(t)$ obtained by deleting every odd-numbered point in an ordinary Poisson process of rate λ. Check that

$$\mathbb{E}M(t) = \frac{2\lambda t + e^{-2\lambda t} - 1}{4}\,, \qquad \mathrm{Var}\ M(t) = \frac{4\lambda t - 8\lambda t e^{-2\lambda t} - e^{-4\lambda t} + 1}{16}\,.$$

Deduce that $M(t)$ is not a doubly-stochastic Poisson process.

27K Principles of Statistics

Random variables $X_1, X_2, \ldots$ are independent and identically distributed from the exponential distribution $\mathcal{E}(\theta)$, with density function

$$p_X(x \mid \theta) = \theta e^{-\theta x} \quad (x > 0),$$

when the parameter Θ takes value $\theta > 0$. The following experiment is performed. First X_1 is observed. Thereafter, if $X_1 = x_1, \ldots, X_i = x_i$ have been observed ($i \geqslant 1$), a coin having probability $\alpha(x_i)$ of landing heads is tossed, where $\alpha : \mathbb{R} \to (0,1)$ is a known function and the coin toss is independent of the X's and previous tosses. If it lands heads, no further observations are made; if tails, X_{i+1} is observed.

Let N be the total number of X's observed, and $\mathbf{X} := (X_1, \ldots, X_N)$. Write down the likelihood function for Θ based on data $\mathbf{X} = (x_1, \ldots, x_n)$, and identify a minimal sufficient statistic. What does the likelihood principle have to say about inference from this experiment?

Now consider the experiment that only records $Y := X_N$. Show that the density function of Y has the form

$$p_Y(y \mid \theta) = \exp\{a(y) - k(\theta) - \theta y\}.$$

Assuming the function $a(\cdot)$ is twice differentiable and that both $p_Y(y \mid \theta)$ and $\partial p_Y(y \mid \theta)/\partial y$ vanish at 0 and ∞, show that $a'(Y)$ is an unbiased estimator of Θ, and find its variance.

Stating clearly any general results you use, deduce that

$$-k''(\theta)\,\mathbb{E}_\theta\{a''(Y)\} \geqslant 1.$$

28K Optimization and Control

An observable scalar state variable evolves as $x_{t+1} = x_t + u_t$, $t = 0, 1, \ldots$. Let controls $u_0, u_1, \ldots$ be determined by a policy π and define

$$C_s(\pi, x_0) = \sum_{t=0}^{s-1} (x_t^2 + 2x_t u_t + 7u_t^2) \quad \text{and} \quad C_s(x_0) = \inf_{\pi} C_s(\pi, x_0)\,.$$

Show that it is possible to express $C_s(x_0)$ in terms of Π_s, which satisfies the recurrence

$$\Pi_s = \frac{6(1 + \Pi_{s-1})}{7 + \Pi_{s-1}}\,, \qquad s = 1, 2, \ldots,$$

with $\Pi_0 = 0$.

Deduce that $C_\infty(x_0) \geqslant 2x_0^2$. [$C_\infty(x_0)$ is defined as $\lim_{s\to\infty} C_s(x_0)$.]

By considering the policy π^* which takes $u_t = -(1/3)(2/3)^t x_0$, $t = 0, 1, \ldots$, show that $C_\infty(x_0) = 2x_0^2$.

Give an alternative description of π^* in closed-loop form.

29J Stochastic Financial Models

First, what is a *Brownian motion?*

(i) The price S_t of an asset evolving in continuous time is represented as

$$S_t = S_0 \, \exp(\sigma W_t + \mu t)\,,$$

where W is a standard Brownian motion, and σ and μ are constants. If riskless investment in a bank account returns a continuously-compounded rate of interest r, derive a formula for the time-0 price of a European call option on the asset S with strike K and expiry T. You may use any general results, but should state them clearly.

(ii) In the same financial market, consider now a derivative which pays

$$Y = \left\{ \exp\big(T^{-1} \int_0^T \log(S_u)\, du \big) - K \right\}^+$$

at time T. Find the time-0 price for this derivative. Show that it is less than the price of the European call option which you derived in (i).

30A Partial Differential Equations

(a) State the local existence theorem of a classical solution of the Cauchy problem

$$a(x_1, x_2, u)\frac{\partial u}{\partial x_1} + b(x_1, x_2, u)\frac{\partial u}{\partial x_2} = c(x_1, x_2, u)\,,$$
$$u|_\Gamma = u_0\,,$$

where Γ is a smooth curve in $\mathbb{R}^2$.

(b) Solve, by using the method of characteristics,

$$2x_1\frac{\partial u}{\partial x_1} + 4x_2\frac{\partial u}{\partial x_2} = u^2\,,$$
$$u(x_1, 2) = h\,,$$

where $h > 0$ is a constant. What is the maximal domain of existence in which u is a solution of the Cauchy problem?

31A Asymptotic Methods

Let

$$I_0 = \int_{C_0} e^{x\phi(z)} dz\,,$$

where $\phi(z)$ is a complex analytic function and C_0 is a steepest descent contour from a simple saddle point of $\phi(z)$ at z_0. Establish the following leading asymptotic approximation, for large real x:

$$I_0 \sim i\sqrt{\frac{\pi}{2\phi''(z_0)x}}\,e^{x\phi(z_0)}\,.$$

Let n be a positive integer, and let

$$I = \int_C e^{-t^2 - 2n\ \ln\ t}\,dt\,,$$

where C is a contour in the upper half t-plane connecting $t = -\infty$ to $t = \infty$, and $\ln t$ is real on the positive t-axis with a branch cut along the negative t-axis. Using the method of steepest descent, find the leading asymptotic approximation to I for large n.

32A Integrable Systems

Let $U(\rho,\tau,\lambda)$ and $V(\rho,\tau,\lambda)$ be matrix-valued functions. Consider the following system of overdetermined linear partial differential equations:

$$\frac{\partial}{\partial\rho}\psi = U\psi\,, \quad \frac{\partial}{\partial\tau}\psi = V\psi\,,$$

where ψ is a column vector whose components depend on (ρ,τ,λ). Using the consistency condition of this system, derive the associated zero curvature representation (ZCR)

$$\frac{\partial}{\partial\tau}U - \frac{\partial}{\partial\rho}V + [U,V] = 0\,, \qquad (*)$$

where $[\cdot\,,\,\cdot]$ denotes the usual matrix commutator.

(i) Let

$$U = \frac{i}{2}\begin{pmatrix} 2\lambda & \partial_\rho\phi \\ \partial_\rho\phi & -2\lambda \end{pmatrix}, \qquad V = \frac{1}{4i\lambda}\begin{pmatrix} \cos\phi & -i\sin\phi \\ i\sin\phi & -\cos\phi \end{pmatrix}.$$

Find a partial differential equation for $\phi = \phi(\rho,\tau)$ which is equivalent to the ZCR $(*)$.

(ii) Assuming that U and V in $(*)$ do not depend on $t := \rho - \tau$, show that the trace of $(U-V)^p$ does not depend on $x := \rho + \tau$, where p is any positive integer. Use this fact to construct a first integral of the ordinary differential equation

$$\phi'' = \sin\phi\,, \qquad \text{where} \quad \phi = \phi(x)\,.$$

33D Principles of Quantum Mechanics

The Pauli matrices $\boldsymbol{\sigma} = (\sigma_x,\ \sigma_y,\ \sigma_z) = (\sigma_1,\ \sigma_2,\ \sigma_3)$, with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are used to represent angular momentum operators with respect to basis states $|\uparrow\rangle$ and $|\downarrow\rangle$ corresponding to spin up and spin down along the z-axis. They satisfy

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k\,.$$

(i) How are $|\uparrow\rangle$ and $|\downarrow\rangle$ represented? How is the spin operator $\mathbf{s}$ related to $\boldsymbol{\sigma}$ and $\hbar$? Check that the commutation relations between the spin operators are as desired. Check that $\mathbf{s}^2$ acting on a spin one-half state has the correct eigenvalue.

What are the states obtained by applying s_x, s_y to the eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of s_z?

(ii) Let V be the space of states for a spin one-half system. Consider a combination of three such systems with states belonging to $V^{(1)} \otimes V^{(2)} \otimes V^{(3)}$ and spin operators acting on each subsystem denoted by $s_x^{(i)}$, $s_y^{(i)}$ with $i = 1, 2, 3$. Find the eigenvalues of the operators

$$s_x^{(1)} s_y^{(2)} s_y^{(3)}\,, \quad s_y^{(1)} s_x^{(2)} s_y^{(3)}\,, \quad s_y^{(1)} s_y^{(2)} s_x^{(3)} \quad \text{and} \quad s_x^{(1)} s_x^{(2)} s_x^{(3)}$$

of the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big[\, |\uparrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3 \ - |\downarrow\rangle_1 |\downarrow\rangle_2 |\downarrow\rangle_3 \,\big]\,.$$

(iii) Consider now whether these outcomes for measurements of particular combinations of the operators $s_x^{(i)}$, $s_y^{(i)}$ in the state $|\Psi\rangle$ could be reproduced by replacing the spin operators with classical variables $\tilde{s}_x^{(i)}$, $\tilde{s}_y^{(i)}$ which take values $\pm\hbar/2$ according to some probabilities. Assume that these variables are identical to the quantum measurements of $s_x^{(1)} s_y^{(2)} s_y^{(3)}$, $s_y^{(1)} s_x^{(2)} s_y^{(3)}$, $s_y^{(1)} s_y^{(2)} s_x^{(3)}$ on $|\Psi\rangle$. Show that classically this implies a unique possibility for

$$\tilde{s}_x^{(1)} \tilde{s}_x^{(2)} \tilde{s}_x^{(3)}\,,$$

and find its value.

State briefly how this result could be used to experimentally test quantum mechanics.

34E Applications of Quantum Mechanics

An electron of mass m moves in a D-dimensional periodic potential that satisfies the periodicity condition

$$V(\boldsymbol{r}+\boldsymbol{l}) = V(\boldsymbol{r}) \quad \forall\, \boldsymbol{l} \in \Lambda\,,$$

where Λ is a D-dimensional Bravais lattice. State Bloch's theorem for the energy eigenfunctions of the electron.

For a one-dimensional potential $V(x)$ such that $V(x+a) = V(x)$, give a full account of how the "nearly free electron model" leads to a band structure for the energy levels.

Explain briefly the idea of a Fermi surface and its rôle in explaining the existence of conductors and insulators.

35D Statistical Physics

A gas of non-interacting particles has energy-momentum relationship $E = A(\hbar k)^\alpha$ for some constants A and α. Determine the density of states $g(E)\,dE$ in a three-dimensional volume V.

Explain why the chemical potential μ satisfies $\mu < 0$ for the Bose–Einstein distribution.

Show that an ideal quantum Bose gas with the energy-momentum relationship above has

$$pV = \frac{\alpha E}{3}\,.$$

If the particles are bosons at fixed temperature T and chemical potential μ, write down an expression for the number of particles that do not occupy the ground state. Use this to determine the values of α for which there exists a Bose–Einstein condensate at sufficiently low temperatures.

Discuss whether a gas of photons can undergo Bose–Einstein condensation.

36C Electrodynamics

Explain how time-dependent distributions of electric charge $\rho(\mathbf{x},t)$ and current $\mathbf{j}(\mathbf{x},t)$ can be combined into a four-vector $j^a(x)$ that obeys $\partial_a j^a = 0$.

This current generates a four-vector potential $A^a(x)$. Explain how to find A^a in the gauge $\partial_a A^a = 0$.

A small circular loop of wire of radius r is centred at the origin. The unit vector normal to the plane of the loop is $\mathbf{n}$. A current $I_o \sin\omega t$ flows in the loop. Find the three-vector potential $\mathbf{A}(\mathbf{x},t)$ to leading order in $r/|\mathbf{x}|$.

37B Fluid Dynamics II

If $A_i(x_j)$ is harmonic, i.e. if $\nabla^2 A_i = 0$, show that

$$u_i = A_i - x_k \frac{\partial A_k}{\partial x_i}\,, \qquad \text{with} \quad p = -2\mu \frac{\partial A_n}{\partial x_n}\,,$$

satisfies the incompressibility condition and the Stokes equation. Show that the stress tensor is

$$\sigma_{ij} = 2\mu \left(\delta_{ij} \frac{\partial A_n}{\partial x_n} - x_k \frac{\partial^2 A_k}{\partial x_i \partial x_j} \right).$$

Consider the Stokes flow corresponding to

$$A_i = V_i \left(1 - \frac{a}{2r}\right)\,,$$

where V_i are the components of a constant vector $\mathbf{V}$. Show that on the sphere $r = a$ the normal component of velocity vanishes and the surface traction $\sigma_{ij} x_j / a$ is in the normal direction. Hence deduce that the drag force on the sphere is given by

$$\mathbf{F} = 4\pi\mu a \mathbf{V}\,.$$

38B Waves

The dispersion relation in a stationary medium is given by $\omega = \Omega_0(\mathbf{k})$, where Ω_0 is a known function. Show that, in the frame of reference where the medium has a uniform velocity $-\mathbf{U}$, the dispersion relation is given by $\omega = \Omega_0(\mathbf{k}) - \mathbf{U} \cdot \mathbf{k}$.

An aircraft flies in a straight line with constant speed Mc_0 through air with sound speed c_0. If $M > 1$ show that, in the reference frame of the aircraft, the steady waves lie behind it on a cone of semi-angle $\sin^{-1}(1/M)$. Show further that the unsteady waves are confined to the interior of the cone.

A small insect swims with constant velocity $\mathbf{U} = (U, 0)$ over the surface of a pool of water. The resultant capillary waves have dispersion relation $\omega^2 = T|\mathbf{k}|^3/\rho$ on stationary water, where T and ρ are constants. Show that, in the reference frame of the insect, steady waves have group velocity

$$\mathbf{c}_g = U\left(\tfrac{3}{2}\cos^2\beta - 1, \tfrac{3}{2}\cos\beta\sin\beta\right),$$

where $\mathbf{k} \propto (\cos\beta, \sin\beta)$. Deduce that the steady wavefield extends in all directions around the insect.

39A Numerical Analysis

(i) The difference equation

$$u_i^{n+1} = u_i^n + \frac{3}{2}\mu\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right) - \frac{1}{2}\mu\left(u_{i-1}^{n-1} - 2u_i^{n-1} + u_{i+1}^{n-1}\right) ,$$

where $\mu = \Delta t/(\Delta x)^2$, is the basic equation used in the second-order Adams–Bashforth method and can be employed to approximate a solution of the diffusion equation $u_t = u_{xx}$. Prove that, as $\Delta t \to 0$ with constant μ, the local error of the method is $O(\Delta t)^2$.

(ii) By applying the Fourier stability test, show that the above method is stable if and only if $\mu \leqslant 1/4$.

(iii) Define the leapfrog scheme to approximate the diffusion equation and prove that it is unstable for every choice of $\mu > 0$.

END OF PAPER