UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 4 June, 2010 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Define the notion of an inner product on a finite-dimensional real vector space V, and the notion of a self-adjoint linear map $\alpha : V \to V$.

Suppose that V is the space of real polynomials of degree at most n in a variable t. Show that

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$$

is an inner product on V, and that the map $\alpha : V \to V$:

$$\alpha(f)(t) = (1 - t^2)f''(t) - 2tf'(t)$$

is self-adjoint.

2H Groups Rings and Modules

Let M be a free $\mathbb{Z}$-module generated by e_1 and e_2. Let a, b be two non-zero integers, and N be the submodule of M generated by $ae_1 + be_2$. Prove that the quotient module M/N is free if and only if a, b are coprime.

3G Analysis II

Let S denote the set of continuous real-valued functions on the interval $[0, 1]$. For $f, g \in S$, set

$$d_1(f, g) = \sup \{|f(x) - g(x)| \,:\, x \in [0, 1]\} \quad \text{and} \quad d_2(f, g) = \int_0^1 |f(x) - g(x)|\, dx\,.$$

Show that both d_1 and d_2 define metrics on S. Does the identity map on S define a continuous map of metric spaces $(S, d_1) \to (S, d_2)$? Does the identity map define a continuous map of metric spaces $(S, d_2) \to (S, d_1)$?

4G Complex Analysis

State the principle of the argument for meromorphic functions and show how it follows from the Residue theorem.

5A Methods

(a) By considering strictly monotonic differentiable functions $\varphi(x)$, such that the zeros satisfy $\varphi(c) = 0$ but $\varphi'(c) \neq 0$, establish the formula

$$\int_{-\infty}^{\infty} f(x)\delta(\varphi(x))dx = \frac{f(c)}{|\varphi'(c)|} .$$

Hence show that for a general differentiable function with only such zeros, labelled by c,

$$\int_{-\infty}^{\infty} f(x)\delta(\varphi(x))\, dx = \sum_c \frac{f(c)}{|\varphi'(c)|} .$$

(b) Hence by changing to plane polar coordinates, or otherwise, evaluate,

$$I = \int_0^{\infty} \int_0^{\infty} (x^3 + y^2 x)\delta(x^2 + y^2 - 1)\, dy dx.$$

6D Quantum Mechanics

Determine the possible values of the energy of a particle free to move inside a cube of side a , confined there by a potential which is infinite outside and zero inside.

What is the degeneracy of the lowest-but-one energy level?

7B Electromagnetism

Give an expression for the force $\mathbf{F}$ on a charge q moving at velocity $\mathbf{v}$ in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Consider a stationary electric circuit C, and let S be a stationary surface bounded by C. Derive from Maxwell's equations the result

$$\mathcal{E} = -\frac{d\Phi}{dt} \qquad (*)$$

where the electromotive force $\mathcal{E} = \oint_C q^{-1}\mathbf{F}\cdot d\mathbf{r}$ and the flux $\Phi = \int_S \mathbf{B}\cdot d\mathbf{S}$.

Now assume that $(*)$ also holds for a moving circuit. A circular loop of wire of radius a and total resistance R, whose normal is in the z-direction, moves at constant speed v in the x-direction in the presence of a magnetic field $\mathbf{B} = (0, 0, B_0\, x/d)$. Find the current in the wire.

8C Numerical Analysis

Suppose $x_0, x_1, \ldots, x_n \in [a,b] \subset \mathbf{R}$ are pointwise distinct and $f(x)$ is continuous on $[a,b]$. For $k = 1, 2, \ldots, n$ define

$$I_{0,k}(x) \;=\; \frac{f(x_0)(x_k - x) - f(x_k)(x_0 - x)}{x_k - x_0}\,,$$

and for $k = 2, 3, \ldots, n$

$$I_{0,\,1,\,\ldots\,,\,k-2,\,k-1,k}(x) \;=\; \frac{I_{0,\,1,\,\ldots\,,\,k-2,\,k-1}(x)(x_k - x) - I_{0,\,1,\,\ldots\,,\,k-2,\,k}(x)(x_{k-1} - x)}{x_k - x_{k-1}}\,.$$

Show that $I_{0,1,\ldots,k-2,k-1,k}(x)$ is a polynomial of order k which interpolates $f(x)$ at $x_0, x_1, \ldots, x_k$.

Given $x_k = \{-1,\ 0,\ 2,\ 5\}$ and $f(x_k) = \{33,\ 5,\ 9,\ 1335\}$, determine the interpolating polynomial.

9E Markov Chains

Consider a Markov chain $(X_n)_{n\geqslant 0}$ with state space $\{a, b, c, d\}$ and transition probabilities given by the following table.

	a	b	c	d
a	1/4	1/4	1/2	0
b	0	1/4	0	3/4
c	1/2	0	1/4	1/4
d	0	1/2	0	1/2

By drawing an appropriate diagram, determine the communicating classes of the chain, and classify them as either open or closed. Compute the following transition and hitting probabilities:

- $\mathbb{P}\,(X_n = b \,|\, X_0 = d)$ for a fixed $n \geqslant 0$,
- $\mathbb{P}\,(X_n = c$ for some $n \geqslant 1 \,|\, X_0 = a)\,$.

SECTION II

10F Linear Algebra

(i) Show that the group $O_n(\mathbb{R})$ of orthogonal $n \times n$ real matrices has a normal subgroup $SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) \mid \det A = 1\}$.

(ii) Show that $O_n(\mathbb{R}) = SO_n(\mathbb{R}) \times \{\pm I_n\}$ if and only if n is odd.

(iii) Show that if n is even, then $O_n(\mathbb{R})$ is not the direct product of $SO_n(\mathbb{R})$ with any normal subgroup.

[*You may assume that the only elements of $O_n(\mathbb{R})$ that commute with all elements of $O_n(\mathbb{R})$ are $\pm I_n$.*]

11H Groups Rings and Modules

Let $V = (\mathbb{Z}/3\mathbb{Z})^2$, a 2-dimensional vector space over the field $\mathbb{Z}/3\mathbb{Z}$, and let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V$.

(1) List all 1-dimensional subspaces of V in terms of e_1, e_2. (For example, there is a subspace $\langle e_1 \rangle$ generated by e_1.)

(2) Consider the action of the matrix group

$$G = GL_2(\mathbb{Z}/3\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,\, b,\, c,\, d \in \mathbb{Z}/3\mathbb{Z},\ ad - bc \neq 0 \right\}$$

on the finite set X of all 1-dimensional subspaces of V. Describe the stabiliser group K of $\langle e_1 \rangle \in X$. What is the order of K? What is the order of G?

(3) Let $H \subset G$ be the subgroup of all elements of G which act trivially on X. Describe H, and prove that G/H is isomorphic to S_4, the symmetric group on four letters.

12G Analysis II

What does it mean to say that a function f on an interval in $\mathbf{R}$ is *uniformly continuous*? Assuming the Bolzano–Weierstrass theorem, show that any continuous function on a finite closed interval is uniformly continuous.

Suppose that f is a continuous function on the real line, and that $f(x)$ tends to finite limits as $x \to \pm\infty$; show that f is uniformly continuous.

If f is a uniformly continuous function on $\mathbf{R}$, show that $f(x)/x$ is bounded as $x \to \pm\infty$. If g is a continuous function on $\mathbf{R}$ for which $g(x)/x \to 0$ as $x \to \pm\infty$, determine whether g is necessarily uniformly continuous, giving proof or counterexample as appropriate.

13H Metric and Topological Spaces

(1) Prove that if X is a compact topological space, then a closed subset Y of X endowed with the subspace topology is compact.

(2) Consider the following equivalence relation on $\mathbb{R}^2$:

$$(x_1, y_1) \sim (x_2, y_2) \iff (x_1 - x_2,\ y_1 - y_2) \in \mathbb{Z}^2.$$

Let $X = \mathbb{R}^2/\sim$ be the quotient space endowed with the quotient topology, and let $p : \mathbb{R}^2 \to X$ be the canonical surjection mapping each element to its equivalence class. Let $Z = \{(x, y) \in \mathbb{R}^2 \mid y = \sqrt{2}x\}$.

(i) Show that X is compact.

(ii) Assuming that $p(Z)$ is dense in X, show that $p|_Z : Z \to p(Z)$ is bijective but not homeomorphic.

14A Complex Methods

A linear system is described by the differential equation

$$y'''(t) - y''(t) - 2\,y'(t) + 2\,y(t) \;=\; f(t)\,,$$

with initial conditions

$$y\,(0) \;=\; 0\,, \qquad y'(0) \;=\; 1\,, \qquad y''(0) \;=\; 1\,.$$

The Laplace transform of $f(t)$ is defined as

$$\mathcal{L}[f(t)] = \tilde{f}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

You may assume the following Laplace transforms,

$$\begin{aligned}
\mathcal{L}[y(t)] &= \tilde{y}(s),\\
\mathcal{L}[y'(t)] &= s\tilde{y}(s) - y(0),\\
\mathcal{L}[y''(t)] &= s^2\tilde{y}(s) - sy(0) - y'(0),\\
\mathcal{L}[y'''(t)] &= s^3\tilde{y}(s) - s^2y(0) - sy'(0) - y''(0).
\end{aligned}$$

(a) Use Laplace transforms to determine the response, $y_1(t)$, of the system to the signal

$$f(t) = -2.$$

(b) Determine the response, $y_2(t)$, given that its Laplace transform is

$$\tilde{y}_2(s) = \frac{1}{s^2(s-1)^2}.$$

(c) Given that

$$y'''(t) - y''(t) - 2y'(t) + 2y(t) \;=\; g(t)$$

leads to the response with Laplace transform

$$\tilde{y}(s) \;=\; \frac{1}{s^2(s-1)^2}\,,$$

determine $g(t)$.

15F Geometry

Suppose that D is the unit disc, with Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{1 - (x^2 + y^2)}.$$

[*Note that this is not a multiple of the Poincaré metric.*] Show that the diameters of D are, with appropriate parametrization, geodesics.

Show that distances between points in D are bounded, but areas of regions in D are unbounded.

16D Variational Principles

A function $\theta(\phi)$ with given values of $\theta(\phi_1)$ and $\theta(\phi_2)$ makes the integral

$$I = \int_{\phi_1}^{\phi_2} \mathcal{L}(\theta, \theta')\, d\phi$$

stationary with respect to small variations of θ which vanish at ϕ_1 and ϕ_2. Show that $\theta(\phi)$ satisfies the first integral of the Euler–Lagrange equation,

$$\mathcal{L}(\theta, \theta') - \theta'(\partial\mathcal{L}/\partial\theta') = C,$$

for some constant C. You may state the Euler–Lagrange equation without proof.

It is desired to tow an iceberg across open ocean from a point on the Antarctic coast (longitude ϕ_1) to a place in Australia (longitude ϕ_2), to provide fresh water for irrigation. The iceberg will melt at a rate proportional to the difference between its temperature (the constant T_0, measured in degrees Celsius and therefore negative) and the sea temperature $T(\theta) > T_0$, where θ is the colatitude (the usual spherical polar coordinate θ). Assume that the iceberg is towed at a constant speed along a path $\theta(\phi)$, where ϕ is the longitude. Given that the infinitesimal arc length on the unit sphere is $(d\theta^2 + \sin^2\theta\ d\phi^2)^{1/2}$, show that the total ice melted along the path from ϕ_1 to ϕ_2 is proportional to

$$I = \int_{\phi_1}^{\phi_2} (T(\theta) - T_0)(\theta'^2 + \sin^2\theta)^{1/2}\, d\phi.$$

Now suppose that in the relevant latitudes, the sea temperature may be approximated by $T(\theta) = T_0(1 + 3\tan\theta)$. (*Note that* $(1 + 3\tan\theta)$ *is negative in the relevant latitudes.*) Show that any smooth path $\theta(\phi)$ which minimizes the total ice melted must satisfy

$$\theta'^2 = \sin^2\theta\,(\tfrac{1}{4}k^2\tan^2\theta\sin^2\theta - 1),$$

and hence that

$$\sin^2\theta = \frac{2}{1 - (1 + k^2)^{1/2}\sin 2(\phi - \phi_0)},$$

where k and ϕ_0 are constants.

[Hint:

$$\int \frac{dx}{x(\alpha^2x^4 + x^2 - 1)^{1/2}} = \frac{1}{2}\arcsin\left[\frac{x^2 - 2}{x^2(1 + 4\alpha^2)^{1/2}}\right].\ ]$$

17B Methods

Defining the function $G_{f3}(\mathbf{r};\mathbf{r}_0) = -1/(4\pi|\mathbf{r}-\mathbf{r}_0|)$, prove Green's third identity for functions $u(\mathbf{r})$ satisfying Laplace's equation in a volume V with surface S, namely

$$u(\mathbf{r}_0) = \int_S \left(u\frac{\partial G_{f3}}{\partial n} - \frac{\partial u}{\partial n}G_{f3} \right) dS.$$

A solution is sought to the Neumann problem for $\nabla^2 u = 0$ in the half plane $z > 0$:

$$u = O(|\mathbf{x}|^{-a}), \qquad \frac{\partial u}{\partial r} = O(|\mathbf{x}|^{-a-1}) \text{ as } |\mathbf{x}| \to \infty, \qquad \frac{\partial u}{\partial z} = p(x,y) \text{ on } z = 0,$$

where $a > 0$. It is assumed that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy = 0$. Explain why this condition is necessary.

Construct an appropriate Green's function $G(\mathbf{r};\mathbf{r}_0)$ satisfying $\partial G/\partial z = 0$ at $z = 0$, using the method of images or otherwise. Hence find the solution in the form

$$u(x_0,y_0,z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) f(x-x_0, y-y_0, z_0)\,dx\,dy,$$

where f is to be determined.

Now let

$$p(x,y) = \begin{cases} x & |x|,|y| < a, \\ 0 & \text{otherwise.} \end{cases}$$

By expanding f in inverse powers of z_0, show that

$$u \to \frac{-2a^4 x_0}{3\pi z_0^3} \qquad \text{as} \quad z_0 \to \infty\,.$$

18B Fluid Dynamics

Write down the velocity potential for a line source flow of strength m located at $(r,\theta)=(d,0)$ in polar coordinates (r,θ) and derive the velocity components u_r, u_θ.

A two-dimensional flow field consists of such a source in the presence of a circular cylinder of radius a $(a<d)$ centred at the origin. Show that the flow field outside the cylinder is the sum of the original source flow, together with that due to a source of the same strength at $(a^2/d,0)$ and another at the origin, of a strength to be determined.

Use Bernoulli's law to find the pressure distribution on the surface of the cylinder, and show that the total force exerted on it is in the x-direction and of magnitude

$$\frac{m^2\rho}{2\pi^2}\int_0^{2\pi}\frac{ad^2\sin^2\theta\,\cos\theta}{(a^2+d^2-2\,ad\,\cos\theta)^2}\;d\theta\,,$$

where ρ is the density of the fluid. Without evaluating the integral, show that it is positive. Comment on the fact that the force on the cylinder is therefore *towards* the source.

19E Statistics

Consider a collection $X_1,\ldots,X_n$ of independent random variables with common density function $f(x;\theta)$ depending on a real parameter θ. What does it mean to say T is a sufficient statistic for θ? Prove that if the joint density of $X_1,\ldots,X_n$ satisfies the factorisation criterion for a statistic T, then T is sufficient for θ.

Let each X_i be uniformly distributed on $[-\sqrt{\theta},\sqrt{\theta}\,]$. Find a two-dimensional sufficient statistic $T=(T_1,T_2)$. Using the fact that $\hat{\theta}=3X_1^2$ is an unbiased estimator of θ, or otherwise, find an unbiased estimator of θ which is a function of T and has smaller variance than $\hat{\theta}$. Clearly state any results you use.

20E Optimization

A factory produces three types of sugar, types X, Y, and Z, from three types of syrup, labelled A, B, and C. The following table contains the number of litres of syrup necessary to make each kilogram of sugar.

	X	Y	Z
A	3	2	1
B	2	3	2
C	4	1	2

For instance, one kilogram of type X sugar requires 3 litres of A, 2 litres of B, and 4 litres of C. The factory can sell each type of sugar for one pound per kilogram. Assume that the factory owner can use no more than 44 litres of A and 51 litres of B, but is required by law to use at least 12 litres of C. If her goal is to maximize profit, how many kilograms of each type of sugar should the factory produce?

END OF PAPER