UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 31 May 2010 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* **three** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold Cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

(a) Find the smallest residue x which equals $28!\,13^{28} \pmod{31}$.

[*You may use any standard theorems provided you state them correctly.*]

(b) Find all integers x which satisfy the system of congruences

$$\begin{aligned} x &\equiv 1 \pmod{2}, \\ 2x &\equiv 1 \pmod{3}, \\ 2x &\equiv 4 \pmod{10}, \\ x &\equiv 10 \pmod{67}. \end{aligned}$$

2E Numbers and Sets

(a) Let r be a real root of the polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, with integer coefficients a_i and leading coefficient 1. Show that if r is rational, then r is an integer.

(b) Write down a series for e. By considering $q!e$ for every natural number q, show that e is irrational.

3B Dynamics and Relativity

A particle of mass m and charge q moves with trajectory $\mathbf{r}(t)$ in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. Write down the Lorentz force on the particle and use Newton's Second Law to deduce that

$$\dot{\mathbf{r}} - \omega\,\mathbf{r} \times \hat{\mathbf{z}} = \mathbf{c}\,,$$

where $\mathbf{c}$ is a constant vector and ω is to be determined. Find $\mathbf{c}$ and hence $\mathbf{r}(t)$ for the initial conditions

$$\mathbf{r}(0) = a\hat{\mathbf{x}} \qquad \text{and} \qquad \dot{\mathbf{r}}(0) = u\hat{\mathbf{y}} + v\hat{\mathbf{z}}$$

where a, u and v are constants. Sketch the particle's trajectory in the case $a\omega + u = 0$.

[*Unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ correspond to a set of Cartesian coordinates.*]

4B Dynamics and Relativity

Let S be an inertial frame with coordinates (t, x) in two-dimensional spacetime. Write down the Lorentz transformation giving the coordinates (t', x') in a second inertial frame S' moving with velocity v relative to S. If a particle has constant velocity u in S, find its velocity u' in S'. Given that $|u| < c$ and $|v| < c$, show that $|u'| < c$.

SECTION II

5E Numbers and Sets

The Fibonacci numbers F_n are defined for all natural numbers n by the rules

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geqslant 3.$$

Prove by induction on k that, for any n,

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n \quad \text{for all } k \geqslant 2.$$

Deduce that

$$F_{2n} = F_n(F_{n+1} + F_{n-1}) \qquad \text{for all } n \geqslant 2.$$

Put $L_1 = 1$ and $L_n = F_{n+1} + F_{n-1}$ for $n > 1$. Show that these (Lucas) numbers L_n satisfy

$$L_1 = 1, \qquad L_2 = 3, \qquad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geqslant 3.$$

Show also that, for all n, the greatest common divisor (F_n, F_{n+1}) is 1, and that the greatest common divisor (F_n, L_n) is at most 2.

6E Numbers and Sets

State and prove Fermat's Little Theorem.

Let p be an odd prime. If $p \neq 5$, show that p divides $10^n - 1$ for infinitely many natural numbers n.

Hence show that p divides infinitely many of the integers

$$5, \qquad 55, \qquad 555, \qquad 5555, \qquad \ldots \quad .$$

7E Numbers and Sets

(a) Let A, B be finite non–empty sets, with $|A| = a,\ |B| = b$. Show that there are b^a mappings from A to B. How many of these are injective ?

(b) State the Inclusion–Exclusion principle.

(c) Prove that the number of surjective mappings from a set of size n onto a set of size k is

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n \qquad \text{for } n \geqslant k \geqslant 1 .$$

Deduce that

$$n! = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n .$$

8E Numbers and Sets

What does it mean for a set to be countable ?

Show that $\mathbb{Q}$ is countable, but $\mathbb{R}$ is not. Show also that the union of two countable sets is countable.

A subset A of $\mathbb{R}$ has the property that, given $\epsilon > 0$ and $x \in \mathbb{R}$, there exist reals a, b with $a \in A$ and $b \notin A$ with $|x - a| < \epsilon$ and $|x - b| < \epsilon$. Can A be countable ? Can A be uncountable ? Justify your answers.

A subset B of $\mathbb{R}$ has the property that given $b \in B$ there exists $\epsilon > 0$ such that if $0 < |b - x| < \epsilon$ for some $x \in \mathbb{R}$, then $x \notin B$. Is B countable ? Justify your answer.

9B Dynamics and Relativity

A sphere of uniform density has mass m and radius a. Find its moment of inertia about an axis through its centre.

A marble of uniform density is released from rest on a plane inclined at an angle α to the horizontal. Let the time taken for the marble to travel a distance ℓ down the plane be: (i) t_1 if the plane is perfectly smooth; or (ii) t_2 if the plane is rough and the marble rolls without slipping.

Explain, with a clear discussion of the forces acting on the marble, whether or not its energy is conserved in each of the cases (i) and (ii). Show that $t_1/t_2 = \sqrt{5/7}$.

Suppose that the original marble is replaced by a new one with the same mass and radius but with a hollow centre, so that its moment of inertia is λma^2 for some constant λ. What is the new value for t_1/t_2?

10B Dynamics and Relativity

A particle of unit mass moves in a plane with polar coordinates (r, θ) and components of acceleration $(\ddot{r} - r\dot{\theta}^2,\ r\ddot{\theta} + 2\dot{r}\dot{\theta})$. The particle experiences a force corresponding to a potential $-Q/r$. Show that

$$E = \frac{1}{2}\dot{r}^2 + U(r) \qquad \text{and} \qquad h = r^2\dot{\theta}$$

are constants of the motion, where

$$U(r) = \frac{h^2}{2r^2} - \frac{Q}{r}.$$

Sketch the graph of $U(r)$ in the cases $Q > 0$ and $Q < 0$.

(a) Assuming $Q > 0$ and $h > 0$, for what range of values of E do bounded orbits exist? Find the minimum and maximum distances from the origin, r_{min} and r_{max}, on such an orbit and show that

$$r_{\text{min}} + r_{\text{max}} = \frac{Q}{|E|}.$$

Prove that the minimum and maximum values of the particle's speed, v_{min} and v_{max}, obey

$$v_{\text{min}} + v_{\text{max}} = \frac{2Q}{h}.$$

(b) Now consider trajectories with $E > 0$ and Q of either sign. Find the distance of closest approach, r_{min}, in terms of the impact parameter, b, and v_∞, the limiting value of the speed as $r \to \infty$. Deduce that if $b \ll |Q|/v_\infty^2$ then, to leading order,

$$r_{\text{min}} \approx \frac{2|Q|}{v_\infty^2} \quad \text{for} \quad Q < 0\,, \qquad\qquad r_{\text{min}} \approx \frac{b^2 v_\infty^2}{2Q} \quad \text{for} \quad Q > 0\,.$$

11B Dynamics and Relativity

Consider a set of particles with position vectors $\mathbf{r}_i(t)$ and masses m_i, where $i = 1, 2, \ldots, N$. Particle i experiences an external force $\mathbf{F}_i$ and an internal force $\mathbf{F}_{ij}$ from particle j, for each $j \neq i$. Stating clearly any assumptions you need, show that

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \quad \text{and} \quad \frac{d\mathbf{L}}{dt} = \mathbf{G},$$

where $\mathbf{P}$ is the total momentum, $\mathbf{F}$ is the total external force, $\mathbf{L}$ is the total angular momentum about a fixed point $\mathbf{a}$, and $\mathbf{G}$ is the total external torque about $\mathbf{a}$.

Does the result $\dfrac{d\mathbf{L}}{dt} = \mathbf{G}$ still hold if the fixed point $\mathbf{a}$ is replaced by the centre of mass of the system? Justify your answer.

Suppose now that the external force on particle i is $-k\dfrac{d\mathbf{r}_i}{dt}$ and that all the particles have the same mass m. Show that

$$\mathbf{L}(t) = \mathbf{L}(0)\, e^{-kt/m}\,.$$

12B Dynamics and Relativity

A particle A of rest mass m is fired at an identical particle B which is stationary in the laboratory. On impact, A and B annihilate and produce two massless photons whose energies are equal. Assuming conservation of four-momentum, show that the angle θ between the photon trajectories is given by

$$\cos\theta = \frac{E - 3mc^2}{E + mc^2}$$

where E is the relativistic energy of A.

Let v be the speed of the incident particle A. For what value of v/c will the photons move in perpendicular directions? If v is very small compared with c, show that

$$\theta \approx \pi - v/c\,.$$

[*All quantities referred to are measured in the laboratory frame.*]

END OF PAPER