UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 9 June, 2011 9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, ..., H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Groups, Rings and Modules

Suppose that A is an integral domain containing a field K and that A is finite-dimensional as a K-vector space. Prove that A is a field.

2E Analysis II

Suppose f is a uniformly continuous mapping from a metric space X to a metric space Y. Prove that $f(x_n)$ is a Cauchy sequence in Y for every Cauchy sequence x_n in X.

Let f be a continuous mapping between metric spaces and suppose that f has the property that $f(x_n)$ is a Cauchy sequence whenever x_n is a Cauchy sequence. Is it true that f must be uniformly continuous? Justify your answer.

3G Metric and Topological Spaces

Let X, Y be topological spaces, and suppose Y is Hausdorff.

(i) Let $f, g : X \to Y$ be two continuous maps. Show that the set

$$E(f,g) := \{x \in X \mid f(x) = g(x)\} \subset X$$

is a closed subset of X.

(ii) Let W be a dense subset of X. Show that a continuous map $f : X \to Y$ is determined by its restriction $f|_W$ to W.

4D Complex Methods

Write down the function $\psi(u, v)$ that satisfies

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = 0\,, \qquad \psi(-\tfrac{1}{2}, v) = -1, \quad \psi(\tfrac{1}{2}, v) = 1\,.$$

The circular arcs $\mathcal{C}_1$ and $\mathcal{C}_2$ in the complex z-plane are defined by

$$|z+1| = 1,\ z \neq 0 \ \text{ and } \ |z-1| = 1,\ z \neq 0\,,$$

respectively. You may assume without proof that the mapping from the complex z-plane to the complex ζ-plane defined by

$$\zeta = \frac{1}{z}$$

takes $\mathcal{C}_1$ to the line $u = -\frac{1}{2}$ and $\mathcal{C}_2$ to the line $u = \frac{1}{2}$, where $\zeta = u + iv$, and that the region $\mathcal{D}$ in the z-plane exterior to both the circles $|z+1| = 1$ and $|z-1| = 1$ maps to the region in the ζ-plane given by $-\frac{1}{2} < u < \frac{1}{2}$.

Use the above mapping to solve the problem

$$\nabla^2 \phi = 0 \quad \text{ in } \mathcal{D}, \qquad \phi = -1 \text{ on } \mathcal{C}_1 \text{ and } \phi = 1 \text{ on } \mathcal{C}_2.$$

5F Geometry

Let $R(x, \theta)$ denote anti-clockwise rotation of the Euclidean plane $\mathbb{R}^2$ through an angle θ about a point x.

Show that $R(x, \theta)$ is a composite of two reflexions.

Assume $\theta, \phi \in (0, \pi)$. Show that the composite $R(y, \phi) \cdot R(x, \theta)$ is also a rotation $R(z, \psi)$. Find z and ψ.

6D Variational Principles

Find, using a Lagrange multiplier, the four stationary points in $\mathbb{R}^3$ of the function $x^2 + y^2 + z^2$ subject to the constraint $x^2 + 2y^2 - z^2 = 1$. By considering the situation geometrically, or otherwise, identify the nature of the constrained stationary points.

How would your answers differ if, instead, the stationary points of the function $x^2 + 2y^2 - z^2$ were calculated subject to the constraint $x^2 + y^2 + z^2 = 1$?

7A Methods

The Fourier transform $\widetilde{h}(k)$ of the function $h(x)$ is defined by

$$\widetilde{h}(k) = \int_{-\infty}^{\infty} h(x) e^{-ikx} dx.$$

(i) State the inverse Fourier transform formula expressing $h(x)$ in terms of $\widetilde{h}(k)$.

(ii) State the convolution theorem for Fourier transforms.

(iii) Find the Fourier transform of the function $f(x) = e^{-|x|}$. Hence show that the convolution of the function $f(x) = e^{-|x|}$ with itself is given by the integral expression

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(1+k^2)^2} dk.$$

8C Quantum Mechanics

A particle of mass m and energy E, incident from $x = -\infty$, scatters off a delta function potential at $x = 0$. The time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\delta(x)\psi = E\psi$$

where U is a positive constant. Find the reflection and transmission probabilities.

9H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain with state space S.

(i) What does it mean to say that $(X_n)_{n\geqslant 0}$ has the strong Markov property? Your answer should include the definition of the term *stopping time.*

(ii) Show that

$$\mathbb{P}(X_n = i \text{ at least } k \text{ times } | X_0 = i) = \left[\mathbb{P}(X_n = i \text{ at least once } | X_0 = i)\right]^k$$

for a state $i \in S$. You may use without proof the fact that $(X_n)_{n\geqslant 0}$ has the strong Markov property.

SECTION II

10G Linear Algebra

(i) Let A be an $n \times n$ complex matrix and $f(X)$ a polynomial with complex coefficients. By considering the Jordan normal form of A or otherwise, show that if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ then the eigenvalues of $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$.

(ii) Let $B = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix}$. Write B as $B = f(A)$ for a polynomial f with $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and find the eigenvalues of B.

[Hint: compute the powers of A.]

11F Groups, Rings and Modules

Suppose that A is a matrix over $\mathbb{Z}$. What does it mean to say that A can be brought to Smith normal form?

Show that the structure theorem for finitely generated modules over $\mathbb{Z}$ (which you should state) follows from the existence of Smith normal forms for matrices over $\mathbb{Z}$.

Bring the matrix $\begin{pmatrix} -4 & -6 \\ 2 & 2 \end{pmatrix}$ to Smith normal form.

Suppose that M is the $\mathbb{Z}$-module with generators e_1, e_2, subject to the relations

$$-4e_1 + 2e_2 = -6e_1 + 2e_2 = 0\,.$$

Describe M in terms of the structure theorem.

12E Analysis II

Consider a map $f : \mathbb{R}^n \to \mathbb{R}^m$.

Assume f is differentiable at x and let $D_x f$ denote the derivative of f at x. Show that

$$D_x f(v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

for any $v \in \mathbb{R}^n$.

Assume now that f is such that for some fixed x and for every $v \in \mathbb{R}^n$ the limit

$$\lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

exists. Is it true that f is differentiable at x? Justify your answer.

Let M_k denote the set of all $k \times k$ real matrices which is identified with $\mathbb{R}^{k^2}$. Consider the function $f : M_k \to M_k$ given by $f(A) = A^3$. Explain why f is differentiable. Show that the derivative of f at the matrix A is given by

$$D_A f(H) = HA^2 + AHA + A^2 H$$

for any matrix $H \in M_k$. State carefully the inverse function theorem and use it to prove that there exist open sets U and V containing the identity matrix such that given $B \in V$ there exists a unique $A \in U$ such that $A^3 = B$.

13E Complex Analysis

Let $g : \mathbb{C} \to \mathbb{C}$ be a continuous function such that

$$\int_\Gamma g(z)\, dz = 0$$

for any closed curve Γ which is the boundary of a rectangle in $\mathbb{C}$ with sides parallel to the real and imaginary axes. Prove that g is analytic.

Let $f : \mathbb{C} \to \mathbb{C}$ be continuous. Suppose in addition that f is analytic at every point $z \in \mathbb{C}$ with non-zero imaginary part. Show that f is analytic at every point in $\mathbb{C}$.

Let $\mathbb{H}$ be the upper half-plane of complex numbers z with positive imaginary part $\Im(z) > 0$. Consider a continuous function $F : \mathbb{H} \cup \mathbb{R} \to \mathbb{C}$ such that F is analytic on $\mathbb{H}$ and $F(\mathbb{R}) \subset \mathbb{R}$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} F(z) & \text{if } \Im(z) \geqslant 0 \\ \overline{F(\overline{z})} & \text{if } \Im(z) \leqslant 0. \end{cases}$$

Show that f is analytic.

14F Geometry

Suppose that $\eta(u) = (f(u), 0, g(u))$ is a unit speed curve in $\mathbb{R}^3$. Show that the corresponding surface of revolution S obtained by rotating this curve about the z-axis has Gaussian curvature $K = -(d^2f/du^2)/f$.

15A Methods

A uniform stretched string of length L, density per unit length μ and tension $T = \mu c^2$ is fixed at both ends. Its transverse displacement is given by $y(x,t)$ for $0 \leqslant x \leqslant L$. The motion of the string is resisted by the surrounding medium with a resistive force per unit length of $-2k\mu\dfrac{\partial y}{\partial t}$.

(i) Show that the equation of motion of the string is

$$\frac{\partial^2 y}{\partial t^2} + 2k\frac{\partial y}{\partial t} - c^2\frac{\partial^2 y}{\partial x^2} = 0$$

provided that the transverse motion can be regarded as small.

(ii) Suppose now that $k = \dfrac{\pi c}{L}$. Find the displacement of the string for $t \geqslant 0$ given the initial conditions

$$y(x,0) = A\sin\left(\frac{\pi x}{L}\right) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = 0.$$

(iii) Sketch the transverse displacement at $x = \dfrac{L}{2}$ as a function of time for $t \geqslant 0$.

16C Quantum Mechanics

For an electron in a hydrogen atom, the stationary state wavefunctions are of the form $\psi(r,\theta,\phi) = R(r)Y_{lm}(\theta,\phi)$, where in suitable units R obeys the radial equation

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + 2\left(E + \frac{1}{r}\right)R = 0\,.$$

Explain briefly how the terms in this equation arise.

This radial equation has bound state solutions of energy $E = E_n$, where $E_n = -\frac{1}{2n^2}$ $(n = 1,2,3,\ldots)$. Show that when $l = n-1$, there is a solution of the form $R(r) = r^\alpha e^{-r/n}$, and determine α. Find the expectation value $\langle r\rangle$ in this state.

What is the total degeneracy of the energy level with energy E_n?

17C Electromagnetism

Show, using the vacuum Maxwell equations, that for any volume V with surface S,

$$\frac{d}{dt}\int_V \left(\frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\right) dV = \int_S \left(-\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)\cdot \mathbf{dS}\,.$$

What is the interpretation of this equation?

A uniform straight wire, with a circular cross section of radius r, has conductivity σ and carries a current I. Calculate $\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ at the surface of the wire, and hence find the flux of $\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ into unit length of the wire. Relate your result to the resistance of the wire, and the rate of energy dissipation.

18D Fluid Dynamics

Water of constant density ρ flows steadily through a long cylindrical tube, the wall of which is elastic. The exterior radius of the tube at a distance z along the tube, $r(z)$, is determined by the pressure in the tube, $p(z)$, according to

$$r(z) = r_0 + b(p(z) - p_0),$$

where r_0 and p_0 are the radius and pressure far upstream ($z \to -\infty$), and b is a positive constant.

The interior radius of the tube is $r(z) - h(z)$, where $h(z)$, the thickness of the wall, is a given slowly varying function of z which is zero at both ends of the pipe. The velocity of the water in the pipe is $u(z)$ and the water enters the pipe at velocity V.

Show that $u(z)$ satisfies

$$H = 1 - v^{-\frac{1}{2}} + \tfrac{1}{4}k(1 - v^2)\,,$$

where $H = \dfrac{h}{r_0}$, $v = \dfrac{u}{V}$ and $k = \dfrac{2b\rho V^2}{r_0}$. Sketch the graph of H against v.

Let H_m be the maximum value of H in the tube. Show that the flow is only possible if H_m does not exceed a certain critical value H_c. Find H_c in terms of k.

Show that, under conditions to be determined (which include a condition on the value of k), the water can leave the pipe with speed less than V.

19B Numerical Analysis

A Gaussian quadrature formula provides an approximation to the integral

$$\int_{-1}^{1}(1-x^2)f(x)\,dx \approx \sum_{k=1}^{\nu} b_k f(c_k)$$

which is exact for all $f(x)$ that are polynomials of degree $\leqslant (2\nu-1)$.

Write down explicit expressions for the b_k in terms of integrals, and explain why it is necessary that the c_k are the zeroes of a (monic) polynomial p_ν of degree ν that satisfies $\int_{-1}^{1}(1-x^2)p_\nu(x)q(x)\,dx = 0$ for any polynomial $q(x)$ of degree less than ν.

The first such polynomials are $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - 1/5$, $p_3 = x^3 - 3x/7$. Show that the Gaussian quadrature formulae for $\nu = 2, 3$ are

$$\begin{aligned}
\nu = 2: &\quad \frac{2}{3}\left[f(-\tfrac{1}{\sqrt{5}}) + f(\tfrac{1}{\sqrt{5}})\right], \\
\nu = 3: &\quad \frac{14}{45}\left[f(-\sqrt{\tfrac{3}{7}}) + f(\sqrt{\tfrac{3}{7}})\right] + \frac{32}{45}f(0).
\end{aligned}$$

Verify the result for $\nu = 3$ by considering $f(x) = 1, x^2, x^4$.

20H Statistics

Consider the general linear model

$$Y = X\beta + \epsilon$$

where X is a known $n \times p$ matrix, β is an unknown $p \times 1$ vector of parameters, and ϵ is an $n \times 1$ vector of independent $N(0, \sigma^2)$ random variables with unknown variance σ^2. Assume the $p \times p$ matrix $X^T X$ is invertible.

(i) Derive the least squares estimator $\widehat{\beta}$ of β.

(ii) Derive the distribution of $\widehat{\beta}$. Is $\widehat{\beta}$ an unbiased estimator of β?

(iii) Show that $\frac{1}{\sigma^2}\|Y - X\widehat{\beta}\|^2$ has the χ^2 distribution with k degrees of freedom, where k is to be determined.

(iv) Let $\widetilde{\beta}$ be an unbiased estimator of β of the form $\widetilde{\beta} = CY$ for some $p \times n$ matrix C. By considering the matrix $\mathbb{E}[(\widehat{\beta} - \widetilde{\beta})(\widehat{\beta} - \beta)^T]$ or otherwise, show that $\widehat{\beta}$ and $\widehat{\beta} - \widetilde{\beta}$ are independent.

[You may use standard facts about the multivariate normal distribution as well as results from linear algebra, including the fact that $I - X(X^T X)^{-1} X^T$ is a projection matrix of rank $n - p$, as long as they are carefully stated.]

21H Optimization

(i) What does it mean to say a set $C \subseteq \mathbb{R}^n$ is convex?

(ii) What does it mean to say z is an extreme point of a convex set C?

Let A be an $m \times n$ matrix, where $n > m$. Let b be an $m \times 1$ vector, and let

$$C = \{x \in \mathbb{R}^n : Ax = b, x \geqslant 0\}$$

where the inequality is interpreted component-wise.

(iii) Show that C is convex.

(iv) Let $z = (z_1, \ldots, z_n)^T$ be a point in C with the property that at least $m+1$ indices i are such that $z_i > 0$. Show that z is not an extreme point of C. [*Hint: If $r > m$, then any set of r vectors in $\mathbb{R}^m$ is linearly dependent.*]

(v) Now suppose that every set of m columns of A is linearly independent. Let $z = (z_1, \ldots, z_n)^T$ be a point in C with the property that at most m indices i are such that $z_i > 0$. Show that z is an extreme point of C.

END OF PAPER