UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II 2007

List of Courses

1/I/1F Number Theory

State the prime number theorem, and Bertrand's postulate.

Let S be a finite set of prime numbers, and write $f_s(x)$ for the number of positive integers no larger than x, all of whose prime factors belong to S. Prove that

$$f_s(x) \leqslant 2^{\#(S)}\sqrt{x},$$

where $\#(S)$ denotes the number of elements in S. Deduce that, if x is a strictly positive integer, we have

$$\pi(x) \geqslant \frac{\log x}{2\log 2}.$$

2/I/1F Number Theory

Let p be an odd prime number. Prove that 2 is a quadratic residue modulo p when $p \equiv 7 \pmod 8$. Deduce that, if q is a prime number strictly greater than 3 with $q \equiv 3 \pmod 4$ such that $2q+1$ is also a prime number, then $2^q - 1$ is necessarily composite. Why does the argument break down for $q = 3$?

3/I/1F Number Theory

Determine the continued fraction of $\sqrt{7}$. Deduce two pairs of solutions in positive integers x, y of the equation

$$x^2 - 7y^2 = 1.$$

3/II/11F **Number Theory**

State the Chinese remainder theorem. Let n be an odd positive integer. If n is divisible by the square of a prime number p, prove that there exists an integer z such that $z^p \equiv 1 \pmod{n}$ but $z \not\equiv 1 \pmod{n}$.

Define the Jacobi symbol

$$\left(\frac{a}{n}\right)$$

for any non-zero integer a. Give a numerical example to show that

$$\left(\frac{a}{n}\right) = +1$$

does not imply in general that a is a square modulo n. State and prove the law of quadratic reciprocity for the Jacobi symbol.

[*You may assume the law of quadratic reciprocity for the Legendre symbol.*]

Assume now that n is divisible by the square of a prime number. Prove that there exists an integer a with $(a, n) = 1$ such that the congruence

$$a^{\frac{n-1}{2}} \equiv \left(\frac{a}{n}\right) \pmod{n}$$

does not hold. Show further that this congruence fails to hold for at least half of all relatively prime residue classes modulo n.

4/I/1F **Number Theory**

Prove Legendre's formula relating $\pi(x)$ and $\pi(\sqrt{x})$ for any positive real number x. Use this formula to compute $\pi(48)$.

4/II/11F **Number Theory**

Let p be a prime number, and let $f(x)$ be a polynomial with integer coefficients, whose leading coefficient is not divisible by p. Prove that the congruence

$$f(x) \equiv 0 \pmod{p}$$

has at most d solutions, where d is the degree of $f(x)$.

Deduce that all coefficients of the polynomial

$$x^{p-1} - 1 - \big((x-1)(x-2)\cdots(x-p+1)\big)$$

must be divisible by p, and prove that:

(i) $(p-1)! + 1 \equiv 0 \pmod{p}$;

(ii) if p is odd, the numerator of the fraction

$$u_p = 1 + \frac{1}{2} + \cdots + \frac{1}{p-1}$$

is divisible by p.

Assume now that $p \geqslant 5$. Show by example that (i) cannot be strengthened to $(p-1)! + 1 \equiv 0 \pmod{p^2}$.

1/I/2F **Topics in Analysis**

Let n be an integer with $n \geqslant 1$. Are the following statements true or false? Give proofs.

(i) There exists a real polynomial T_n of degree n such that

$$T_n(\cos t) = \cos nt$$

for all real t.

(ii) There exists a real polynomial R_n of degree n such that

$$R_n(\cosh t) = \cosh nt$$

for all real t.

(iii) There exists a real polynomial S_n of degree n such that

$$S_n(\cos t) = \sin nt$$

for all real t.

2/II/12F **Topics in Analysis**

(i) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Prove the theorem of Bernstein which states that, if we write

$$f_m(t) = \sum_{r=0}^{m} \binom{m}{r} f(r/m) t^r (1-t)^{m-r},$$

for $0 \leqslant t \leqslant 1$, then $f_m \to f$ uniformly as $m \to \infty$.

(ii) Let $n \geqslant 1$, $a_{1,n}, a_{2,n}, \ldots, a_{n,n} \in \mathbb{R}$ and let $x_{1,n}, x_{2,n}, \ldots, x_{n,n}$ be distinct points in $[0,1]$. We write

$$I_n(g) = \sum_{j=1}^{n} a_{j,n} g(x_{j,n})$$

for every continuous function $g : [0,1] \to \mathbb{R}$. Show that, if

$$I_n(P) = \int_0^1 P(t)\, dt,$$

for all polynomials P of degree $2n-1$ or less, then $a_{j,n} \geqslant 0$ for all $1 \leqslant j \leqslant n$ and $\sum_{j=1}^{n} a_{j,n} = 1$.

(iii) If I_n satisfies the conditions set out in (ii), show that

$$I_n(f) \to \int_0^1 f(t)\, dt$$

as $n \to \infty$ whenever $f : [0,1] \to \mathbb{R}$ is continuous.

2/I/2F **Topics in Analysis**

Write

$$P^+ = \{(x,y) \in \mathbb{R}^2 : x,\, y > 0\}.$$

Suppose that K is a convex, compact subset of $\mathbb{R}^2$ with $K \cap P^+ \neq \emptyset$. Show that there is a unique point $(x_0, y_0) \in K \cap P^+$ such that

$$xy \leqslant x_0 y_0$$

for all $(x,y) \in K \cap P^+$.

3/II/12F Topics in Analysis

(i) State and prove Liouville's theorem on approximation of algebraic numbers by rationals.

(ii) Consider the continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ldots}}}}$$

where the a_j are strictly positive integers. You may assume the following algebraic facts about the nth convergent p_n/q_n.

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Show that

$$\left| \frac{p_n}{q_n} - x \right| \leqslant \frac{1}{q_n q_{n+1}}.$$

Give explicit values for a_n so that x is transcendental and prove that you have done so.

3/I/2F Topics in Analysis

State a version of Runge's theorem and use it to prove the following theorem:

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and define $f : D \to \mathbb{C}$ by the condition

$$f(re^{i\theta}) = r^{3/2} e^{3i\theta/2}$$

for all $0 \leqslant r < 1$ and all $0 \leqslant \theta < 2\pi$. (We take $r^{1/2}$ to be the positive square root.) Then there exists a sequence of analytic functions $f_n : D \to \mathbb{C}$ such that $f_n(z) \to f(z)$ for each $z \in D$ as $n \to \infty$.

4/I/2F Topics in Analysis

State Brouwer's fixed point theorem for a triangle in two dimensions.

Let $A = (a_{ij})$ be a 3×3 matrix with real positive entries and such that all its columns are non-zero vectors. Show that A has an eigenvector with positive entries.

1/I/3G Geometry of Group Actions

Show that there are two ways to embed a regular tetrahedron in a cube C so that the vertices of the tetrahedron are also vertices of C. Show that the symmetry group of C permutes these tetrahedra and deduce that the symmetry group of C is isomorphic to the Cartesian product $S_4 \times C_2$ of the symmetric group S_4 and the cyclic group C_2.

1/II/12G Geometry of Group Actions

Define the Hausdorff d-dimensional measure $\mathcal{H}^d(C)$ and the Hausdorff dimension of a subset C of $\mathbb{R}$.

Set $s = \log 2/\log 3$. Define the Cantor set C and show that its Hausdorff s-dimensional measure is at most 1.

Let (X_n) be independent Bernoulli random variables that take the values 0 and 2, each with probability $\frac{1}{2}$. Define

$$\xi = \sum_{n=1}^{\infty} \frac{X_n}{3^n} .$$

Show that ξ is a random variable that takes values in the Cantor set C.

Let U be a subset of $\mathbb{R}$ with $3^{-(k+1)} \leqslant \mathrm{diam}(U) < 3^{-k}$. Show that $\mathbb{P}(\xi \in U) \leqslant 2^{-k}$ and deduce that, for any set $U \subset \mathbb{R}$, we have

$$\mathbb{P}(\xi \in U) \leqslant 2(\mathrm{diam}(U))^s .$$

Hence, or otherwise, prove that $\mathcal{H}^s(C) \geqslant \frac{1}{2}$ and that the Cantor set has Hausdorff dimension s.

2/I/3G Geometry of Group Actions

Explain what is meant by a lattice in the Euclidean plane $\mathbb{R}^2$. Prove that such a lattice is either $\mathbb{Z}\boldsymbol{w}$ for some vector $\boldsymbol{w} \in \mathbb{R}^2$ or else $\mathbb{Z}\boldsymbol{w}_1 + \mathbb{Z}\boldsymbol{w}_2$ for two linearly independent vectors $\boldsymbol{w}_1, \boldsymbol{w}_2$ in $\mathbb{R}^2$.

3/I/3G Geometry of Group Actions

Let G be a 2-dimensional Euclidean crystallographic group. Define the lattice and point group corresponding to G.

Prove that any non-trivial rotation in the point group of G must have order $2, 3, 4$ or 6.

4/I/3G **Geometry of Group Actions**

Let Γ be a circle on the Riemann sphere. Explain what it means to say that two points of the sphere are inverse points for the circle Γ. Show that, for each point z on the Riemann sphere, there is a unique point z' with z, z' inverse points. Define inversion in Γ.

Prove that the composition of an even number of inversions is a Möbius transformation.

4/II/12G **Geometry of Group Actions**

Explain what it means to say that a group G is a Kleinian group. What is the definition of the limit set for the group G? Prove that a fixed point of a parabolic element in G must lie in the limit set.

Show that the matrix $\begin{pmatrix} 1+aw & -aw^2 \\ a & 1-aw \end{pmatrix}$ represents a parabolic transformation for any non-zero choice of the complex numbers a and w. Find its fixed point.

The *Gaussian integers* are $\mathbb{Z}[i] = \{m + in : m, n \in \mathbb{Z}\}$. Let G be the set of Möbius transformations $z \mapsto \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{Z}[i]$ and $ad - bc = 1$. Prove that G is a Kleinian group. For each point $w = \dfrac{p+iq}{r}$ with p, q, r non-zero integers, find a parabolic transformation $T \in G$ that fixes w. Deduce that the limit set for G is all of the Riemann sphere.

1/I/4G **Coding and Cryptography**

Let Σ_1 and Σ_2 be alphabets of sizes m and a. What does it mean to say that $f : \Sigma_1 \to \Sigma_2^*$ is a decipherable code? State the inequalities of Kraft and Gibbs, and deduce that if letters are drawn from Σ_1 with probabilities $p_1, \ldots, p_m$ then the expected word length is at least $H(p_1, \ldots, p_m)/\log a$.

2/I/4G **Coding and Cryptography**

Briefly explain how and why a signature scheme is used. Describe the El Gamal scheme.

1/II/11G **Coding and Cryptography**

Define the bar product $C_1|C_2$ of linear codes C_1 and C_2, where C_2 is a subcode of C_1. Relate the rank and minimum distance of $C_1|C_2$ to those of C_1 and C_2. Show that if $C^\perp$ denotes the dual code of C, then

$$(C_1|C_2)^\perp = C_2^\perp|C_1^\perp .$$

Using the bar product construction, or otherwise, define the Reed–Muller code $RM(d,r)$ for $0 \leqslant r \leqslant d$. Show that if $0 \leqslant r \leqslant d-1$, then the dual of $RM(d,r)$ is again a Reed–Muller code.

3/I/4G **Coding and Cryptography**

Compute the rank and minimum distance of the cyclic code with generator polynomial $g(X) = X^3 + X + 1$ and parity-check polynomial $h(X) = X^4 + X^2 + X + 1$. Now let α be a root of $g(X)$ in the field with 8 elements. We receive the word $r(X) = X^5 + X^3 + X \pmod{X^7 - 1}$. Verify that $r(\alpha) = \alpha^4$, and hence decode $r(X)$ using minimum-distance decoding.

2/II/11G **Coding and Cryptography**

Define the capacity of a discrete memoryless channel. State Shannon's second coding theorem and use it to show that the discrete memoryless channel with channel matrix

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

has capacity $\log 5 - 2$.

4/I/4G **Coding and Cryptography**

What is a linear feedback shift register? Explain the Berlekamp–Massey method for recovering the feedback polynomial of a linear feedback shift register from its output. Illustrate in the case when we observe output

$$1\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ \ldots .$$

1/I/5I **Statistical Modelling**

According to the *Independent* newspaper (London, 8 March 1994) the Metropolitan Police in London reported 30475 people as missing in the year ending March 1993. For those aged 18 or less, 96 of 10527 missing males and 146 of 11363 missing females were still missing a year later. For those aged 19 and above, the values were 157 of 5065 males and 159 of 3520 females. This data is summarised in the table below.

	age	gender	still	total
1	Kid	M	96	10527
2	Kid	F	146	11363
3	Adult	M	157	5065
4	Adult	F	159	3520

Explain and interpret the R commands and (slightly abbreviated) output below. You should describe the model being fitted, explain how the standard errors are calculated, and comment on the hypothesis tests being described in the summary. In particular, what is the worst of the four categories for the probability of remaining missing a year later?

```
> fit <- glm(still/total ~ age + gender, family = binomial,
+            weights = total)
> summary(fit)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -3.06073    0.07216 -42.417  < 2e-16 ***
ageKid      -1.27079    0.08698 -14.610  < 2e-16 ***
genderM     -0.37211    0.08671  -4.291 1.78e-05 ***

Residual deviance:   0.06514  on 1  degrees of freedom
```

For a person who was missing in the year ending in March 1993, find a formula, as a function of age and gender, for the estimated expected probability that they are still missing a year later.

1/II/13I **Statistical Modelling**

This problem deals with data collected as the number of each of two different strains of *Ceriodaphnia* organisms are counted in a controlled environment in which reproduction is occurring among the organisms. The experimenter places into the containers a varying concentration of a particular component of jet fuel that impairs reproduction. Hence it is anticipated that as the concentration of jet fuel grows, the mean number of organisms should decrease.

The table below gives a subset of the data. The full dataset has $n = 70$ rows. The first column provides the number of organisms, the second the concentration of jet fuel (in grams per litre) and the third specifies the strain of the organism.

```
number fuel  strain

82     0     1

58     0     0

45     0.5   1

27     0.5   0

29     0.75  1

15     1.25  1

6      1.25  1

8      1.5   0

4      1.75  0

.      .     .

.      .     .
```

Explain and interpret the `R` commands and (slightly abbreviated) output below. In particular, you should describe the model being fitted, explain how the standard errors are calculated, and comment on the hypothesis tests being described in the summary.

```
> fit1 <- glm(number ~ fuel + strain + fuel:strain,family = poisson)

> summary(fit1)

Coefficients:

            Estimate Std. Error z value Pr(>|z|)

(Intercept)  4.14443    0.05101  81.252  < 2e-16 ***

fuel        -1.47253    0.07007 -21.015  < 2e-16 ***

strain       0.33667    0.06704   5.022 5.11e-07 ***

fuel:strain -0.12534    0.09385  -1.336    0.182
```

The following `R` code fits two very similar models. Briefly explain the difference between these models and the one above. Motivate the fitting of these models in light of the summary from the fit of the one above.

```
> fit2 <- glm(number ~ fuel + strain, family = poisson)
> fit3 <- glm(number ~ fuel, family = poisson)
```

Denote by H_1, H_2, H_3 the three hypotheses being fitted in sequence above.

Explain the hypothesis tests, including an approximate test of the fit of H_1, that can be performed using the output from the following `R` code. Use these numbers to comment on the most appropriate model for the data.

```
> c(fit1$dev, fit2$dev, fit3$dev)
[1]  84.59557  86.37646 118.99503
> qchisq(0.95, df = 1)
[1] 3.841459
```

2/I/5I **Statistical Modelling**

Consider the linear regression setting where the responses Y_i, $i = 1, \ldots, n$ are assumed independent with means $\mu_i = x_i^{\mathrm{T}}\beta$. Here x_i is a vector of known explanatory variables and β is a vector of unknown regression coefficients.

Show that if the response distribution is Laplace, i.e.,

$$Y_i \sim f(y_i; \mu_i, \sigma) = (2\sigma)^{-1} \exp\left\{-\frac{|y_i - \mu_i|}{\sigma}\right\}, \qquad i = 1, \ldots, n; \quad y_i, \mu_i \in \mathbb{R}; \ \sigma \in (0, \infty);$$

then the maximum likelihood estimate $\hat{\beta}$ of β is obtained by minimising

$$S_1(\beta) = \sum_{i=1}^{n} |Y_i - x_i^{\mathrm{T}}\beta|.$$

Obtain the maximum likelihood estimate for σ in terms of $S_1(\hat{\beta})$.

Briefly comment on why the Laplace distribution cannot be written in exponential dispersion family form.

3/I/5I **Statistical Modelling**

Consider two possible experiments giving rise to observed data y_{ij} where $i = 1, \ldots, I,\ j = 1, \ldots, J$.

1. The data are realizations of independent Poisson random variables, i.e.,

$$Y_{ij} \sim f_1(y_{ij}; \mu_{ij}) = \frac{\mu_{ij}^{y_{ij}}}{y_{ij}!} \exp\{-\mu_{ij}\}$$

   where $\mu_{ij} = \mu_{ij}(\beta)$, with β an unknown (possibly vector) parameter. Write $\hat{\beta}$ for the maximum likelihood estimator (m.l.e.) of β and $\hat{y}_{ij} = \mu_{ij}(\hat{\beta})$ for the (i, j)th fitted value under this model.

2. The data are components of a realization of a multinomial random 'vector'

$$Y \sim f_2((y_{ij}); n, (p_{ij})) = n! \prod_{i=1}^{I} \prod_{j=1}^{J} \frac{p_{ij}^{y_{ij}}}{y_{ij}!}$$

   where the y_{ij} are non-negative integers with

$$\sum_{i=1}^{I} \sum_{j=1}^{J} y_{ij} = n \quad \text{and} \quad p_{ij}(\beta) = \frac{\mu_{ij}(\beta)}{n}\,.$$

   Write β^* for the m.l.e. of β and $y^*_{ij} = np_{ij}(\beta^*)$ for the (i, j)th fitted value under this model.

Show that, if

$$\sum_{i=1}^{I} \sum_{j=1}^{J} \hat{y}_{ij} = n\,,$$

then $\hat{\beta} = \beta^*$ and $\hat{y}_{ij} = y^*_{ij}$ for all i, j. Explain the relevance of this result in the context of fitting multinomial models within a generalized linear model framework.

4/I/5I **Statistical Modelling**

Consider the normal linear model $Y = X\beta + \varepsilon$ in vector notation, where

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}, \quad X = \begin{pmatrix} x_1^{\mathrm{T}} \\ \vdots \\ x_n^{\mathrm{T}} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}, \quad \varepsilon_i \sim \text{i.i.d. } N(0, \sigma^2),$$

where $x_i^{\mathrm{T}} = (x_{i1}, \ldots, x_{ip})$ is known and X is of full rank ($p < n$). Give expressions for maximum likelihood estimators $\hat{\beta}$ and $\hat{\sigma}^2$ of β and σ^2 respectively, and state their joint distribution.

Suppose that there is a new pair (x^*, y^*), independent of $(x_1, y_1), \ldots, (x_n, y_n)$, satisfying the relationship

$$y^* = x^{*\mathrm{T}}\beta + \varepsilon^*, \qquad \text{where} \qquad \varepsilon^* \sim N(0, \sigma^2).$$

We suppose that x^* is known, and estimate y^* by $\tilde{y} = x^{*\mathrm{T}}\hat{\beta}$. State the distribution of

$$\frac{\tilde{y} - y^*}{\tilde{\sigma}\tau}, \qquad \text{where} \qquad \tilde{\sigma}^2 = \frac{n}{n-p}\hat{\sigma}^2 \qquad \text{and} \qquad \tau^2 = x^{*\mathrm{T}}(X^{\mathrm{T}}X)^{-1}x^* + 1.$$

Find the form of a $(1-\alpha)$–level prediction interval for y^*.

4/II/13I **Statistical Modelling**

Let Y have a Gamma distribution with density

$$f(y; \alpha, \lambda) = \frac{\lambda^{\alpha} y^{\alpha - 1}}{\Gamma(\alpha)}\, e^{-\lambda y}\,.$$

Show that the Gamma distribution is of exponential dispersion family form. Deduce directly the corresponding expressions for $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$ in terms of α and λ. What is the canonical link function?

Let $p < n$. Consider a generalised linear model (g.l.m.) for responses y_i, $i = 1, \ldots, n$ with random component defined by the Gamma distribution with canonical link $g(\mu)$, so that $g(\mu_i) = \eta_i = x_i^{\mathrm{T}}\beta$, where $\beta = (\beta_1, \ldots, \beta_p)^{\mathrm{T}}$ is the vector of unknown regression coefficients and $x_i = (x_{i1}, \ldots, x_{ip})^{\mathrm{T}}$ is the vector of known values of the explanatory variables for the ith observation, $i = 1, \ldots, n$.

Obtain expressions for the score function and Fisher information matrix and explain how these can be used in order to approximate $\hat{\beta}$, the maximum likelihood estimator (m.l.e.) of β.

[*Use the canonical link function and assume that the dispersion parameter is known.*]

Finally, obtain an expression for the deviance for a comparison of the full (saturated) model to the g.l.m. with canonical link using the m.l.e. $\hat{\beta}$ (or estimated mean $\hat{\mu} = X\hat{\beta}$).

1/I/6B **Mathematical Biology**

A chemostat is a well-mixed tank of given volume V_0 that contains water in which lives a population $N(t)$ of bacteria that consume nutrient whose concentration is $C(t)$ per unit volume. An inflow pipe supplies a solution of nutrient at concentration C_0 and at a constant flow rate of Q units of volume per unit time. The mixture flows out at the same rate through an outflow pipe. The bacteria consume the nutrient at a rate $NK(C)$, where

$$K(C) = \frac{K_{\max} C}{K_0 + C},$$

and the bacterial population grows at a rate $\gamma NK(C)$, where $0 < \gamma < 1$.

Write down the differential equations for $N(t), C(t)$ and show that they can be rescaled into the following form:

$$\begin{aligned} \frac{dn}{d\tau} &= \alpha \frac{cn}{1+c} - n, \\ \frac{dc}{d\tau} &= -\frac{cn}{1+c} - c + \beta, \end{aligned}$$

where α, β are positive constants, to be found.

Show that this system of equations has a non-trivial steady state if $\alpha > 1$ and $\beta > \dfrac{1}{\alpha - 1}$, and that it is stable.

2/I/6B **Mathematical Biology**

A field contains X_n seed-producing poppies in August of year n. On average each poppy produces γ seeds, a number that is assumed not to vary from year to year. A fraction σ of seeds survive the winter and a fraction α of those germinate in May of year $n+1$. A fraction β of those that survive the next winter germinate in year $n+2$. Show that X_n satisfies the following difference equation:

$$X_{n+1} = \alpha\sigma\gamma X_n + \beta\sigma^2(1-\alpha)\gamma X_{n-1}.$$

Write down the general solution of this equation, and show that the poppies in the field will eventually die out if

$$\sigma\gamma[(1-\alpha)\beta\sigma + \alpha] < 1.$$

2/II/13B Mathematical Biology

Show that the concentration $C(\mathbf{x}, t)$ of a diffusible chemical substance in a stationary medium satisfies the partial differential equation

$$\frac{\partial C}{\partial t} = \nabla \cdot (D \nabla C) + F \, ,$$

where D is the diffusivity and $F(\mathbf{x}, t)$ is the rate of supply of the chemical.

A finite amount of the chemical, $4\pi M$, is supplied at the origin at time $t = 0$, and spreads out in a spherically symmetric manner, so that $C = C(r, t)$ for $r > 0, t > 0$, where r is the radial coordinate. The diffusivity is given by $D = kC$, for constant k . Show, by dimensional analysis or otherwise, that it is appropriate to seek a similarity solution in which

$$C = \frac{M^{\alpha}}{(k\,t)^{\beta}} f(\xi) \, , \quad \xi = \frac{r}{(Mkt)^{\gamma}} \quad \text{and} \quad \int_0^{\infty} \xi^2 f(\xi) \, d\xi = 1 \, ,$$

where α, β, γ are constants to be determined, and derive the ordinary differential equation satisfied by $f(\xi)$.

Solve this ordinary differential equation, subject to appropriate boundary conditions, and deduce that the chemical occupies a finite spherical region of radius

$$r_0(t) = (75Mkt)^{1/5} \, .$$

[*Note: in spherical polar coordinates*

$$\nabla C \equiv \left(\frac{\partial C}{\partial r} \, , 0, 0 \right) \quad \textit{and} \quad \nabla \cdot (V(r,t), 0, 0) \equiv \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V) \, .$$]

3/I/6B Mathematical Biology

Consider a birth and death process in which births always give rise to two offspring, with rate λ, while the death rate per individual is β.

Write down the master equation (or probability balance equation) for this system.

Show that the population mean is given by

$$\langle n \rangle = \frac{2\lambda}{\beta}(1 - e^{-\beta t}) + n_0 e^{-\beta t}$$

where n_0 is the initial population mean, and that the population variance satisfies

$$\sigma^2 \to 3\lambda/\beta \quad \text{as} \quad t \to \infty \, .$$

3/II/13B **Mathematical Biology**

The number density of a population of cells is $n(\mathbf{x}, t)$. The cells produce a chemical whose concentration is $C(\mathbf{x}, t)$ and respond to it chemotactically. The equations governing n and C are

$$\frac{\partial n}{\partial t} = \gamma n(n_0 - n) + D_n \nabla^2 n - \chi \nabla \cdot (n \nabla C)$$
$$\frac{\partial C}{\partial t} = \alpha n - \beta C + D_c \nabla^2 C.$$

(i) Give a biological interpretation of each term in these equations, where you may assume that α, β, γ, n_0, D_n, D_c and χ are all positive.

(ii) Show that there is a steady-state solution that is stable to spatially invariant disturbances.

(iii) Analyse small, spatially-varying perturbations to the steady state that satisfy $\nabla^2 \phi = -k^2 \phi$ for any variable ϕ, and show that a chemotactic instability is possible if

$$\chi \alpha n_0 > \beta D_n + \gamma n_0 D_c + (4\beta\gamma n_0 D_n D_c)^{1/2} .$$

(iv) Find the critical value of k at which the instability first appears as χ is increased.

4/I/6B **Mathematical Biology**

The non-dimensional equations for two competing populations are

$$\frac{du}{dt} = u(1 - v) - \epsilon_1 u^2,$$
$$\frac{dv}{dt} = \alpha \left[v(1 - u) - \epsilon_2 v^2 \right] .$$

Explain the meaning of each term in these equations.

Find all the fixed points of this system when $\alpha > 0$, $0 < \epsilon_1 < 1$ and $0 < \epsilon_2 < 1$, and investigate their stability.

1/I/7E **Dynamical Systems**

Given a non-autonomous kth-order differential equation

$$\frac{d^k y}{dt^k} = g\left(t, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \ldots, \frac{d^{k-1} y}{dt^{k-1}}\right)$$

with $y \in \mathbb{R}$, explain how it may be written in the autonomous first-order form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ for suitably chosen vectors $\mathbf{x}$ and $\mathbf{f}$.

Given an autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$, define the corresponding *flow* $\boldsymbol{\phi}_t(\mathbf{x})$. What is $\boldsymbol{\phi}_s(\boldsymbol{\phi}_t(\mathbf{x}))$ equal to? Define the *orbit* $\mathcal{O}(\mathbf{x})$ through $\mathbf{x}$ and the *limit set* $\omega(\mathbf{x})$ of $\mathbf{x}$. Define a *homoclinic orbit*.

3/II/14E **Dynamical Systems**

The Lorenz equations are

$$\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz
\end{aligned}$$

where r, σ and b are positive constants and $(x, y, z) \in \mathbb{R}^3$.

(i) Show that the origin is globally asymptotically stable for $0 < r < 1$ by considering a function $V(x, y, z) = \frac{1}{2}(x^2 + Ay^2 + Bz^2)$ with a suitable choice of constants A and B.

(ii) State, without proof, the Centre Manifold Theorem.

Show that the fixed point at the origin is nonhyperbolic at $r = 1$. What are the dimensions of the linear stable and (non-extended) centre subspaces at this point?

(iii) Let $\sigma = 1$ from now on. Make the substitutions $u = x + y$, $v = x - y$ and $\mu = r - 1$ and derive the resulting equations for $\dot{u}, \dot{v}$ and $\dot{z}$.

The extended centre manifold is given by

$$v = V(u, \mu), \qquad z = Z(u, \mu)$$

where V and Z can be expanded as power series about $u = \mu = 0$. What is known about V and Z from the Centre Manifold Theorem? Assuming that $\mu = O(u^2)$, determine Z correct to $O(u^2)$ and V to $O(u^3)$. Hence obtain the evolution equation on the extended centre manifold correct to $O(u^3)$, and identify the type of bifurcation.

2/I/7E **Dynamical Systems**

Find and classify the fixed points of the system

$$\dot{x} = (1 - x^2)y \ ,$$
$$\dot{y} = x(1 - y^2) \ .$$

What are the values of their Poincaré indices? Prove that there are no periodic orbits. Sketch the phase plane.

4/II/14E **Dynamical Systems**

Consider the one-dimensional map $F : \mathbb{R} \to \mathbb{R}$ defined by

$$x_{i+1} = F(x_i) = x_i(ax_i^2 + bx_i + \mu),$$

where a and b are constants, μ is a parameter and $a \neq 0$.

(i) Find the fixed points of F and determine the linear stability of $x = 0$. Hence show that there are bifurcations at $\mu = 1$, at $\mu = -1$ and, if $b \neq 0$, at $\mu = 1 + b^2/4a$.

Sketch the bifurcation diagram for each of the cases:

$$(1)\ a > b = 0, \quad (2)\ a, b > 0 \quad \text{and} \quad (3)\ a, b < 0.$$

In each case show the locus and stability of the fixed points in the (μ, x)-plane, and state the type of each bifurcation. [*Assume that there are no further bifurcations in the region sketched.*]

(ii) For the case $F(x) = x(\mu - x^2)$ (i.e. $a = -1$, $b = 0$), you may assume that

$$F^2(x) = x + x(\mu - 1 - x^2)(\mu + 1 - x^2)(1 - \mu x^2 + x^4) \ .$$

Show that there are at most three 2-cycles and determine when they exist. By considering $F'(x_i)F'(x_{i+1})$, or otherwise, show further that one 2-cycle is always unstable when it exists and that the others are unstable when $\mu > \sqrt{5}$. Sketch the bifurcation diagram showing the locus and stability of the fixed points and 2-cycles. State briefly what you would expect to occur in the region $\mu > \sqrt{5}$.

3/I/7E **Dynamical Systems**

State the Poincaré–Bendixson Theorem for a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$.

Prove that if $k^2 < 4$ then the system

$$\dot{x} = x - y - x^3 - xy^2 - k^2xy^2$$
$$\dot{y} = y + x - x^2y - y^3 - k^2x^2y$$

has a periodic orbit in the region $2/(2+k^2) \leqslant x^2 + y^2 \leqslant 1$.

4/I/7E **Dynamical Systems**

By considering the binary representation of the sawtooth map, $F(x) = 2x$ [mod 1] for $x \in [0,1)$, show that:

(i) F has sensitive dependence on initial conditions on $[0,1)$.

(ii) F has topological transitivity on $[0,1)$.

(iii) Periodic points are dense in $[0,1)$.

Find all the 4-cycles of F and express them as fractions.

1/I/8B **Further Complex Methods**

The coefficients $p(z)$ and $q(z)$ of the differential equation

$$w''(z)+p(z)w'(z)+q(z)w(z)=0 \qquad (*)$$

are analytic in the punctured disc $0<|z|<R$, and $w_1(z)$ and $w_2(z)$ are linearly independent solutions in the neighbourhood of the point z_0 in the disc. By considering the effect of analytically continuing w_1 and w_2, show that the equation $(*)$ has a non-trivial solution of the form

$$w(z) = z^{\sigma}\sum_{n=-\infty}^{\infty} c_n z^n .$$

2/I/8B **Further Complex Methods**

The function $I(z)$ is defined by

$$I(z)=\frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t+1}dt .$$

For what values of z is $I(z)$ analytic?

By considering $I(z)-\zeta(z)$, where $\zeta(z)$ is the Riemann zeta function which you may assume is given by

$$\zeta(z)=\frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t-1}dt \qquad (\operatorname{Re} z>1)\,,$$

show that $I(z)=\left(1-2^{1-z}\right)\zeta(z)$. Deduce from this result that the analytic continuation of $I(z)$ is an entire function. [*You may use properties of $\zeta(z)$ without proof.*]

3/I/8B **Further Complex Methods**

Let $w_1(z)$ and $w_2(z)$ be any two linearly independent branches of the P-function

$$\left\{ \begin{matrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\},$$

where $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$, and let $W(z)$ be the Wronskian of $w_1(z)$ and $w_2(z)$.

(i) How is $W(z)$ related to the Wronskian of the principal branches of the P-function at $z = 0$?

(ii) Show that $z^{-\alpha-\alpha'+1}(1-z)^{-\gamma-\gamma'+1}W(z)$ is an entire function.

(iii) Given that $z^{\beta+\beta'+1}W(z)$ is bounded as $z \to \infty$, show that

$$W(z) = Az^{\alpha+\alpha'-1}(1-z)^{\gamma+\gamma'-1},$$

where A is a non-zero constant.

1/II/14B **Further Complex Methods**

The function $J(z)$ is defined by

$$J(z) = \int_{\mathcal{P}} t^{z-1}(1-t)^{b-1}dt$$

where b is a constant (which is not an integer). The path of integration, $\mathcal{P}$, is the Pochhammer contour, defined as follows. It starts at a point A on the axis between 0 and 1, then it circles the points 1 and 0 in the negative sense, then it circles the points 1 and 0 in the positive sense, returning to A. At the start of the path, $\arg(t) = \arg(1-t) = 0$ and the integrand is defined at other points on $\mathcal{P}$ by analytic continuation along $\mathcal{P}$.

(i) For what values of z is $J(z)$ analytic? Give brief reasons for your answer.

(ii) Show that, in the case $\operatorname{Re} z > 0$ and $\operatorname{Re} b > 0$,

$$J(z) = -4e^{-\pi i(z+b)}\sin(\pi z)\sin(\pi b)\,\mathrm{B}(z,b)\ ,$$

where $\mathrm{B}(z,b)$ is the Beta function.

(iii) Deduce that the only singularities of $\mathrm{B}(z,b)$ are simple poles. Explain carefully what happens if z is a positive integer.

4/I/8B **Further Complex Methods**

The hypergeometric function $F(a,b;c;z)$ is defined by

$$F(a,b;c;z) = K\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$

where $|\arg(1-tz)| < \pi$ and K is a constant determined by the condition $F(a,b;c;0) = 1$.

(i) Express K in terms of Gamma functions.

(ii) By considering the nth derivative $F^{(n)}(a,b;c;0)$, show that $F(a,b;c;z) = F(b,a;c;z)$.

2/II/14B **Further Complex Methods**

Show that the equation

$$zw'' - (1+z)w' + 2(1-z)w = 0$$

has solutions of the form $w(z) = \int_\gamma e^{zt} f(t)\, dt$, where

$$f(t) = \frac{1}{(t-2)(t+1)^2}\ ,$$

provided that γ is suitably chosen.

Hence find the general solution, evaluating the integrals explicitly. Show that the general solution is entire, but that there is no solution that satisfies $w(0) = 0$ and $w'(0) \neq 0$.

1/I/9C **Classical Dynamics**

The action for a system with generalized coordinates, $q_i(t)$, for a time interval $[t_1, t_2]$ is given by

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt \ ,$$

where L is the Lagrangian, and where the end point values $q_i(t_1)$ and $q_i(t_2)$ are fixed at specified values. Derive Lagrange's equations from the principle of least action by considering the variation of S for all possible paths.

What is meant by the statement that a particular coordinate q_j is ignorable? Show that there is an associated constant of the motion, to be specified in terms of L.

A particle of mass m is constrained to move on the surface of a sphere of radius a under a potential, $V(\theta)$, for which the Lagrangian is given by

$$L = \frac{m}{2} a^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - V(\theta) \ .$$

Identify an ignorable coordinate and find the associated constant of the motion, expressing it as a function of the generalized coordinates. Evaluate the quantity

$$H = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

in terms of the same generalized coordinates, for this case. Is H also a constant of the motion? If so, why?

2/II/15C **Classical Dynamics**

(a) A Hamiltonian system with n degrees of freedom is described by the phase space coordinates $(q_1, q_2, ..., q_n)$ and momenta $(p_1, p_2, ..., p_n)$. Show that the phase-space volume element

$$d\tau \;=\; dq_1 dq_2 .....dq_n dp_1 dp_2 .....dp_n$$

is conserved under time evolution.

(b) The Hamiltonian, H , for the system in part (a) is independent of time. Show that if $F(q_1, ..., q_n, p_1, ..., p_n)$ is a constant of the motion, then the Poisson bracket $[F, H]$ vanishes. Evaluate $[F, H]$ when

$$F \;=\; \sum_{k=1}^{n} p_k$$

and

$$H \;=\; \sum_{k=1}^{n} p_k^2 + V(q_1, q_2, ..., q_n) \; ,$$

where the potential V depends on the q_k $(k = 1, 2, ..., n)$ only through quantities of the form $q_i - q_j$ for $i \neq j$.

(c) For a system with one degree of freedom, state what is meant by the transformation

$$(q, p) \to \big(Q(q, p),\; P(q, p)\big)$$

being canonical. Show that the transformation is canonical if and only if the Poisson bracket $[Q, P] = 1$.

2/I/9C **Classical Dynamics**

The Lagrangian for a particle of mass m and charge e moving in a magnetic field with position vector $\mathbf{r} = (x, y, z)$ is given by

$$L = \tfrac{1}{2} m \dot{\mathbf{r}}^2 + e \frac{\dot{\mathbf{r}} \cdot \mathbf{A}}{c} ,$$

where the vector potential $\mathbf{A}(\mathbf{r})$, which does not depend on time explicitly, is related to the magnetic field $\mathbf{B}$ through

$$\mathbf{B} = \nabla \times \mathbf{A} .$$

Write down Lagrange's equations and use them to show that the equation of motion of the particle can be written in the form

$$m \ddot{\mathbf{r}} = e \frac{\dot{\mathbf{r}} \times \mathbf{B}}{c} .$$

Deduce that the kinetic energy, T, is constant.

When the magnetic field is of the form $\mathbf{B} = (0, 0, dF/dx)$ for some specified function $F(x)$, show further that

$$\dot{x}^2 = \frac{2T}{m} - \frac{\big(eF(x) + C\big)^2}{m^2 c^2} + D ,$$

where C and D are constants.

3/I/9C **Classical Dynamics**

A particle of mass m_1 is constrained to move in the horizontal (x, y) plane, around a circle of fixed radius r_1 whose centre is at the origin of a Cartesian coordinate system (x, y, z). A second particle of mass m_2 is constrained to move around a circle of fixed radius r_2 that also lies in a horizontal plane, but whose centre is at $(0, 0, a)$. It is given that the Lagrangian L of the system can be written as

$$L = \frac{m_1}{2} r_1^2 \dot{\phi}_1^2 + \frac{m_2}{2} r_2^2 \dot{\phi}_2^2 + \omega^2 r_1 r_2 \cos(\phi_2 - \phi_1) ,$$

using the particles' cylindrical polar angles ϕ_1 and ϕ_2 as generalized coordinates. Deduce the equations of motion and use them to show that $m_1 r_1^2 \dot{\phi}_1 + m_2 r_2^2 \dot{\phi}_2$ is constant, and that $\psi = \phi_2 - \phi_1$ obeys an equation of the form

$$\ddot{\psi} = -k^2 \sin \psi ,$$

where k is a constant to be determined.

Find two values of ψ corresponding to equilibria, and show that one of the two equilibria is stable. Find the period of small oscillations about the stable equilibrium.

4/II/15C **Classical Dynamics**

The Hamiltonian for an oscillating particle with one degree of freedom is

$$H = \frac{p^2}{2m} + V(q, \lambda) \ .$$

The mass m is a constant, and λ is a function of time t alone. Write down Hamilton's equations and use them to show that

$$\frac{dH}{dt} = \frac{\partial H}{\partial \lambda} \frac{d\lambda}{dt} \ .$$

Now consider a case in which λ is constant and the oscillation is exactly periodic. Denote the constant value of H in that case by E. Consider the quantity $I = (2\pi)^{-1} \oint p\, dq$, where the integral is taken over a single oscillation cycle. For any given function $V(q, \lambda)$ show that I can be expressed as a function of E and λ alone, namely

$$I = I(E, \lambda) = \frac{(2m)^{1/2}}{2\pi} \oint \big(E - V(q, \lambda) \big)^{1/2} dq \ ,$$

where the sign of the integrand alternates between the two halves of the oscillation cycle. Let τ be the period of oscillation. Show that the function $I(E, \lambda)$ has partial derivatives

$$\frac{\partial I}{\partial E} = \frac{\tau}{2\pi} \qquad \text{and} \qquad \frac{\partial I}{\partial \lambda} = -\frac{1}{2\pi} \oint \frac{\partial V}{\partial \lambda} dt \ .$$

You may assume without proof that $\partial/\partial E$ and $\partial/\partial\lambda$ may be taken inside the integral.

Now let λ change very slowly with time t, by a negligible amount during an oscillation cycle. Assuming that, to sufficient approximation,

$$\frac{d\langle H \rangle}{dt} = \frac{\partial \langle H \rangle}{\partial \lambda} \frac{d\lambda}{dt}$$

where $\langle H \rangle$ is the average value of H over an oscillation cycle, and that

$$\frac{dI}{dt} = \frac{\partial I}{\partial E} \frac{d\langle H \rangle}{dt} + \frac{\partial I}{\partial \lambda} \frac{d\lambda}{dt} \ ,$$

deduce that $dI/dt = 0$, carefully explaining your reasoning.

When

$$V(q, \lambda) = \lambda q^{2n}$$

with n a positive integer and λ positive, deduce that

$$\langle H \rangle = C \lambda^{1/(n+1)}$$

for slowly-varying λ, where C is a constant.

[*Do not try to solve Hamilton's equations. Rather, consider the form taken by* I.]

4/I/9C **Classical Dynamics**

(a) Show that the principal moments of inertia for the oblate spheroid of mass M defined by

$$\frac{(x_1^2 + x_2^2)}{a^2} + \frac{x_3^2}{a^2(1-e^2)} \leqslant 1$$

are given by $(I_1, I_2, I_3) = \frac{2}{5} M a^2 \left(1 - \frac{1}{2}e^2,\, 1 - \frac{1}{2}e^2,\, 1\right)$. Here a is the semi-major axis and e is the eccentricity.

[*You may assume that a sphere of radius a has principal moments of inertia $\frac{2}{5}Ma^2$.*]

(b) The spheroid in part (a) rotates about an axis that is not a principal axis. Euler's equations governing the angular velocity $(\omega_1, \omega_2, \omega_3)$ as viewed in the body frame are

$$I_1 \frac{d\omega_1}{dt} = (I_2 - I_3)\omega_2\omega_3 \ ,$$

$$I_2 \frac{d\omega_2}{dt} = (I_3 - I_1)\omega_3\omega_1 \ ,$$

and

$$I_3 \frac{d\omega_3}{dt} = (I_1 - I_2)\omega_1\omega_2 \ .$$

Show that ω_3 is constant. Show further that the angular momentum vector precesses around the x_3 axis with period

$$P = \frac{2\pi(2 - e^2)}{e^2 \omega_3} \ .$$

1/I/10A **Cosmology**

Describe the motion of light rays in an expanding universe with scale factor $a(t)$, and derive the redshift formula

$$1+z=\frac{a(t_0)}{a(t_{\mathrm{e}})}\,,$$

where the light is emitted at time t_{e} and observed at time t_0.

A galaxy at comoving position $\mathbf{x}$ is observed to have a redshift z. Given that the galaxy emits an amount of energy L per unit time, show that the total energy per unit time crossing a sphere centred on the galaxy and intercepting the earth is $L/(1+z)^2$. Hence, show that the energy per unit time per unit area passing the earth is

$$\frac{L}{(1+z)^2}\,\frac{1}{4\pi|\mathbf{x}|^2a^2(t_0)}\,.$$

2/I/10A **Cosmology**

The number density of photons in thermal equilibrium at temperature T takes the form

$$n=\frac{8\pi}{c^3}\int\frac{\nu^2d\nu}{\exp(h\nu/kT)-1}\,.$$

At time $t=t_{\mathrm{dec}}$ and temperature $T=T_{\mathrm{dec}}$, photons decouple from thermal equilibrium. By considering how the photon frequency redshifts as the universe expands, show that the form of the equilibrium frequency distribution is preserved, with the temperature for $t>t_{\mathrm{dec}}$ defined by

$$T\equiv\frac{a(t_{\mathrm{dec}})}{a(t)}T_{\mathrm{dec}}\,.$$

Show that the photon number density n and energy density ϵ can be expressed in the form

$$n=\alpha T^3\,,\qquad\qquad \epsilon=\xi T^4\,,$$

where the constants α and ξ need not be evaluated explicitly.

1/II/15A **Cosmology**

In a homogeneous and isotropic universe, the scale factor $a(t)$ obeys the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho\,,$$

where ρ is the matter density, which, together with the pressure P, satisfies

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P/c^2\right)\,.$$

Here, k is a constant curvature parameter. Use these equations to show that the rate of change of the Hubble parameter $H = \dot{a}/a$ satisfies

$$\dot{H} + H^2 = -\frac{4\pi G}{3}\left(\rho + 3P/c^2\right)\,.$$

Suppose that an *expanding* Friedmann universe is filled with radiation (density ρ_R and pressure $P_R = \rho_R c^2/3$) as well as a "dark energy" component (density ρ_Λ and pressure $P_\Lambda = -\rho_\Lambda c^2$). Given that the energy densities of these two components are measured today ($t = t_0$) to be

$$\rho_{R0} = \beta\,\frac{3H_0^2}{8\pi G} \quad \text{and} \quad \rho_{\Lambda 0} = \frac{3H_0^2}{8\pi G} \quad \text{with constant } \beta > 0 \quad \text{and} \quad a(t_0) = 1\,,$$

show that the curvature parameter must satisfy $kc^2 = \beta H_0^2$. Hence derive the following relations for the Hubble parameter and its time derivative:

$$H^2 = \frac{H_0^2}{a^4}\left(\beta - \beta a^2 + a^4\right)\,,$$

$$\dot{H} = -\beta\,\frac{H_0^2}{a^4}\left(2 - a^2\right)\,.$$

Show qualitatively that universes with $\beta > 4$ will recollapse to a Big Crunch in the future. [*Hint: Sketch a^4H^2 and $a^4\dot{H}$ versus a^2 for representative values of β.*]

For $\beta = 4$, find an explicit solution for the scale factor $a(t)$ satisfying $a(0) = 0$. Find the limiting behaviours of this solution for large and small t. Comment briefly on their significance.

3/I/10A **Cosmology**

The number density of a non-relativistic species in thermal equilibrium is given by

$$n = g_s \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \exp\left[(\mu - mc^2)/kT\right] .$$

Suppose that thermal and chemical equilibrium is maintained between protons p (mass m_{p}, degeneracy $g_s = 2$), neutrons n (mass $m_{\mathrm{n}} \approx m_{\mathrm{p}}$, degeneracy $g_s = 2$) and helium-4 nuclei ^{4}He (mass $m_{\mathrm{He}} \approx 4m_{\mathrm{p}}$, degeneracy $g_s = 1$) via the interaction

$$2\mathrm{p} + 2\mathrm{n} \leftrightarrow {}^4\mathrm{He} + \gamma\,,$$

where you may assume the photons γ have zero chemical potential $\mu_\gamma = 0$. Given that the binding energy of helium-4 obeys $B_{\mathrm{He}}/c^2 \equiv 2m_{\mathrm{p}} + 2n_{\mathrm{n}} - m_{\mathrm{He}} \ll m_{\mathrm{He}}$, show that the ratio of the number densities can be written as

$$\frac{n_{\mathrm{p}}^2\, n_{\mathrm{n}}^2}{n_{\mathrm{He}}} = 2\left(\frac{2\pi m_{\mathrm{p}}kT}{h^2}\right)^{9/2} \exp(-B_{\mathrm{He}}/kT)\,. \qquad (\dagger)$$

Explain briefly why the baryon-to-photon ratio $\eta \equiv n_B/n_\gamma$ remains constant during the expansion of the universe, where $n_B \approx n_{\mathrm{p}} + n_{\mathrm{n}} + 4n_{\mathrm{He}}$ and $n_\gamma \approx (16\pi/(hc)^3)(kT)^3$.

By considering the fractional densities $X_i \equiv n_i/n_B$ of the species i, re-express the ratio ($\dagger$) in the form

$$\frac{X_{\mathrm{p}}^2 X_{\mathrm{n}}^2}{X_{\mathrm{He}}} = \eta^{-3}\frac{1}{32}\left(\frac{\pi}{2}\right)^{3/2}\left(\frac{m_{\mathrm{p}}c^2}{kT}\right)^{9/2} \exp\left(-B_{\mathrm{He}}/kT\right) .$$

Given that $B_{\mathrm{He}} \approx 30$MeV, verify (very approximately) that this ratio approaches unity when $kT \approx 0.3$MeV. In reality, helium-4 is not formed until after deuterium production at a considerably lower temperature. Explain briefly the reason for this delay.

4/I/10A **Cosmology**

The equation governing density perturbation modes $\delta_{\mathbf{k}}(t)$ in a matter-dominated universe (with $a(t) = (t/t_0)^{2/3}$) is

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 \delta_{\mathbf{k}} = 0\,,$$

where $\mathbf{k}$ is the comoving wavevector. Find the general solution for the perturbation, showing that there is a growing mode such that

$$\delta_{\mathbf{k}}(t) \approx \frac{a(t)}{a(t_i)}\delta_{\mathbf{k}}(t_i) \quad (t \gg t_i)\,.$$

Show that the physical wavelength corresponding to the comoving wavenumber $k = |\mathbf{k}|$ crosses the Hubble radius cH^{-1} at a time t_k given by

$$\frac{t_k}{t_0} = \left(\frac{k_0}{k}\right)^3\,, \qquad \text{where } \ k_0 = \frac{2\pi}{cH_0^{-1}}\,.$$

According to inflationary theory, the amplitude of the variance at horizon-crossing is constant, that is, $\langle|\delta_{\mathbf{k}}(t_k)|^2\rangle = AV^{-1}/k^3$ where A and V (the volume) are constants. Given this amplitude and the results obtained above, deduce that the power spectrum today takes the form

$$P(k) \equiv V\langle|\delta_{\mathbf{k}}(t_0)|^2\rangle = \frac{A}{k_0^4}\,k\,.$$

3/II/15A **Cosmology**

A spherically symmetric star with outer radius R has mass density $\rho(r)$ and pressure $P(r)$, where r is the distance from the centre of the star. Show that hydrostatic equilibrium implies the pressure support equation,

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,, \qquad (\dagger)$$

where $m(r)$ is the mass inside radius r. State without proof any results you may need.

Write down an integral expression for the total gravitational potential energy $E_{\rm grav}$ of the star. Hence use ($\dagger$) to deduce the virial theorem

$$E_{\rm grav} = -3\langle P\rangle V\,, \qquad (*)$$

where $\langle P\rangle$ is the average pressure and V is the volume of the star.

Given that a non-relativistic ideal gas obeys $P = 2E_{\rm kin}/3V$ and that an ultra-relativistic gas obeys $P = E_{\rm kin}/3V$, where $E_{\rm kin}$ is the kinetic energy, discuss briefly the gravitational stability of a star in these two limits.

At zero temperature, the number density of particles obeying the Pauli exclusion principle is given by

$$n = \frac{4\pi g_s}{h^3}\int_0^{p_{\rm F}} p^2\,dp = \frac{4\pi g_s}{3}\left(\frac{p_{\rm F}}{h}\right)^3\,,$$

where $p_{\rm F}$ is the Fermi momentum, g_s is the degeneracy and h is Planck's constant. Deduce that the non-relativistic internal energy $E_{\rm kin}$ of these particles is

$$E_{\rm kin} = \frac{4\pi g_s V h^2}{10 m_p}\left(\frac{p_{\rm F}}{h}\right)^5\,,$$

where m_p is the mass of a particle. Hence show that the non-relativistic Fermi degeneracy pressure satisfies

$$P \;\sim\; \frac{h^2}{m_p}\,n^{5/3}\,.$$

Use the virial theorem ($*$) to estimate that the radius R of a star supported by Fermi degeneracy pressure is approximately

$$R \;\sim\; \frac{h^2 M^{-1/3}}{G m_p^{8/3}}\,,$$

where M is the total mass of the star.

[*Hint: Assume* $\rho(r) = m_p n(r) \sim m_p\langle n\rangle$ *and note that* $M \approx (4\pi R^3/3)\,m_p\langle n\rangle$.]

1/II/16G **Set Theory and Logic**

By a *directed set* in a poset $(P, \leqslant)$, we mean a nonempty subset D such that any pair $\{x, y\}$ of elements of D has an upper bound in D. We say $(P, \leqslant)$ is *directed-complete* if each directed subset $D \subseteq P$ has a least upper bound in P. Show that a poset is complete if and only if it is directed-complete and has joins for all its finite subsets. Show also that, for any two sets A and B, the set $[A \rightharpoonup B]$ of partial functions from A to B, ordered by extension, is directed-complete.

Let $(P, \leqslant)$ be a directed-complete poset, and $f : P \to P$ an order-preserving map which is *inflationary*, i.e. satisfies $x \leqslant f(x)$ for all $x \in P$. We define a subset $C \subseteq P$ to be *closed* if it satisfies $(x \in C) \to (f(x) \in C)$, and is also closed under joins of directed sets (i.e., $D \subseteq C$ and D directed imply $\bigvee D \in C$). We write $x \ll y$ to mean that every closed set containing x also contains y. Show that $\ll$ is a partial order on P, and that $x \ll y$ implies $x \leqslant y$. Now consider the set H of all functions $h : P \to P$ which are order-preserving and satisfy $x \ll h(x)$ for all x. Show that H is closed under composition of functions, and deduce that, for each $x \in P$, the set $H_x = \{h(x) \mid h \in H\}$ is directed. Defining $h_0(x) = \bigvee H_x$ for each x, show that the function h_0 belongs to H, and deduce that $h_0(x)$ is the least fixed point of f lying above x, for each $x \in P$.

2/II/16G **Set Theory and Logic**

Explain carefully what is meant by a *deduction* in the propositional calculus. State the completeness theorem for the propositional calculus, and deduce the compactness theorem.

Let P, Q, R be three pairwise-disjoint sets of primitive propositions, and suppose given compound propositions $s \in \mathcal{L}(P \cup Q)$ and $t \in \mathcal{L}(Q \cup R)$ such that $(s \vdash t)$ holds. Let U denote the set

$$\{u \in \mathcal{L}(Q) \mid \ (s \vdash u)\} \ .$$

If $v : Q \to 2$ is any valuation making all the propositions in U true, show that the set

$$\{s\} \cup \{q \mid q \in Q, v(q) = 1\} \cup \{\neg q \mid q \in Q, v(q) = 0\}$$

is consistent. Deduce that $U \cup \{\neg t\}$ is inconsistent, and hence show that there exists $u \in \mathcal{L}(Q)$ such that $(s \vdash u)$ and $(u \vdash t)$ both hold.

3/II/16G Set Theory and Logic

Write down the recursive definitions of ordinal addition, multiplication and exponentiation. Prove carefully that $\omega^\alpha \geqslant \alpha$ for all α, and hence show that for each non-zero ordinal α there exists a unique $\alpha_0 \leqslant \alpha$ such that

$$\omega^{\alpha_0} \leqslant \alpha < \omega^{\alpha_0+1} .$$

Deduce that any non-zero ordinal α has a unique representation of the form

$$\omega^{\alpha_0} \cdot a_0 + \omega^{\alpha_1} \cdot \alpha_1 + \cdots + \omega^{\alpha_n} \cdot a_n$$

where $\alpha \geqslant \alpha_0 > \alpha_1 > \cdots > \alpha_n$ and $a_0, a_1, \ldots, a_n$ are non-zero natural numbers.

Two ordinals β, γ are said to be *commensurable* if we have neither $\beta + \gamma = \gamma$ nor $\gamma + \beta = \beta$. Show that β and γ are commensurable if and only if there exists α such that both β and γ lie in the set

$$\{\delta \mid \omega^\alpha \leqslant \delta < \omega^{\alpha+1}\} .$$

4/II/16G Set Theory and Logic

Explain what is meant by a *well-founded* binary relation on a set.

Given a set a, we say that a mapping $f : a \to \mathcal{P}a$ is *recursive* if, given any set b equipped with a mapping $g \colon \mathcal{P}b \to b$, there exists a unique $h \colon a \to b$ such that $h = g \circ h_* \circ f$, where $h_* \colon \mathcal{P}a \to \mathcal{P}b$ denotes the mapping $a' \mapsto \{h(x) \mid x \in a'\}$. Show that f is recursive if and only if the relation $\{\langle x, y\rangle \mid x \in f(y)\}$ is well-founded.

[*If you need to use any form of the recursion theorem, you should prove it.*]

1/II/17H **Graph Theory**

Let G be a connected cubic graph drawn in the plane with each edge in the boundary of two distinct faces. Show that the associated map is 4-colourable if and only if G is 3-edge colourable.

Is the above statement true if the plane is replaced by the torus and all faces are required to be simply connected? Give a proof or a counterexample.

2/II/17H **Graph Theory**

The Ramsey number $R(G)$ of a graph G is the smallest n such that in any red/blue colouring of the edges of K_n there is a monochromatic copy of G.

Show that $R(K_t) \leqslant \binom{2t-2}{t-1}$ for every $t \geqslant 3$.

Let H be the graph on four vertices obtained by adding an edge to a triangle. Show that $R(H) = 7$.

3/II/17H **Graph Theory**

Let G be a bipartite graph with vertex classes X and Y, each of size n. State and prove Hall's theorem giving a necessary and sufficient condition for G to contain a perfect matching.

A vertex $x \in X$ is *flexible* if every edge from x is contained in a perfect matching. Show that if $|\Gamma(A)| > |A|$ for every subset A of X with $\emptyset \neq A \neq X$, then every $x \in X$ is flexible.

Show that whenever G contains a perfect matching, there is at least one flexible $x \in X$.

Give an example of such a G where no $x \in X$ of minimal degree is flexible.

4/II/17H **Graph Theory**

Let G be a graph with n vertices and m edges. Show that if G contains no C_4, then $m \leqslant \frac{n}{4}(1+\sqrt{4n-3})$.

Let $C_4(G)$ denote the number of subgraphs of G isomorphic to C_4. Show that if $m \geqslant \frac{n(n-1)}{4}$, then G contains at least $\frac{n(n-1)(n-3)}{8}$ paths of length 2. By considering the numbers $r_1, r_2, \ldots, r_{\binom{n}{2}}$ of vertices joined to each pair of vertices of G, deduce that

$$C_4(G) \geqslant \tfrac{1}{2}\binom{n}{2}\binom{(n-3)/4}{2}.$$

Now let $G = G(n, 1/2)$ be the random graph on $\{1, 2, \ldots, n\}$ in which each pair of vertices is joined independently with probability $1/2$. Find the expectation $\mathbb{E}(C_4(G))$ of $C_4(G)$. Deduce that if $0 < \epsilon < 1/2$, then

$$\Pr\left(C_4(G) \leqslant (1+2\epsilon)\frac{3}{16}\binom{n}{4}\right) \geqslant \epsilon.$$

1/II/18F Galois Theory

Let $L/K/M$ be field extensions. Define the *degree* $[K : M]$ of the field extension K/M, and state and prove the tower law.

Now let K be a finite field. Show $\#K = p^n$, for some prime p and positive integer n. Show also that K contains a subfield of order p^m if and only if $m|n$.

If $f \in K[x]$ is an irreducible polynomial of degree d over the finite field K, determine its Galois group.

2/II/18F Galois Theory

Let $L = K(\xi_n)$, where ξ_n is a primitive nth root of unity and $G = \mathrm{Aut}(L/K)$. Prove that there is an injective group homomorphism $\chi : G \to (\mathbb{Z}/n\mathbb{Z})^*$.

Show that, if M is an intermediate subfield of $K(\xi_n)/K$, then M/K is Galois. State carefully any results that you use.

Give an example where G is non-trivial but χ is not surjective. Show that χ is surjective when $K = \mathbb{Q}$ and n is a prime.

Determine all the intermediate subfields M of $\mathbb{Q}(\xi_7)$ and the automorphism groups $\mathrm{Aut}(\mathbb{Q}\ (\xi_7)/M)$. Write the quadratic subfield in the form $\mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Q}$.

3/II/18F Galois Theory

(i) Let K be the splitting field of the polynomial $x^4 - 3$ over $\mathbb{Q}$. Describe the field K, the Galois group $G = \mathrm{Aut}(K/\mathbb{Q})$, and the action of G on K.

(ii) Let K be the splitting field of the polynomial $x^4 + 4x^2 + 2$ over $\mathbb{Q}$. Describe the field K and determine $\mathrm{Aut}(K/\mathbb{Q})$.

4/II/18F Galois Theory

Let $f(x) \in K[x]$ be a monic polynomial, L a splitting field for f, $\alpha_1, \ldots, \alpha_n$ the roots of f in L. Let $\triangle(f) = \prod_{i<j}(\alpha_i - \alpha_j)^2$ be the *discriminant* of f. Explain why $\triangle(f)$ is a polynomial function in the coefficients of f, and determine $\triangle(f)$ when $f(x) = x^3 + px + q$.

Compute the Galois group of the polynomial $x^3 - 3x + 1 \in \mathbb{Q}[x]$.

1/II/19H Representation Theory

A finite group G has seven conjugacy classes $C_1 = \{e\}, C_2, \ldots, C_7$ and the values of five of its irreducible characters are given in the following table.

C_1	C_2	C_3	C_4	C_5	C_6	C_7
1	1	1	1	1	1	1
1	1	1	1	-1	-1	-1
4	0	1	-1	2	-1	0
4	0	1	-1	-2	1	0
5	1	-1	0	1	1	-1

Calculate the number of elements in the various conjugacy classes and complete the character table.

[*You may not identify G with any known group, unless you justify doing so.*]

2/II/19H Representation Theory

Let G be a finite group and let Z be its centre. Show that if ρ is a complex irreducible representation of G, assumed to be faithful (that is, the kernel of ρ is trivial), then Z is cyclic.

Now assume that G is a p-group (that is, the order of G is a power of the prime p), and assume that Z is cyclic. If ρ is a faithful representation of G, show that some irreducible component of ρ is faithful.

[*You may use without proof the fact that, since G is a p-group, Z is non-trivial and any non-trivial normal subgroup of G intersects Z non-trivially.*]

Deduce that a finite p-group has a faithful irreducible representation if and only if its centre is cyclic.

3/II/19H Representation Theory

Let G be a finite group with a permutation action on the set X. Describe the corresponding permutation character π_X. Show that the multiplicity in π_X of the principal character 1_G equals the number of orbits of G on X.

Assume that G is transitive on X, with $|X| > 1$. Show that G contains an element g which is fixed-point-free on X, that is, $g\alpha \neq \alpha$ for all α in X.

Assume that $\pi_X = 1_G + m\chi$, with χ an irreducible character of G, for some natural number m. Show that $m = 1$.

[*You may use without proof any facts about algebraic integers, provided you state them correctly.*]

Explain how the action of G on X induces an action of G on X^2. Assume that G has r orbits on X^2. If now

$$\pi_X = 1_G + m_2\chi_2 + \ldots + m_k\chi_k,$$

with $1_G, \chi_2, \ldots, \chi_k$ distinct irreducible characters of G, and $m_2, \ldots, m_k$ natural numbers, show that $r = 1 + m_2^2 + \ldots + m_k^2$. Deduce that, if $r \leqslant 5$, then $k = r$ and $m_2 = \ldots = m_k = 1$.

4/II/19H Representation Theory

Write an essay on the representation theory of SU_2.

Your answer should include a description of each irreducible representation and an explanation of how to decompose arbitrary representations into a direct sum of these.

1/II/20H **Number Fields**

Let $K = \mathbb{Q}(\sqrt{-26})$.

(a) Show that $\mathcal{O}_K = \mathbb{Z}[\sqrt{-26}]$ and that the discriminant d_K is equal to -104.

(b) Show that 2 ramifies in $\mathcal{O}_K$ by showing that $[2] = \mathfrak{p}_2^2$, and that $\mathfrak{p}_2$ is not a principal ideal. Show further that $[3] = \mathfrak{p}_3\bar{\mathfrak{p}}_3$ with $\mathfrak{p}_3 = [3, 1 - \sqrt{-26}]$. Deduce that neither $\mathfrak{p}_3$ nor $\mathfrak{p}_3^2$ is a principal ideal, but $\mathfrak{p}_3^3 = [1 - \sqrt{-26}]$.

(c) Show that 5 splits in $\mathcal{O}_K$ by showing that $[5] = \mathfrak{p}_5\bar{\mathfrak{p}}_5$, and that

$$N_{K/\mathbb{Q}}(2 + \sqrt{-26}) = 30.$$

Deduce that $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5$ has trivial class in the ideal class group of K. Conclude that the ideal class group of K is cyclic of order six.

[*You may use the fact that* $\frac{2}{\pi}\sqrt{104} \approx 6.492$.]

2/II/20H **Number Fields**

Let $K = \mathbb{Q}(\sqrt{10})$ and put $\varepsilon = 3 + \sqrt{10}$.

(a) Show that 2, 3 and $\varepsilon + 1$ are irreducible elements in $\mathcal{O}_K$. Deduce from the equation

$$6 = 2 \cdot 3 = (\varepsilon + 1)(\bar{\varepsilon} + 1)$$

that $\mathcal{O}_K$ is not a principal ideal domain.

(b) Put $\mathfrak{p}_2 = [2, \varepsilon + 1]$ and $\mathfrak{p}_3 = [3, \varepsilon + 1]$. Show that

$$[2] = \mathfrak{p}_2^2\,, \quad [3] = \mathfrak{p}_3\bar{\mathfrak{p}}_3\,, \quad \mathfrak{p}_2\mathfrak{p}_3 = [\varepsilon + 1]\,, \quad \mathfrak{p}_2\bar{\mathfrak{p}}_3 = [\varepsilon - 1]\,.$$

Deduce that K has class number 2.

(c) Show that ε is the fundamental unit of K. Hence prove that all solutions in integers x, y of the equation $x^2 - 10y^2 = 6$ are given by

$$x + \sqrt{10}y = \pm\varepsilon^n(\varepsilon + (-1)^n)\,, \;\; n = 0, 1, 2, \ldots$$

4/II/20H **Number Fields**

Let K be a finite extension of $\mathbb{Q}$ and let $\mathcal{O} = \mathcal{O}_K$ be its ring of integers. We will assume that $\mathcal{O} = \mathbb{Z}[\theta]$ for some $\theta \in \mathcal{O}$. The minimal polynomial of θ will be denoted by g. For a prime number p let

$$\bar{g}(X) = \bar{g}_1(X)^{e_1} \cdot \ldots \cdot \bar{g}_r(X)^{e_r}$$

be the decomposition of $\bar{g}(X) = g(X) + p\mathbb{Z}[X] \in (\mathbb{Z}/p\mathbb{Z})[X]$ into distinct irreducible monic factors $\bar{g}_i(X) \in (\mathbb{Z}/p\mathbb{Z})[X]$. Let $g_i(X) \in \mathbb{Z}[X]$ be a polynomial whose reduction modulo p is $\bar{g}_i(X)$. Show that

$$\mathfrak{p}_i = [p, g_i(\theta)]\,, \quad i = 1, \ldots, r\,,$$

are the prime ideals of $\mathcal{O}$ containing p, that these are pairwise different, and

$$[p] = \mathfrak{p}_1^{e_1} \cdot \ldots \cdot \mathfrak{p}_r^{e_r}\,.$$

1/II/21H **Algebraic Topology**

(i) Compute the fundamental group of the Klein bottle. Show that this group is not abelian, for example by defining a suitable homomorphism to the symmetric group S_3.

(ii) Let X be the closed orientable surface of genus 2. How many (connected) double coverings does X have? Show that the fundamental group of X admits a homomorphism onto the free group on 2 generators.

2/II/21H **Algebraic Topology**

State the Mayer–Vietoris sequence for a simplicial complex X which is a union of two subcomplexes A and B. Define the homomorphisms in the sequence (but do *not* check that they are well-defined). Prove exactness of the sequence at the term $H_i(A \cap B)$.

3/II/20H **Algebraic Topology**

Define what it means for a group G to act on a topological space X. Prove that, if G acts freely, in a sense that you should specify, then the quotient map $X \to X/G$ is a covering map and there is a surjective group homomorphism from the fundamental group of X/G to G.

4/II/21H **Algebraic Topology**

Compute the homology of the space obtained from the torus $S^1 \times S^1$ by identifying $S^1 \times \{p\}$ to a point and $S^1 \times \{q\}$ to a point, for two distinct points p and q in S^1.

1/II/22G Linear Analysis

Let X be a normed vector space over $\mathbb{R}$. Define the dual space X^* and show directly that X^* is a Banach space. Show that the map $\phi : X \to X^{**}$ defined by $\phi(x)v = v(x)$, for all $x \in X$, $v \in X^*$, is a linear map. Using the Hahn–Banach theorem, show that ϕ is injective and $|\phi(x)| = |x|$.

Give an example of a Banach space X for which ϕ is not surjective. Justify your answer.

2/II/22G Linear Analysis

Let X be a Banach space, Y a normed vector space, and $T : X \to Y$ a bounded linear map. Assume that $T(X)$ is of second category in Y. Show that T is surjective and $T(\mathcal{U})$ is open whenever $\mathcal{U}$ is open. Show that, if T is also injective, then T^{-1} exists and is bounded.

Give an example of a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R})$ is of second category in $\mathbb{R}$ but f is not surjective. Give an example of a continuous surjective map $f : \mathbb{R} \to \mathbb{R}$ which does not take open sets to open sets.

3/II/21G Linear Analysis

State and prove the Arzela–Ascoli theorem.

Let N be a positive integer. Consider the subset $\mathcal{S}_N \subset C([0,1])$ consisting of all thrice differentiable solutions to the differential equation

$$f'' = f + (f')^2 \quad \text{with} \quad |f(0)| \leqslant N\,, \quad |f(1)| \leqslant N\,, \quad |f'(0)| \leqslant N\,, \quad |f'(1)| \leqslant N\,.$$

Show that this set is totally bounded as a subset of $C([0,1])$.

[*It may be helpful to consider interior maxima.*]

4/II/22G Linear Analysis

Let X be a Banach space and $T : X \to X$ a bounded linear map. Define the *spectrum* $\sigma(T)$, *point spectrum* $\sigma_p(T)$, *resolvent* $R_T(\lambda)$, and *resolvent set* $\rho(T)$. Show that the spectrum is a closed and bounded subset of $\mathbb{C}$. Is the point spectrum always closed? Justify your answer.

Now suppose H is a Hilbert space, and $T : H \to H$ is self-adjoint. Show that the point spectrum $\sigma_p(T)$ is real.

1/II/23F Riemann Surfaces

Define a complex structure on the unit sphere $S^2 \subset \mathbb{R}^3$ using stereographic projection charts φ, ψ. Let $U \subset \mathbb{C}$ be an open set. Show that a continuous non-constant map $F : U \to S^2$ is holomorphic if and only if $\varphi \circ F$ is a meromorphic function. Deduce that a non-constant rational function determines a holomorphic map $S^2 \to S^2$. Define what is meant by a rational function taking the value $a \in \mathbb{C} \cup \{\infty\}$ with multiplicity m at infinity.

Define the degree of a rational function. Show that any rational function f satisfies $(\deg f) - 1 \leqslant \deg f' \leqslant 2 \deg f$ and give examples to show that the bounds are attained. Is it true that the product $f.g$ satisfies $\deg(f.g) = \deg f + \deg g$, for any non-constant rational functions f and g? Justify your answer.

2/II/23F Riemann Surfaces

A function ψ is defined for $z \in \mathbb{C}$ by

$$\psi(z) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i \left(n + \tfrac{1}{2}\right)^2 \tau + 2\pi i \left(n + \tfrac{1}{2}\right)\left(z + \tfrac{1}{2}\right)\right)$$

where τ is a complex parameter with $\operatorname{Im}(\tau) > 0$. Prove that this series converges uniformly on the subsets $\{|\operatorname{Im}(z)| \leqslant R\}$ for $R > 0$ and deduce that ψ is holomorphic on $\mathbb{C}$.

You may assume without proof that

$$\psi(z+1) = -\psi(z) \qquad \text{and} \qquad \psi(z+\tau) = -\exp(-\pi i \tau - 2\pi i z)\,\psi(z)$$

for all $z \in \mathbb{C}$. Let $\ell(z)$ be the logarithmic derivative $\ell(z) = \dfrac{\psi'(z)}{\psi(z)}$. Show that

$$\ell(z+1) = \ell(z) \qquad \text{and} \qquad \ell(z+\tau) = -2\pi i + \ell(z)$$

for all $z \in \mathbb{C}$. Deduce that ψ has only one zero in the parallelogram P with vertices $\frac{1}{2}(\pm 1 \pm \tau)$. Find all of the zeros of ψ.

Let Λ be the lattice in $\mathbb{C}$ generated by 1 and τ. Show that, for $\lambda_j, a_j \in \mathbb{C}$ $(j = 1, \ldots, n)$, the formula

$$f(z) = \lambda_1 \frac{\psi'(z-a_1)}{\psi(z-a_1)} + \ldots + \lambda_n \frac{\psi'(z-a_n)}{\psi(z-a_n)}$$

gives a Λ-periodic meromorphic function f if and only if $\lambda_1 + \ldots + \lambda_n = 0$. Deduce that $\dfrac{d}{dz}\left(\dfrac{\psi'(z-a)}{\psi(z-a)}\right)$ is Λ-periodic.

3/II/22F Riemann Surfaces

(i) Let R and S be compact connected Riemann surfaces and $f : R \to S$ a non-constant holomorphic map. Define the branching order $v_f(p)$ at $p \in R$ showing that it is well defined. Prove that the set of ramification points $\{p \in R : v_f(p) > 1\}$ is finite. State the Riemann–Hurwitz formula.

Now suppose that R and S have the same genus g. Prove that, if $g > 1$, then f is biholomorphic. In the case when $g = 1$, write down an example where f is not biholomorphic.

[*The inverse mapping theorem for holomorphic functions on domains in* $\mathbb{C}$ *may be assumed without proof if accurately stated.*]

(ii) Let Y be a non-singular algebraic curve in $\mathbb{C}^2$. Describe, without detailed proofs, a family of charts for Y, so that the restrictions to Y of the first and second projections $\mathbb{C}^2 \to \mathbb{C}$ are holomorphic maps. Show that the algebraic curve

$$Y \;=\; \{(s,t) \in \mathbb{C}^2 : t^4 = (s^2-1)(s-4)\}$$

is non-singular. Find all the ramification points of the map $f : Y \to \mathbb{C}; (s,t) \mapsto s$.

4/II/23F Riemann Surfaces

Let R be a Riemann surface, $\widetilde{R}$ a topological surface, and $p : \widetilde{R} \to R$ a continuous map. Suppose that every point $x \in \widetilde{R}$ admits a neighbourhood $\widetilde{U}$ such that p maps $\widetilde{U}$ homeomorphically onto its image. Prove that $\widetilde{R}$ has a complex structure such that p is a holomorphic map.

A holomorphic map $\pi : Y \to X$ between Riemann surfaces is called a *covering map* if every $x \in X$ has a neighbourhood V with $\pi^{-1}(V)$ a disjoint union of open sets W_k in Y, so that $\pi : W_k \to V$ is biholomorphic for each W_k. Suppose that a Riemann surface Y admits a holomorphic covering map from the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Prove that any holomorphic map $\mathbb{C} \to Y$ is constant.

[*You may assume any form of the monodromy theorem and basic results about the lifts of paths, provided that these are accurately stated.*]

1/II/24H **Differential Geometry**

Let $f : X \to Y$ be a smooth map between manifolds without boundary. Recall that f is a *submersion* if $df_x : T_xX \to T_{f(x)}Y$ is surjective for all $x \in X$. The *canonical submersion* is the standard projection of $\mathbb{R}^k$ onto $\mathbb{R}^l$ for $k \geqslant l$, given by

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_l).$$

(i) Let f be a submersion, $x \in X$ and $y = f(x)$. Show that there exist local coordinates around x and y such that f, in these coordinates, is the canonical submersion.
[*You may assume the inverse function theorem.*]

(ii) Show that submersions map open sets to open sets.

(iii) If X is compact and Y connected, show that every submersion is surjective. Are there submersions of compact manifolds into Euclidean spaces $\mathbb{R}^k$ with $k \geqslant 1$?

2/II/24H **Differential Geometry**

(i) What is a minimal surface? Explain why minimal surfaces always have non-positive Gaussian curvature.

(ii) A smooth map $f : S_1 \to S_2$ between two surfaces in 3-space is said to be *conformal* if

$$\langle df_p(v_1), df_p(v_2) \rangle = \lambda(p) \langle v_1, v_2 \rangle$$

for all $p \in S_1$ and all $v_1, v_2 \in T_pS_1$, where $\lambda(p) \neq 0$ is a number which depends only on p.

Let S be a surface without umbilical points. Prove that S is a minimal surface if and only if the Gauss map $N : S \to S^2$ is conformal.

(iii) Show that isothermal coordinates exist around a non-planar point in a minimal surface.

3/II/23H **Differential Geometry**

(i) Let $f : X \to Y$ be a smooth map between manifolds without boundary. Define critical point, critical value and regular value. State Sard's theorem.

(ii) Explain how to define the degree modulo 2 of a smooth map f, indicating clearly the hypotheses on X and Y. Show that a smooth map with non-zero degree modulo 2 must be surjective.

(iii) Let S be the torus of revolution obtained by rotating the circle $(y-2)^2 + z^2 = 1$ in the yz-plane around the z-axis. Describe the critical points and the critical values of the Gauss map N of S. Find the degree modulo 2 of N. Justify your answer by means of a sketch or otherwise.

4/II/24H **Differential Geometry**

(i) What is a geodesic? Show that geodesics are critical points of the energy functional.

(ii) Let S be a surface which admits a parametrization $\phi(u,v)$ defined on an open subset W of $\mathbb{R}^2$ such that $E = G = U + V$ and $F = 0$, where $U = U(u)$ is a function of u alone and $V = V(v)$ is a function of v alone. Let $\gamma : I \to \phi(W)$ be a geodesic and write $\gamma(t) = \phi(u(t), v(t))$. Show that

$$\big[U(u(t)) + V(v(t))\big]\big[V(v(t))\dot{u}^2 - U(u(t))\dot{v}^2\big]$$

is independent of t.

1/II/25J Probability and Measure

Let E be a set and $\mathcal{E} \subseteq \mathcal{P}(E)$ be a set system.

(a) Explain what is meant by a π-system, a d-system and a σ-algebra.

(b) Show that $\mathcal{E}$ is a σ-algebra if and only if $\mathcal{E}$ is a π-system and a d-system.

(c) Which of the following set systems $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ are π-systems, d-systems or σ-algebras? Justify your answers. ($\#(A)$ denotes the number of elements in A.)

$E_1 = \{1, 2, \ldots, 10\}$ and $\mathcal{E}_1 = \{A \subseteq E_1 : \#(A) \text{ is even}\}$,
$E_2 = \mathbb{N} = \{1, 2, \ldots\}$ and $\mathcal{E}_2 = \{A \subseteq E_2 : \#(A) \text{ is even or } \#(A) = \infty\}$,
$E_3 = \mathbb{R}$ and $\mathcal{E}_3 = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{\emptyset\}$.

(d) State and prove the theorem on the uniqueness of extension of a measure.

[*You may use standard results from the lectures without proof, provided they are clearly stated.*]

2/II/25J Probability and Measure

(a) State and prove the first Borel–Cantelli lemma. State the second Borel–Cantelli lemma.

(b) Let $X_1, X_2, \ldots$ be a sequence of independent random variables that converges in probability to the limit X. Show that X is almost surely constant.

A sequence $X_1, X_2, \ldots$ of random variables is said to be *completely convergent* to X if

$$\sum_{n \in \mathbb{N}} \mathbb{P}\big(A_n(\epsilon)\big) < \infty \quad \text{for all } \epsilon > 0 , \quad \text{where } A_n(\epsilon) = \big\{|X_n - X| > \epsilon\big\} .$$

(c) Show that complete convergence implies almost sure convergence.

(d) Show that, for sequences of independent random variables, almost sure convergence also implies complete convergence.

(e) Find a sequence of (dependent) random variables that converges almost surely but does not converge completely.

3/II/24J **Probability and Measure**

Let $(E, \mathcal{E}, \mu)$ be a finite measure space, i.e. $\mu(E) < \infty$, and let $1 \leqslant p \leqslant \infty$.

(a) Define the L^p-norm $\|f\|_p$ of a measurable function $f : E \to \overline{\mathbb{R}}$, define the space $L^p(E, \mathcal{E}, \mu)$ and define convergence in L^p.

In the following you may use inequalities from the lectures without proof, provided they are clearly stated.

(b) Let $f, f_1, f_2, \ldots \in L^p(E, \mathcal{E}, \mu)$. Show that $f_n \to f$ in L^p implies $\|f_n\|_p \to \|f\|_p$.

(c) Let $f : E \to \mathbb{R}$ be a bounded measurable function with $\|f\|_\infty > 0$. Let

$$M_n = \int_E |f|^n d\mu \,.$$

Show that $M_n \in (0, \infty)$ and $M_{n+1} M_{n-1} \geqslant M_n^2$.

By using Jensen's inequality, or otherwise, show that

$$\mu(E)^{-1/n} \|f\|_n \leqslant M_{n+1}/M_n \leqslant \|f\|_\infty .$$

Prove that $\lim_{n \to \infty} M_{n+1}/M_n = \|f\|_\infty$.

$\left[\textit{Observe that } |f| \geqslant \mathbf{1}_{\{|f| > \|f\|_\infty - \epsilon\}} \big(\|f\|_\infty - \epsilon \big). \right]$

4/II/25J **Probability and Measure**

Let $(E, \mathcal{E}, \mu)$ be a measure space with $\mu(E) < \infty$ and let $\theta : E \to E$ be measurable.

(a) Define an invariant set $A \in \mathcal{E}$ and an invariant function $f : E \to \mathbb{R}$.
What is meant by saying that θ is measure-preserving?
What is meant by saying that θ is ergodic?

(b) Which of the following functions θ_1 to θ_4 is ergodic? Justify your answer.

On the measure space $\big([0,1], \mathcal{B}([0,1]), \mu\big)$ with Lebesgue measure μ consider

$$\theta_1(x) = 1 + x \ , \qquad \theta_2(x) = x^2 \ , \qquad \theta_3(x) = 1 - x \ .$$

On the discrete measure space $\big(\{-1,1\}, \mathcal{P}(\{-1,1\}), \tfrac{1}{2}\delta_{-1} + \tfrac{1}{2}\delta_1\big)$ consider

$$\theta_4(x) = -x \ .$$

(c) State Birkhoff's almost everywhere ergodic theorem.

(d) Let θ be measure-preserving and let $f : E \to \mathbb{R}$ be bounded.
Prove that $\dfrac{1}{n}\big(f + f \circ \theta + \ldots + f \circ \theta^{n-1}\big)$ converges in L^p for all $p \in [1, \infty)$.

1/II/26J **Applied Probability**

An open air rock concert is taking place in beautiful Pine Valley, and enthusiastic fans from the entire state of Alifornia are heading there long before the much anticipated event. The arriving cars have to be directed to one of three large (practically unlimited) parking lots, a, b and c situated near the valley entrance. The traffic cop at the entrance to the valley decides to direct every third car (in the order of their arrival) to a particular lot. Thus, cars 1, 4, 7, 10 and so on are directed to lot a, cars 2, 5, 8, 11 to lot b and cars 3, 6, 9, 12 to lot c.

Suppose that the total arrival process $N(t)$, $t \geqslant 0$, at the valley entrance is Poisson, of rate $\lambda > 0$ (the initial time $t = 0$ is taken to be considerably ahead of the actual event). Consider the processes $X^a(t)$, $X^b(t)$ and $X^c(t)$ where $X^i(t)$ is the number of cars arrived in lot i by time t, $i = a, b, c$. Assume for simplicity that the time to reach a parking lot from the entrance is negligible so that the car enters its specified lot at the time it crosses the valley entrance.

(a) Give the probability density function of the time of the first arrival in each of the processes $X^a(t)$, $X^b(t)$, $X^c(t)$.

(b) Describe the distribution of the time between two subsequent arrivals in each of these processes. Are these times independent? Justify your answer.

(c) Which of these processes are delayed renewal processes (where the distribution of the first arrival time differs from that of the inter-arrival time)?

(d) What are the corresponding equilibrium renewal processes?

(e) Describe how the direction rule should be changed for $X^a(t)$, $X^b(t)$ and $X^c(t)$ to become Poisson processes, of rate $\lambda/3$. Will these Poisson processes be independent? Justify your answer.

2/II/26J Applied Probability

In this question we work with a continuous-time Markov chain where the rate of jump $i \to j$ may depend on j but not on i. A virus can be in one of s strains $1, \ldots, s$, and it mutates to strain j with rate $r_j \geqslant 0$ from each strain $i \neq j$. (Mutations are caused by the chemical environment.) Set $R = r_1 + \ldots + r_s$.

(a) Write down the Q-matrix (the generator) of the chain (X_t) in terms of r_j and R.

(b) If $R = 0$, that is, $r_1 = \ldots = r_s = 0$, what are the communicating classes of the chain (X_t)?

(c) From now on assume that $R > 0$. State and prove a necessary and sufficient condition, in terms of the numbers r_j, for the chain (X_t) to have a single communicating class (which therefore should be closed).

(d) In general, what is the number of closed communicating classes in the chain (X_t)? Describe all open communicating classes of (X_t).

(e) Find the equilibrium distribution of (X_t). Is the chain (X_t) reversible? Justify your answer.

(f) Write down the transition matrix $\widehat{P} = (\widehat{p}_{ij})$ of the discrete-time jump chain for (X_t) and identify its equilibrium distribution. Is the jump chain reversible? Justify your answer.

3/II/25J **Applied Probability**

For a discrete-time Markov chain, if the probability of transition $i \to j$ does not depend on i then the chain is reduced to a sequence of independent random variables (states). In this case, the chain forgets about its initial state and enters equilibrium after a single transition. In the continuous-time case, a Markov chain whose rates q_{ij} of transition $i \to j$ depend on j but not on $i \neq j$ still 'remembers' its initial state and reaches equilibrium only in the limit as the time grows indefinitely. This question is an illustration of this property.

A protean sea sponge may change its colour among s varieties $1, \ldots, s$, under the influence of the environment. The rate of transition from colour i to j equals $r_j \geqslant 0$ and does not depend on i, $i \neq j$. Consider a Q-matrix $Q = (q_{ij})$ with entries

$$q_{ij} = \begin{cases} r_j, & i \neq j, \\ -R + r_i, & i = j, \end{cases}$$

where $R = r_1 + \ldots + r_s$. Assume that $R > 0$ and let (X_t) be the continuous-time Markov chain with generator Q. Given $t \geqslant 0$, let $P(t) = (p_{ij}(t))$ be the matrix of transition probabilities in time t in chain (X_t).

(a) State the exponential relation between the matrices Q and $P(t)$.

(b) Set $\pi_j = r_j/R$, $j = 1, \ldots, s$. Check that $\pi_1, \ldots, \pi_s$ are equilibrium probabilities for the chain (X_t). Is this a unique equilibrium distribution? What property of the vector with entries π_j relative to the matrix Q is involved here?

(c) Let $\mathbf{x}$ be a vector with components $x_1, \ldots, x_s$ such that $x_1 + \ldots + x_s = 0$. Show that $\mathbf{x}^{\mathrm{T}} Q = -R\mathbf{x}^{\mathrm{T}}$. Compute $\mathbf{x}^{\mathrm{T}} P(t)$.

(d) Now let δ_i denote the (column) vector whose entries are 0 except for the ith one which equals 1. Observe that the ith row of $P(t)$ is $\delta_i^{\mathrm{T}} P(t)$. Prove that $\delta_i^{\mathrm{T}} P(t) = \pi^{\mathrm{T}} + e^{-tR}(\delta_i^{\mathrm{T}} - \pi^{\mathrm{T}})$.

(e) Deduce the expression for transition probabilities $p_{ij}(t)$ in terms of rates r_j and their sum R.

4/II/26J **Applied Probability**

A population of rare Monarch butterflies functions as follows. At the times of a Poisson process of rate λ a caterpillar is produced from an egg. After an exponential time, the caterpillar is transformed into a pupa which, after an exponential time, becomes a butterfly. The butterfly lives for another exponential time and then dies. (The Poissonian assumption reflects the fact that butterflies lay a huge number of eggs most of which do not develop.) Suppose that all lifetimes are independent (of the arrival process and of each other) and let their rate be μ. Assume that the population is in an equilibrium and let C be the number of caterpillars, R the number of pupae and B the number of butterflies (so that the total number of insects, in any metamorphic form, equals $N = C + R + B$). Let $\pi_{(c,r,b)}$ be the equilibrium probability $\mathbb{P}(C = c, R = r, B = b)$ where $c, r, b = 0, 1, \ldots$.

(a) Specify the rates of transitions $(c, r, b) \to (c', r', b')$ for the resulting continuous-time Markov chain (X_t) with states (c, r, b). (The rates are non-zero only when $c' = c$ or $c' = c \pm 1$ and similarly for other co-ordinates.) Check that the holding rate for state (c, r, b) is $\lambda + \mu n$ where $n = c + r + b$.

(b) Let Q be the Q-matrix from (a). Consider the invariance equation $\pi Q = 0$. Verify that the only solution is

$$\pi_{(c,r,b)} = \frac{(3\lambda/\mu)^n}{3^n c!r!b!} \exp\left(-\frac{3\lambda}{\mu}\right), \quad n = c + r + b.$$

(c) Derive the marginal equilibrium probabilities $\mathbb{P}(N = n)$ and the conditional equilibrium probabilities $\mathbb{P}(C = c,\ R = r,\ B = b \mid N = n)$.

(d) Determine whether the chain (X_t) is positive recurrent, null-recurrent or transient.

(e) Verify that the equilibrium probabilities $\mathbb{P}(N = n)$ are the same as in the corresponding $M/GI/\infty$ system (with the correct specification of the arrival rate and the service-time distribution).

1/II/27I Principles of Statistics

Suppose that X has density $f(\cdot|\theta)$ where $\theta \in \Theta$. What does it mean to say that statistic $T \equiv T(X)$ is *sufficient* for θ?

Suppose that $\theta = (\psi, \lambda)$, where ψ is the parameter of interest, and λ is a nuisance parameter, and that the sufficient statistic T has the form $T = (C, S)$. What does it mean to say that the statistic S is *ancillary*? If it is, how (according to the conditionality principle) do we test hypotheses on ψ? Assuming that the set of possible values for X is discrete, show that S is ancillary if and only if the density (probability mass function) $f(x|\psi, \lambda)$ factorises as

$$f(x|\psi, \lambda) = \varphi_0(x)\, \varphi_C(C(x), S(x), \psi)\, \varphi_S(S(x), \lambda) \qquad (*)$$

for some functions φ_0, φ_C, and φ_S with the properties

$$\sum_{x \in C^{-1}(c) \cap S^{-1}(s)} \varphi_0(x) = 1 = \sum_s \varphi_S(s, \lambda) = \sum_s \sum_c \varphi_C(c, s, \psi)$$

for all c, s, ψ, and λ.

Suppose now that $X_1, \ldots, X_n$ are independent observations from a $\Gamma(a, b)$ distribution, with density

$$f(x|a, b) = (bx)^{a-1} e^{-bx} b I_{\{x>0\}} / \Gamma(a).$$

Assuming that the criterion $(*)$ holds also for observations which are not discrete, show that it is not possible to find $(C(X), S(X))$ sufficient for (a, b) such that S is ancillary when b is regarded as a nuisance parameter, and a is the parameter of interest.

2/II/27I Principles of Statistics

(i) State Wilks' likelihood ratio test of the null hypothesis $H_0 : \theta \in \Theta_0$ against the alternative $H_1 : \theta \in \Theta_1$, where $\Theta_0 \subset \Theta_1$. Explain when this test may be used.

(ii) Independent identically-distributed observations $X_1, \ldots, X_n$ take values in the set $S = \{1, \ldots, K\}$, with common distribution which under the null hypothesis is of the form

$$P(X_1 = k|\theta) = f(k|\theta) \qquad (k \in S)$$

for some $\theta \in \Theta_0$, where Θ_0 is an open subset of some Euclidean space $\mathbb{R}^d$, $d < K - 1$. Under the alternative hypothesis, the probability mass function of the X_i is unrestricted.

Assuming sufficient regularity conditions on f to guarantee the existence and uniqueness of a maximum-likelihood estimator $\hat{\theta}_n(X_1, \ldots, X_n)$ of θ for each n, show that for large n the Wilks' likelihood ratio test statistic is approximately of the form

$$\sum_{j=1}^{K} (N_j - n\hat{\pi}_j)^2 / N_j,$$

where $N_j = \sum_{i=1}^n I_{\{X_i = j\}}$, and $\hat{\pi}_j = f(j|\hat{\theta}_n)$. What is the asymptotic distribution of this statistic?

3/II/26I Principles of Statistics

(i) In the context of decision theory, what is a *Bayes rule* with respect to a given loss function and prior? What is an *extended Bayes rule*?

Characterise the Bayes rule with respect to a given prior in terms of the posterior distribution for the parameter given the observation. When $\Theta = \mathcal{A} = \mathbb{R}^d$ for some d, and the loss function is $L(\theta, a) = \|\theta - a\|^2$, what is the Bayes rule?

(ii) Suppose that $\mathcal{A} = \Theta = \mathbb{R}$, with loss function $L(\theta, a) = (\theta - a)^2$ and suppose further that under P_θ, $X \sim N(\theta, 1)$.

Supposing that a $N(0, \tau^{-1})$ prior is taken over θ, compute the Bayes risk of the decision rule $d_\lambda(X) = \lambda X$. Find the posterior distribution of θ given X, and confirm that its mean is of the form $d_\lambda(X)$ for some value of λ which you should identify. Hence show that the decision rule d_1 is an extended Bayes rule.

4/II/27I **Principles of Statistics**

Assuming sufficient regularity conditions on the likelihood $f(x|\theta)$ for a univariate parameter $\theta \in \Theta$, establish the Cramér–Rao lower bound for the variance of an unbiased estimator of θ.

If $\hat{\theta}(X)$ is an unbiased estimator of θ whose variance attains the Cramér–Rao lower bound for every value of $\theta \in \Theta$, show that the likelihood function is an exponential family.

1/II/28J Stochastic Financial Models

(i) What does it mean to say that a process $(M_t)_{t\geqslant 0}$ is a *martingale*? What does the *martingale convergence theorem* tell us when applied to *positive martingales*?

(ii) What does it mean to say that a process $(B_t)_{t\geqslant 0}$ is a *Brownian motion*? Show that $\sup_{t\geqslant 0} B_t = \infty$ with probability one.

(iii) Suppose that $(B_t)_{t\geqslant 0}$ is a Brownian motion. Find μ such that

$$S_t = \exp\left(x_0 + \sigma B_t + \mu t\right)$$

is a martingale. Discuss the limiting behaviour of S_t and $\mathbb{E}\left(S_t\right)$ for this μ as $t \to \infty$.

2/II/28J Stochastic Financial Models

In the context of a single-period financial market with n traded assets, what is an arbitrage? What is an equivalent martingale measure?

Fix $\epsilon \in (0,1)$ and consider the following single-period market with 3 assets:

Asset 1 is a riskless bond and pays no interest.

Asset 2 is a stock with initial price £1 per share; its possible final prices are $u = 1+\epsilon$ with probability $3/5$ and $d = 1-\epsilon$ with probability $2/5$.

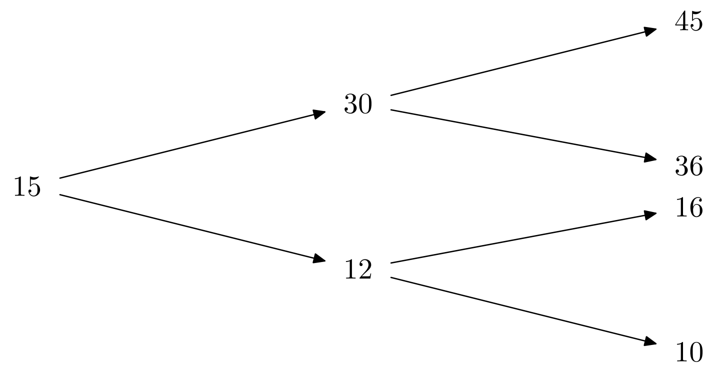
Asset 3 is another stock that behaves like an independent copy of asset 2.

Find all equivalent martingale measures for the problem and characterise all contingent claims that can be replicated.

Consider a contingent claim Y that pays 1 if both risky assets move in the same direction and zero otherwise. Show that the lower arbitrage bound, simply obtained by calculating all possible prices as the pricing measure ranges over all equivalent martingale measures, is zero. Why might someone pay for such a contract?

3/II/27J **Stochastic Financial Models**

Suppose that over two periods a stock price moves on a binomial tree



(i) Determine for what values of the riskless rate r there is no arbitrage. From here on, fix $r = 1/4$ and determine the equivalent martingale measure.

(ii) Find the time-zero price and replicating portfolio for a European put option with strike 15 and expiry 2.

(iii) Find the time-zero price and optimal exercise policy for an American put option with the same strike and expiry.

(iv) Deduce the corresponding (European and American) call option prices for the same strike and expiry.

4/II/28J **Stochastic Financial Models**

Briefly describe the Black–Scholes model. Consider a "cash-or-nothing" option with strike price K , i.e. an option whose payoff at maturity is

$$f\left(S_T\right) = \begin{cases} 1 & \text{if} \quad S_T > K \ , \\ 0 & \text{if} \quad S_T \leqslant K \ . \end{cases}$$

It can be interpreted as a bet that the stock will be worth at least K at time T. Find a formula for its value at time t, in terms of the spot price S_t . Find a formula for its Delta (i.e. its hedge ratio). How does the Delta behave as $t \to T$? Why is it difficult, in practice, to hedge such an instrument?

2/II/29I Optimization and Control

State Pontryagin's maximum principle in the case where both the terminal time and the terminal state are given.

Show that π is the minimum value taken by the integral

$$\tfrac{1}{2}\int_0^1 (u_t^2 + v_t^2)\,dt$$

subject to the constraints $x_0 = y_0 = z_0 = x_1 = y_1 = 0$ and $z_1 = 1$, where

$$\dot{x}_t = u_t, \quad \dot{y}_t = v_t, \quad \dot{z}_t = u_t y_t - v_t x_t, \quad 0 \leqslant t \leqslant 1.$$

[*You may find it useful to note the fact that the problem is rotationally symmetric about the z-axis, so that the angle made by the initial velocity $(\dot{x}_0, \dot{y}_0)$ with the positive x-axis may be chosen arbitrarily.*]

3/II/28I Optimization and Control

Let P be a discrete-time controllable dynamical system (or Markov decision process) with countable state-space S and action-space A. Consider the n-horizon dynamic optimization problem with instantaneous costs $c(k,x,a)$, on choosing action a in state x at time $k \leqslant n-1$, with terminal cost $C(x)$, in state x at time n. Explain what is meant by a Markov control and how the choice of a control gives rise to a time-inhomogeneous Markov chain.

Suppose we can find a bounded function V and a Markov control u^* such that

$$V(k,x) \leqslant (c + PV)(k,x,a), \quad 0 \leqslant k \leqslant n-1, \quad x \in S, \quad a \in A,$$

with equality when $a = u^*(k,x)$, and such that $V(n,x) = C(x)$ for all x. Here $PV(k,x,a)$ denotes the expected value of $V(k+1, X_{k+1})$, given that we choose action a in state x at time k. Show that u^* is an optimal Markov control.

A well-shuffled pack of cards is placed face-down on the table. The cards are turned over one by one until none are left. Exactly once you may place a bet of £1000 on the event that the next *two* cards will be red. How should you choose the moment to bet? Justify your answer.

4/II/29I **Optimization and Control**

Consider the scalar controllable linear system, whose state X_n evolves by

$$X_{n+1} = X_n + U_n + \varepsilon_{n+1},$$

with observations Y_n given by

$$Y_{n+1} = X_n + \eta_{n+1}.$$

Here, U_n is the control variable, which is to be determined on the basis of the observations up to time n, and ε_n, η_n are independent $N(0,1)$ random variables. You wish to minimize the long-run average expected cost, where the instantaneous cost at time n is $X_n^2 + U_n^2$. You may assume that the optimal control in equilibrium has the form $U_n = -K\hat{X}_n$, where $\hat{X}_n$ is given by a recursion of the form

$$\hat{X}_{n+1} = \hat{X}_n + U_n + H(Y_{n+1} - \hat{X}_n),$$

and where H is chosen so that $\Delta_n = X_n - \hat{X}_n$ is independent of the observations up to time n. Show that $K = H = (\sqrt{5}-1)/2 = 2/(\sqrt{5}+1)$, and determine the minimal long-run average expected cost. You are not expected to simplify the arithmetic form of your answer but should show clearly how you have obtained it.

1/II/29A **Partial Differential Equations**

(i) Consider the problem of solving the equation

$$\sum_{j=1}^{n} a_j(\mathbf{x}) \frac{\partial u}{\partial x_j} = b(\mathbf{x}, u)$$

for a C^1 function $u = u(\mathbf{x}) = u(x_1, \ldots, x_n)$, with data specified on a C^1 hypersurface $\mathcal{S} \subset \mathbb{R}^n$

$$u(\mathbf{x}) = \phi(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathcal{S}.$$

Assume that $a_1, \ldots, a_n, \phi, b$ are C^1 functions. Define the characteristic curves and explain what it means for the non-characteristic condition to hold at a point on $\mathcal{S}$. State a local existence and uniqueness theorem for the problem.

(ii) Consider the case $n = 2$ and the equation

$$\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} = x_2 u$$

with data $u(x_1, 0) = \phi(x_1, 0) = f(x_1)$ specified on the axis $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = 0\}$. Obtain a formula for the solution.

(iii) Consider next the case $n = 2$ and the equation

$$\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} = 0$$

with data $u(\mathbf{g}(s)) = \phi(\mathbf{g}(s)) = f(s)$ specified on the hypersurface $\mathcal{S}$, which is given parametrically as $\mathcal{S} \equiv \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{g}(s)\}$ where $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2$ is defined by

$$\mathbf{g}(s) = (s, 0), \qquad s < 0,$$

$$\mathbf{g}(s) = (s, s^2), \qquad s \geqslant 0.$$

Find the solution u and show that it is a global solution. (Here "global" means u is C^1 on all of $\mathbb{R}^2$.)

(iv) Consider next the equation

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0$$

to be solved with the same data given on the same hypersurface as in (iii). Explain, with reference to the characteristic curves, why there is generally no global C^1 solution. Discuss the existence of local solutions defined in some neighbourhood of a given point $\mathbf{y} \in \mathcal{S}$ for various $\mathbf{y}$. [*You need not give formulae for the solutions.*]

2/II/30A **Partial Differential Equations**

Define (i) the Fourier transform of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^3)$, and (ii) the convolution $T * g$ of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^3)$ and a Schwartz function $g \in \mathcal{S}(\mathbb{R}^3)$. Give a formula for the Fourier transform of $T * g$ ("convolution theorem").

Let $t > 0$. Compute the Fourier transform of the tempered distribution $A_t \in \mathcal{S}'(\mathbb{R}^3)$ defined by

$$\langle A_t, \phi \rangle = \int_{\|y\|=t} \phi(y) d\Sigma(y), \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R}^3),$$

and deduce the Kirchhoff formula for the solution $u(t,x)$ of

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0,$$

$$u(0,x) = 0, \qquad \frac{\partial u}{\partial t}(0,x) = g(x), \quad g \in \mathcal{S}(\mathbb{R}^3)\ .$$

Prove, by consideration of the quantities $e = \frac{1}{2}(u_t^2 + |\nabla u|^2)$ and $p = -u_t \nabla u$, that any C^2 solution is also given by the Kirchhoff formula (uniqueness).

Prove a corresponding uniqueness statement for the initial value problem

$$\frac{\partial^2 w}{\partial t^2} - \Delta w + V(x) w = 0,$$

$$w(0,x) = 0, \qquad \frac{\partial w}{\partial t}(0,x) = g(x), \quad g \in \mathcal{S}(\mathbb{R}^3)$$

where V is a smooth positive real-valued function of $x \in \mathbb{R}^3$ only.

3/II/29A **Partial Differential Equations**

Write down the formula for the solution $u = u(t,x)$ for $t > 0$ of the initial value problem for the heat equation in one space dimension

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0\,,$$

$$u(0,x) = g(x)\,,$$

for $g : \mathbb{R} \to \mathbb{C}$ a given smooth bounded function.

Define the distributional derivative of a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$. Define a fundamental solution of a constant-coefficient linear differential operator P, and show that the distribution defined by the function $\frac{1}{2}e^{-|x|}$ is a fundamental solution for the operator $P = -\dfrac{d^2}{dx^2} + 1$.

For the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = e^t \phi(x), \qquad (*)$$

where $\phi \in \mathcal{S}(\mathbb{R})$, prove that there is a unique solution of the form $e^t v(x)$ with $v \in \mathcal{S}(\mathbb{R})$. Hence write down the solution of $(*)$ with general initial data $u(0,x) = f(x)$ and describe the large time behaviour.

4/II/30A **Partial Differential Equations**

State and prove the mean value property for harmonic functions on $\mathbb{R}^3$.

Obtain a generalization of the mean value property for sub-harmonic functions on $\mathbb{R}^3$, i.e. C^2 functions for which

$$-\Delta u(x) \leqslant 0$$

for all $x \in \mathbb{R}^3$.

Let $\phi \in C^2(\mathbb{R}^3; \mathbb{C})$ solve the equation

$$-\Delta \phi + iV(x)\phi = 0\,,$$

where V is a real-valued continuous function. By considering the function $w(x) = |\phi(x)|^2$ show that, on any ball $B(y,R) = \{x : \|x-y\| < R\} \subset \mathbb{R}^3$,

$$\sup_{x \in B(y,R)} |\phi(x)| \leqslant \sup_{\|x-y\|=R} |\phi(x)|.$$

1/II/30B **Asymptotic Methods**

State Watson's lemma, describing the asymptotic behaviour of the integral

$$I(\lambda) = \int_0^A e^{-\lambda t} f(t) dt, \quad A > 0,$$

as $\lambda \to \infty$, given that $f(t)$ has the asymptotic expansion

$$f(t) \;\sim\; t^{\alpha} \sum_{n=0}^{\infty} a_n \, t^{n\beta}$$

as $t \to 0_+$, where $\beta > 0$ and $\alpha > -1$.

Give an account of Laplace's method for finding asymptotic expansions of integrals of the form

$$J(z) = \int_{-\infty}^{\infty} e^{-zp(t)} \; q(t) \; dt$$

for large real z, where $p(t)$ is real for real t.

Deduce the following asymptotic expansion of the contour integral

$$\int_{-\infty - i\pi}^{\infty + i\pi} \exp\left(z \cosh t\right) dt \;=\; 2^{1/2} i e^{z} \, \Gamma\left(\tfrac{1}{2}\right) \left[z^{-1/2} + \tfrac{1}{8} \, z^{-3/2} + O\left(z^{-5/2}\right)\right]$$

as $z \to \infty$.

3/II/30B **Asymptotic Methods**

Explain the method of stationary phase for determining the behaviour of the integral

$$I(x) = \int_a^b du \; e^{ixf(u)}$$

for large x. Here, the function $f(u)$ is real and differentiable, and a, b and x are all real.

Apply this method to show that the first term in the asymptotic behaviour of the function

$$\Gamma(m+1) = \int_0^{\infty} du \; u^m \; e^{-u} \; ,$$

where $m = i\,n$ with $n > 0$ and real, is

$$\Gamma(i\,n+1) \quad \sim \quad \sqrt{2\pi} \; e^{-i\,n} \exp\left[\left(i\,n + \tfrac{1}{2}\right)\left(\frac{i\pi}{2} + \log n\right)\right]$$

as $n \to \infty$.

4/II/31B **Asymptotic Methods**

Consider the time-independent Schrödinger equation

$$\frac{d^2\psi}{dx^2} + \lambda^2 q(x)\psi(x) = 0,$$

where $\lambda \gg 1$ denotes $\hbar^{-1}$ and $q(x)$ denotes $2m[E - V(x)]$. Suppose that

$$\begin{aligned} & q(x) > 0 \quad \text{for} \quad a < x < b, \\ \text{and} \quad & q(x) < 0 \quad \text{for} \quad -\infty < x < a \;\text{ and }\; b < x < \infty \end{aligned}$$

and consider a bound state $\psi(x)$. Write down the possible Liouville–Green approximate solutions for $\psi(x)$ in each region, given that $\psi \to 0$ as $|x| \to \infty$.

Assume that $q(x)$ may be approximated by $q'(a)(x-a)$ near $x = a$, where $q'(a) > 0$, and by $q'(b)(x - b)$ near $x = b$, where $q'(b) < 0$. The Airy function $\mathrm{Ai}(z)$ satisfies

$$\frac{d^2(\mathrm{Ai})}{dz^2} - z(\mathrm{Ai}) = 0$$

and has the asymptotic expansions

$$\mathrm{Ai}(z) \quad \sim \quad \tfrac{1}{2}\pi^{-1/2}z^{-1/4}\exp\left(-\frac{2}{3}z^{3/2}\right) \qquad \text{as} \qquad z \to +\infty\,,$$

and

$$\mathrm{Ai}(z) \quad \sim \quad \pi^{-1/2}|z|^{-1/4}\cos\left[\left(\frac{2}{3}|z|^{3/2}\right) - \frac{\pi}{4}\right] \qquad \text{as} \qquad z \to -\infty\,.$$

Deduce that the energies E of bound states are given approximately by the WKB condition:

$$\lambda\int_a^b q^{1/2}(x)\,dx = \left(n + \tfrac{1}{2}\right)\pi \qquad (n = 0, 1, 2, \ldots).$$

1/II/31E **Integrable Systems**

(i) Using the Cole–Hopf transformation

$$u = -\frac{2\nu}{\phi}\frac{\partial \phi}{\partial x},$$

map the Burgers equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = \nu\,\frac{\partial^2 u}{\partial x^2}$$

to the heat equation

$$\frac{\partial \phi}{\partial t} = \nu\,\frac{\partial^2 \phi}{\partial x^2}\,.$$

(ii) Given that the solution of the heat equation on the infinite line $\mathbb{R}$ with initial condition $\phi(x,0) = \Phi(x)$ is given by

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} \Phi(\xi)\, e^{-\frac{(x-\xi)^2}{4\nu t}}\, d\xi\,,$$

show that the solution of the analogous problem for the Burgers equation with initial condition $u(x,0) = U(x)$ is given by

$$u = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-\xi}{t}\, e^{-\frac{1}{2\nu}G(x,\xi,t)}\, d\xi}{\displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2\nu}G(x,\xi,t)}\, d\xi}\,,$$

where the function G is to be determined in terms of U.

(iii) Determine the ODE characterising the scaling reduction of the spherical modified Korteweg – de Vries equation

$$\frac{\partial u}{\partial t} + 6u^2\,\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \frac{u}{t} = 0\,.$$

2/II/31E **Integrable Systems**

Solve the following linear singular equation

$$\left(t + t^{-1}\right)\phi(t) + \frac{\left(t - t^{-1}\right)}{\pi i} ⨍_C \frac{\phi(\tau)}{\tau - t}\, d\tau - \frac{\left(t + t^{-1}\right)}{2\pi i}\oint_C \left(\tau + 2\tau^{-1}\right)\phi(\tau)\, d\tau = 2t^{-1}\,,$$

where C denotes the unit circle, $t \in C$ and $⨍_C$ denotes the principal value integral.

3/II/31E **Integrable Systems**

Find a Lax pair formulation for the linearised NLS equation

$$iq_t + q_{xx} = 0\,.$$

Use this Lax pair formulation to show that the initial value problem on the infinite line of the linearised NLS equation is associated with the following Riemann–Hilbert problem

$$M^+(x,t,k) = M^-(x,t,k)\begin{pmatrix}1 & e^{ikx-ik^2t}\hat{q}_0(k)\\ 0 & 1\end{pmatrix}, \quad k \in \mathbb{R}\,,$$

$$M = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + O\left(\frac{1}{k}\right), \qquad k \to \infty\,.$$

By deforming the above problem obtain the Riemann–Hilbert problem and hence the linear integral equation associated with the following system of nonlinear evolution PDEs

$$iq_t + q_{xx} - 2\vartheta q^2 = 0\,,$$
$$-i\vartheta_t + \vartheta_{xx} - 2\vartheta^2 q = 0\,.$$

1/II/32D **Principles of Quantum Mechanics**

A particle in one dimension has position and momentum operators $\hat{x}$ and $\hat{p}$ whose eigenstates obey

$$\langle x|x'\rangle = \delta(x-x') \ , \qquad \langle p|p'\rangle = \delta(p-p') \ , \qquad \langle x|p\rangle = (2\pi\hbar)^{-1/2}e^{ixp/\hbar} \ .$$

Given a state $|\psi\rangle$, define the corresponding position-space and momentum-space wavefunctions $\psi(x)$ and $\tilde{\psi}(p)$ and show how each of these can be expressed in terms of the other. Derive the form taken in momentum space by the time-independent Schrödinger equation

$$\Big(\frac{\hat{p}^2}{2m} + V(\hat{x})\Big)|\psi\rangle = E|\psi\rangle$$

for a general potential V.

Now let $V(x) = -(\hbar^2\lambda/m)\delta(x)$ with λ a positive constant. Show that the Schrödinger equation can be written

$$\Big(\frac{p^2}{2m} - E\Big)\tilde{\psi}(p) = \frac{\hbar\lambda}{2\pi m}\int_{-\infty}^{\infty} dp'\,\tilde{\psi}(p')$$

and verify that it has a solution $\tilde{\psi}(p) = N/(p^2+\alpha^2)$ for unique choices of α and E, to be determined (you need not find the normalisation constant, N). Check that this momentum space wavefunction can also be obtained from the position space solution $\psi(x) = \sqrt{\lambda}e^{-\lambda|x|}$.

2/II/32D **Principles of Quantum Mechanics**

Let $|s\,m\rangle$ denote the combined spin eigenstates for a system of two particles, each with spin 1. Derive expressions for all states with $m = s$ in terms of product states.

Given that the particles are identical, and that the spatial wavefunction describing their relative position has definite orbital angular momentum ℓ, show that $\ell + s$ must be even. Suppose that this two-particle state is known to arise from the decay of a single particle, X, also of spin 1. Assuming that total angular momentum and parity are conserved in this process, find the values of ℓ and s that are allowed, depending on whether the intrinsic parity of X is even or odd.

[*You may set* $\hbar = 1$ *and use* $J_\pm|\,j\,m\,\rangle = \sqrt{(j\mp m)(j\pm m+1)}\;|\,j\;m{\pm}1\,\rangle$.]

3/II/32D **Principles of Quantum Mechanics**

Let

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^\dagger) \ , \qquad \hat{p} = \left(\frac{\hbar m\omega}{2}\right)^{1/2} i(a^\dagger - a)$$

be the position and momentum operators for a one-dimensional harmonic oscillator of mass m and frequency ω. Write down the commutation relations obeyed by a and $a^\dagger$ and give an expression for the oscillator Hamiltonian $H(\hat{x}, \hat{p})$ in terms of them. Prove that the only energies allowed are $E_n = \hbar\omega(n + \frac{1}{2})$ with $n = 0, 1, 2, \ldots$ and give, without proof, a formula for a general normalised eigenstate $|n\rangle$ in terms of $|0\rangle$.

A three-dimensional oscillator with charge is subjected to a weak electric field so that its total Hamiltonian is

$$H_1 \ + \ H_2 \ + \ H_3 \ + \ \lambda m\omega^2(\,\hat{x}_1\hat{x}_2 + \hat{x}_2\hat{x}_3 + \hat{x}_3\hat{x}_1\,)$$

where $H_i = H(\hat{x}_i, \hat{p}_i)$ for $i = 1, 2, 3$ and λ is a small, dimensionless parameter. Express the general eigenstate for the Hamiltonian with $\lambda = 0$ in terms of one-dimensional oscillator states, and give the corresponding energy eigenvalue. Use perturbation theory to compute the changes in energies of states in the lowest two levels when $\lambda \neq 0$, working to the leading order at which non-vanishing corrections occur.

4/II/32D **Principles of Quantum Mechanics**

The Hamiltonian for a particle of spin $\frac{1}{2}$ in a magnetic field $\mathbf{B}$ is

$$H = -\frac{1}{2}\hbar\gamma\mathbf{B}\cdot\boldsymbol{\sigma} \qquad \text{where} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and γ is a constant (the motion of the particle in space can be ignored). Consider a magnetic field which is independent of time. Writing $\mathbf{B} = B\mathbf{n}$, where $\mathbf{n}$ is a unit vector, calculate the time evolution operator and show that if the particle is initially in a state $|\chi\rangle$ the probability of measuring it to be in an orthogonal state $|\chi'\rangle$ after a time t is

$$|\langle\chi'|\mathbf{n}\cdot\boldsymbol{\sigma}|\chi\rangle|^2 \sin^2\frac{\gamma Bt}{2} .$$

Evaluate this to find the probability for a transition from a state of spin up along the z direction to one of spin down along the z direction when $\mathbf{B} = (B_x, 0, B_z)$.

Now consider a magnetic field whose x and y components are time-dependent but small:

$$\mathbf{B} = (\, A\cos\alpha t,\ A\sin\alpha t,\ B_z\,) .$$

Show that the probability for a transition from a spin-up state at time zero to a spin-down state at time t (with spin measured along the z direction, as before) is approximately

$$\left(\frac{\gamma A}{\gamma B_z+\alpha}\right)^2 \sin^2\frac{(\gamma B_z+\alpha)t}{2} ,$$

where you may assume $|A| \ll |B_z+\alpha\gamma^{-1}|$. Comment on how this compares, when $\alpha = 0$, with the result for a time-independent field.

[*The first-order transition amplitude due to a perturbation $V(t)$ is*

$$-\frac{i}{\hbar}\int_0^t dt' e^{i(E'-E)t'/\hbar}\langle\chi'|V(t')|\chi\rangle$$

where $|\chi\rangle$ and $|\chi'\rangle$ are orthogonal eigenstates of the unperturbed Hamiltonian with eigenvalues E and E' respectively.]

1/II/33A Applications of Quantum Mechanics

In a certain spherically symmetric potential, the radial wavefunction for particle scattering in the $l = 0$ sector (S-wave), for wavenumber k and $r \gg 0$, is

$$R(r,k) = \frac{A}{kr}\left(g(-k)e^{-ikr} - g(k)e^{ikr}\right)$$

where

$$g(k) = \frac{k + i\kappa}{k - i\alpha}$$

with κ and α real, positive constants. Scattering in sectors with $l \neq 0$ can be neglected. Deduce the formula for the S-matrix in this case and show that it satisfies the expected symmetry and reality properties. Show that the phase shift is

$$\delta(k) = \tan^{-1}\frac{k(\kappa+\alpha)}{k^2 - \kappa\alpha} \ .$$

What is the scattering length for this potential?

From the form of the radial wavefunction, deduce the energies of the bound states, if any, in this system. If you were given only the S-matrix as a function of k, and no other information, would you reach the same conclusion? Are there any resonances here?

[*Hint: Recall that* $S(k) = e^{2i\delta(k)}$ *for real* k, *where* $\delta(k)$ *is the phase shift.*]

2/II/33A Applications of Quantum Mechanics

Describe the variational method for estimating the ground state energy of a quantum system. Prove that an error of order ϵ in the wavefunction leads to an error of order ϵ^2 in the energy.

Explain how the variational method can be generalized to give an estimate of the energy of the first excited state of a quantum system.

Using the variational method, estimate the energy of the first excited state of the anharmonic oscillator with Hamiltonian

$$H = -\frac{d^2}{dx^2} + x^2 + x^4 \ .$$

How might you improve your estimate?

[*Hint: If* $I_{2n} = \int_{-\infty}^{\infty} x^{2n}\, e^{-ax^2} dx$ *then*

$$I_0 = \sqrt{\frac{\pi}{a}}, \quad I_2 = \sqrt{\frac{\pi}{a}}\,\frac{1}{2a}, \quad I_4 = \sqrt{\frac{\pi}{a}}\,\frac{3}{4a^2}, \quad I_6 = \sqrt{\frac{\pi}{a}}\,\frac{15}{8a^3}. \ \Big]$$

3/II/33A **Applications of Quantum Mechanics**

Consider the Hamiltonian

$$H = \mathbf{B}(t) \cdot \mathbf{S}$$

for a particle of spin $\frac{1}{2}$ fixed in space, in a rotating magnetic field, where

$$S_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\mathbf{B}(t) = B(\sin\alpha\cos\omega t,\ \sin\alpha\sin\omega t,\ \cos\alpha)$$

with B, α and ω constant, and $B > 0$, $\omega > 0$.

There is an exact solution of the time-dependent Schrödinger equation for this Hamiltonian,

$$\chi(t) = \left(\cos\left(\tfrac{1}{2}\lambda t\right) - i\frac{B - \omega\cos\alpha}{\lambda}\sin\left(\tfrac{1}{2}\lambda t\right)\right) e^{-i\omega t/2}\,\chi_+ + i\left(\frac{\omega}{\lambda}\sin\alpha\sin\left(\tfrac{1}{2}\lambda t\right)\right) e^{i\omega t/2}\,\chi_-$$

where $\lambda \equiv (\omega^2 - 2\omega B\cos\alpha + B^2)^{1/2}$ and

$$\chi_+ = \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\omega t}\sin\frac{\alpha}{2} \end{pmatrix}, \quad \chi_- = \begin{pmatrix} e^{-i\omega t}\sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2} \end{pmatrix}.$$

Show that, for $\omega \ll B$, this exact solution simplifies to a form consistent with the adiabatic approximation. Find the dynamic phase and the geometric phase in the adiabatic regime. What is the Berry phase for one complete cycle of $\mathbf{B}$?

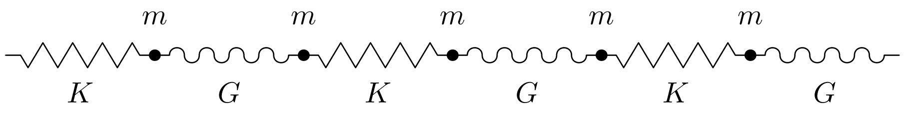
The Berry phase can be calculated as an integral of the form

$$\Gamma = i\oint \langle\psi|\nabla_{\mathbf{R}}\psi\rangle \cdot d\mathbf{R}\ .$$

Evaluate Γ for the adiabatic evolution described above.

4/II/33A **Applications of Quantum Mechanics**

Consider a 1-dimensional chain of $2N$ atoms of mass m (with N large and with periodic boundary conditions). The interactions between neighbouring atoms are modelled by springs with alternating spring constants K and G, with $K > G$.



In equilibrium, the separation of the atoms is a, the natural length of the springs.

Find the frequencies of the longitudinal modes of vibration for this system, and show that they are labelled by a wavenumber q that is restricted to a Brillouin zone. Identify the acoustic and optical bands of the vibration spectrum, and determine approximations for the frequencies near the centre of the Brillouin zone. What is the frequency gap between the acoustic and optical bands at the zone boundary?

Describe briefly the properties of the phonons in this system.

2/II/34D **Statistical Physics**

Derive the Maxwell relation

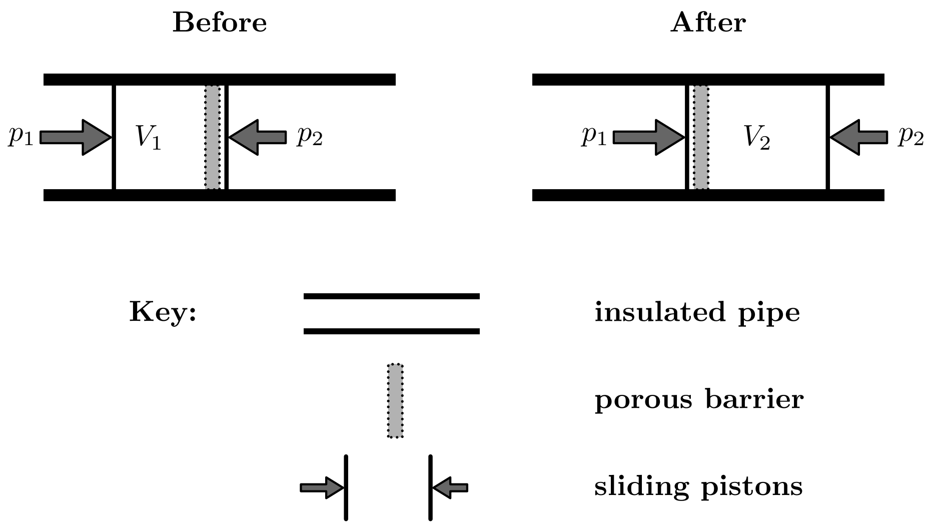
$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p .$$

The diagram below illustrates the Joule–Thomson throttling process for a porous barrier. A gas of volume V_1, initially on the left-hand side of a thermally insulated pipe, is forced by a piston to go through the barrier using constant pressure p_1. As a result the gas flows to the right-hand side, resisted by a piston which applies a constant pressure p_2 (with $p_2 < p_1$). Eventually all of the gas occupies a volume V_2 on the right-hand side. Show that this process conserves enthalpy.

The Joule–Thomson coefficient μ_{JT} is the change in temperature with respect to a change in pressure during a process that conserves enthalpy H. Express the Joule–Thomson coefficient, $\mu_{\mathrm{JT}} \equiv \left(\frac{\partial T}{\partial p}\right)_H$, in terms of T, V, the heat capacity at constant pressure C_p, and the volume coefficient of expansion $\alpha \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$.

What is μ_{JT} for an ideal gas?

If one wishes to use the Joule–Thomson process to cool a real (non-ideal) gas, what must the sign of μ_{JT} be?



3/II/34D **Statistical Physics**

For a **2-dimensional** gas of N nonrelativistic, non-interacting, spinless bosons, find the density of states $g(\varepsilon)$ in the neighbourhood of energy ε. [*Hint: consider the gas in a box of size* $L \times L$ *which has periodic boundary conditions. Work in the thermodynamic limit* $N \to \infty$, $L \to \infty$, *with* N/L^2 *held finite.*]

Calculate the number of particles per unit area at a given temperature and chemical potential.

Explain why Bose–Einstein condensation does not occur in this gas at any temperature.

[*Recall that*

$$\frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1} dx}{z^{-1} e^x - 1} = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^n} \, . \Bigg]$$

4/II/34D **Statistical Physics**

Consider a classical gas of diatomic molecules whose orientation is fixed by a strong magnetic field. The molecules are not free to rotate, but they are free to vibrate. Assuming that the vibrations are approximately harmonic, calculate the contribution to the partition function due to vibrations.

Evaluate the free energy $F = -kT \ln Z$, where Z is the total partition function for the gas, and hence calculate the entropy.

[*Note that* $\int_{-\infty}^{\infty} \exp(-au^2) du = \sqrt{\pi/a}$ *and* $\int_0^\infty u^2 \exp(-au^2) du = \sqrt{\pi}/4a^{3/2}$. *You may approximate* $\ln N!$ *by* $N \ln N - N$.]

1/II/34E **Electrodynamics**

Frame $\mathcal{S}'$ is moving with uniform speed v in the z-direction relative to a laboratory frame $\mathcal{S}$. Using Cartesian coordinates and units such that $c = 1$, the relevant Lorentz transformation is

$$t' = \gamma(t - vz), \quad x' = x, \quad y' = y, \quad z' = \gamma(z - vt)\,,$$

where $\gamma = 1/\sqrt{1 - v^2}$. A straight thin wire of infinite extent lies along the z-axis and carries charge and current line densities σ and J per unit length, as measured in $\mathcal{S}$. Stating carefully your assumptions show that the corresponding quantities in $\mathcal{S}'$ are given by

$$\sigma' = \gamma(\sigma - vJ), \qquad J' = \gamma(J - v\sigma)\,.$$

Using cylindrical polar coordinates, and the integral forms of the Maxwell equations $\nabla \cdot \mathbf{E} = \mu_0 \rho$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, derive the electric and magnetic fields outside the wire in both frames.

In a standard notation the Lorentz transformation for the electric and magnetic fields is

$$\mathbf{E}_{\|}{}' = \mathbf{E}_{\|}, \qquad \mathbf{B}_{\|}{}' = \mathbf{B}_{\|}, \qquad \mathbf{E}_{\perp}{}' = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}), \qquad \mathbf{B}_{\perp}{}' = \gamma(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E}_{\perp}).$$

Is your result consistent with this?

3/II/35E Electrodynamics

Consider a particle of charge q moving with 3-velocity $\mathbf{v}$. If the particle is moving slowly then Larmor's formula asserts that the instantaneous radiated power is

$$\mathcal{P} = \frac{\mu_0}{6\pi} q^2 \left| \frac{d\mathbf{v}}{dt} \right|^2 .$$

Suppose, however, that the particle is moving relativistically. Give reasons why one should conclude that $\mathcal{P}$ is a Lorentz invariant. Writing the 4-velocity as $U^a = (\gamma, \gamma\mathbf{v})$ where $\gamma = 1/\sqrt{1-|\mathbf{v}|^2}$ and $c = 1$, show that

$$\dot{U}^a = (\gamma^3\alpha, \gamma^3\alpha\mathbf{v} + \gamma\dot{\mathbf{v}})$$

where $\alpha = \mathbf{v}\cdot\dot{\mathbf{v}}$ and $\dot{f} = df/ds$ where s is the particle's proper time. Show also that

$$\dot{U}^a\dot{U}_a = -\gamma^4\alpha^2 - \gamma^2|\dot{\mathbf{v}}|^2.$$

Deduce the relativistic version of Larmor's formula.

Suppose the particle moves in a circular orbit perpendicular to a uniform magnetic field $\mathbf{B}$. Show that

$$\mathcal{P} = \frac{\mu_0}{6\pi}\frac{q^4}{m^2}(\gamma^2 - 1)|\mathbf{B}|^2,$$

where m is the mass of the particle, and comment briefly on the slow motion limit.

4/II/35E Electrodynamics

An action

$$S[\varphi] = \int d^4x \, L(\varphi, \varphi_{,a})$$

is given, where $\varphi(x)$ is a scalar field. Explain heuristically how to compute the functional derivative $\delta S/\delta\varphi$.

Consider the action for electromagnetism,

$$S[A_a] = -\int d^4x \left\{ \frac{1}{4\mu_0} F^{ab}F_{ab} + J^a A_a \right\} .$$

Here J^a is the 4-current density, A_a is the 4-potential and $F_{ab} = A_{b,a} - A_{a,b}$ is the Maxwell field tensor. Obtain Maxwell's equations in 4-vector form.

Another action that is sometimes suggested is

$$\widehat{S}[A_a] = -\int d^4x \left\{ \frac{1}{2\mu_0} A^{a,b}A_{a,b} + J^a A_a \right\}.$$

Under which additional assumption can Maxwell's equations be obtained using this action?

Using this additional assumption establish the relationship between the actions S and $\widehat{S}$.

1/II/35A **General Relativity**

Starting from the Riemann tensor for a metric g_{ab}, define the Ricci tensor R_{ab} and the scalar curvature R.

The Riemann tensor obeys

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0\,.$$

Deduce that

$$\nabla^a R_{ab} = \tfrac{1}{2}\nabla_b R\,. \qquad (*)$$

Write down Einstein's field equations in the presence of a matter source, with energy-momentum tensor T_{ab}. How is the relation $(*)$ important for the consistency of Einstein's equations?

Show that, for a scalar function ϕ, one has

$$\nabla^2 \nabla_a \phi = \nabla_a \nabla^2 \phi + R_{ab} \nabla^b \phi\ .$$

Assume that

$$R_{ab} = \nabla_a \nabla_b \phi$$

for a scalar field ϕ. Show that the quantity

$$R + \nabla^a \phi \nabla_a \phi$$

is then a constant.

2/II/35A **General Relativity**

The symbol ∇_a denotes the covariant derivative defined by the Christoffel connection $\Gamma^a{}_{bc}$ for a metric g_{ab}. Explain briefly why

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\phi = 0,$$
$$(\nabla_a \nabla_b - \nabla_b \nabla_a)v_c \neq 0,$$

in general, where ϕ is a scalar field and v_c is a covariant vector field.

A Killing vector field v_a satisfies the equation

$$S_{ab} \;\equiv\; \nabla_a v_b + \nabla_b v_a \;=\; 0\,.$$

By considering the quantity $\nabla_a S_{bc} + \nabla_b S_{ac} - \nabla_c S_{ab}$, show that

$$\nabla_a \nabla_b v_c \;=\; -R^d{}_{abc} v_d\,.$$

Find all Killing vector fields v_a in the case of flat Minkowski space-time.

For a metric of the form

$$\mathrm{d}s^2 \;=\; -f(\mathbf{x})\,\mathrm{d}t^2 \;+\; g_{ij}(\mathbf{x})\,\mathrm{d}x^i \mathrm{d}x^j\,, \qquad i,j = 1,2,3\,,$$

where $\mathbf{x}$ denotes the coordinates x^i, show that $\Gamma^0{}_{00} = \Gamma^0{}_{ij} = 0$ and that $\Gamma^0{}_{0i} = \Gamma^0{}_{i0} = \frac{1}{2}\,(\partial_i f)\,/f$. Deduce that the vector field $v^a = (1,0,0,0)$ is a Killing vector field.

[*You may assume the standard symmetries of the Riemann tensor.*]

4/II/36A **General Relativity**

Consider a particle on a trajectory $x^a(\lambda)$. Show that the geodesic equations, with affine parameter λ, coincide with the variational equations obtained by varying the integral

$$I = \int_{\lambda_0}^{\lambda_1} g_{ab}(x) \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \, \mathrm{d}\lambda \,,$$

the end-points being fixed.

In the case that $f(r) = 1 - 2GMu$, show that the space-time metric is given in the form

$$\mathrm{d}s^2 \;=\; -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \; \mathrm{d}\phi^2)\,,$$

for a certain function $f(r)$. Assuming the particle motion takes place in the plane $\theta = \dfrac{\pi}{2}$ show that

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} \;=\; \frac{h}{r^2} \;, \qquad\qquad \frac{\mathrm{d}t}{\mathrm{d}\lambda} \;=\; \frac{E}{f(r)} \;,$$

for h, E constants. Writing $u = 1/r$, obtain the equation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + f(r)\,u^2 \;=\; -\frac{k}{h^2}\,f(r) + \frac{E^2}{h^2}\,,$$

where k can be chosen to be 1 or 0, according to whether the particle is massive or massless. In the case that $f(r) = 1 - GMu$, show that

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u \;=\; k\,\frac{GM}{h^2} + 3GMu^2\,.$$

In the massive case, show that there is an approximate solution of the form

$$u \;=\; \frac{1}{\ell}\,\bigl(1 + e\cos\,(\alpha\phi)\bigr)\,,$$

where

$$1 - \alpha \;=\; \frac{3GM}{\ell}\,.$$

What is the interpretation of this solution?

1/II/36B **Fluid Dynamics II**

Discuss how the methods of lubrication theory may be used to find viscous fluid flows in thin layers or narrow gaps, explaining carefully what inequalities need to hold in order that the theory may apply.

Viscous fluid of kinematic viscosity ν flows under the influence of gravity g, down an inclined plane making an angle $\alpha \ll 1$ with the horizontal. The fluid layer lies between $y = 0$ and $y = h(x,t)$, where x, y are distances measured down the plane and perpendicular to it, and $|\partial h/\partial x|$ is of the same order as α. Give conditions involving h, α, ν and g that ensure that lubrication theory can be used, and solve the lubrication equations, together with the equation of mass conservation, to obtain an equation for h in the form

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(-Ah^3 + Bh^3\frac{\partial h}{\partial x}\right),$$

where A, B are constants to be determined. Show that there is a steady solution with $\partial h/\partial x = k =$ constant, and interpret this physically. Show also that a solution of this equation exists in the form of a *front*, with $h(x,t) = F(\xi)$, where $\xi = x - ct$, $F(0) = 0$, and $F(\xi) \to h_0$ as $\xi \to -\infty$. Determine c in terms of h_0, find the shape of the front implicitly in the form $\xi = G(h)$, and show that $h \propto (-\xi)^{1/3}$ as $\xi \to 0$ from below.

2/II/36B **Fluid Dynamics II**

Viscous fluid is extracted through a small hole in the tip of the cone given by $\theta = \alpha$ in spherical polar coordinates (R, θ, ϕ). The total volume flux through the hole takes the constant value Q. It is given that there is a steady solution of the Navier–Stokes equations for the fluid velocity $\mathbf{u}$. For small enough R, the velocity $\mathbf{u}$ is well approximated by $\mathbf{u} \sim (-A/R^2, 0, 0)$, where $A = Q/[2\pi(1 - \cos\alpha)]$ except in thin boundary layers near $\theta = \alpha$.

(i) Verify that the volume flux through the hole is approximately Q.

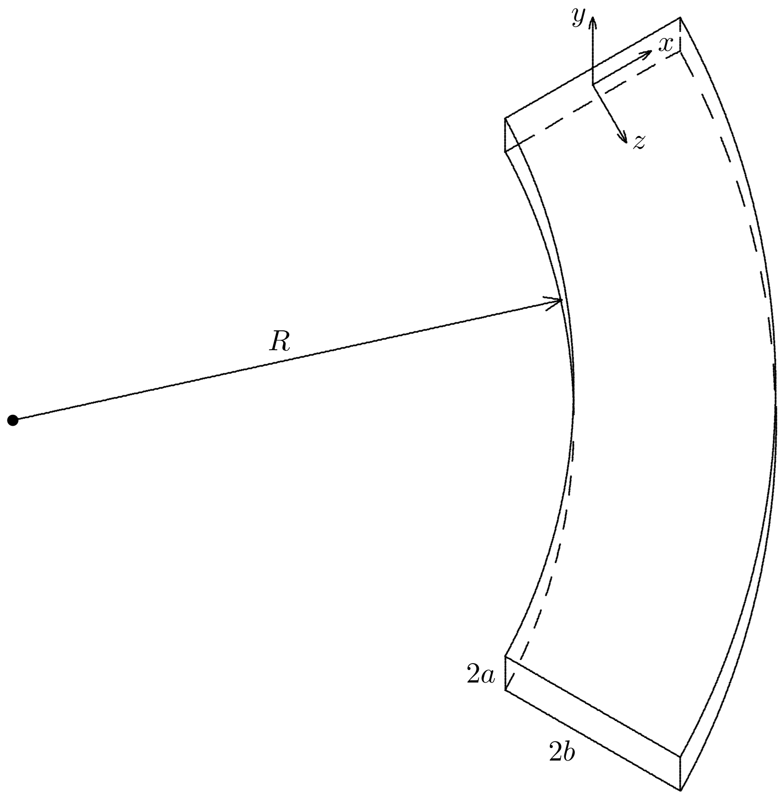
(ii) Construct a Reynolds number (depending on R) in terms of Q and the kinematic viscosity ν, and thus give an estimate of the value of R below which solutions of this type will appear.

(iii) Assuming that there is a boundary layer near $\theta = \alpha$, write down the boundary layer equations in the usual form, using local Cartesian coordinates x and y parallel and perpendicular to the boundary. Show that the boundary layer thickness $\delta(x)$ is proportional to $x^{\frac{3}{2}}$, and show that the x component of the velocity u_x may be written in the form

$$u_x = -\frac{A}{x^2} F'(\eta), \quad \text{where} \quad \eta = \frac{y}{\delta(x)}.$$

Derive the equation and boundary conditions satisfied by F. Give an expression, in terms of F, for the volume flux through the boundary layer, and use this to derive the R-dependence of the first correction to the flow outside the boundary layer.

3/II/36B **Fluid Dynamics II**



Viscous fluid of kinematic viscosity ν and density ρ flows in a curved pipe of constant rectangular cross section and constant curvature. The cross-section has height $2a$ and width $2b$ (in the radial direction) with $b \gg a$, and the radius of curvature of the inner wall is R, with $R \gg b$. A uniform pressure gradient $-G$ is applied along the pipe.

(i) Assume to a first approximation that the pipe is straight, and ignore variation in the x-direction, where (x, y, z) are Cartesian coordinates referred to an origin at the centre of the section, with x increasing radially and z measured along the pipe. Find the flow field along the pipe in the form $\mathbf{u} = (0, 0, w(y))$.

(ii) It is given that the largest component of the inertial acceleration $\mathbf{u} \cdot \nabla \mathbf{u}$ due to the curvature of the pipe is $-w^2/R$ in the x direction. Consider the secondary flow $\mathbf{u}_s$ induced in the x, y plane, again ignoring variations in x and any end effects (except for the requirement that there be zero total mass flux in the x direction). Show that $\mathbf{u}_s$ takes the form $\mathbf{u}_s = (u(y), 0, 0)$, where

$$u(y) = \frac{G^2}{120\rho^2\nu^3 R}\left(5a^2y^4 - y^6\right) + \frac{C}{2}y^2 + D,$$

and write down two equations determining the constants C and D. [*It is not necessary to solve these equations.*]

Give conditions on the parameters that ensure that $|u| \ll |w|$.

4/II/37B **Fluid Dynamics II**

(i) Assuming that axisymmetric incompressible flow $\mathbf{u} = (u_R, u_\theta, 0)$, with vorticity $(0, 0, \omega)$ in spherical polar coordinates (R, θ, ϕ) satisfies the equations

$$\mathbf{u} = \nabla \times \left(0, 0, \frac{\Psi}{R \sin\theta}\right), \quad \omega = -\frac{1}{R \sin\theta} D^2 \Psi,$$

where

$$D^2 \equiv \frac{\partial^2}{\partial R^2} + \frac{\sin\theta}{R^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\right),$$

show that for Stokes flow Ψ satisfies the equation

$$D^4 \Psi = 0. \tag{$*$}$$

(ii) A rigid sphere of radius a moves at velocity $U\hat{\mathbf{z}}$ through viscous fluid of density ρ and dynamic viscosity μ which is at rest at infinity. Assuming Stokes flow and by applying the boundary conditions at $R = a$ and as $R \to \infty$, verify that $\Psi = (AR + B/R)\sin^2\theta$ is the appropriate solution to $(*)$ for this flow, where A and B are to be determined.

(iii) Hence find the velocity field outside the sphere. Without direct calculation, explain why the drag is in the z direction and has magnitude proportional to U.

(iv) A second identical sphere is introduced into the flow, at a distance $b \gg a$ from the first, and moving at the same velocity. Justify the assertion that, when the two spheres are at the same height, or when one is vertically above the other, the drag on each sphere is the same. Calculate the leading correction to the drag in each case, to leading order in a/b.

[*You may quote without proof the fact that, for an axisymmetric function* $F(R, \theta)$,

$$\nabla \times (0, 0, F) = \left(\frac{1}{R \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta F),\ -\frac{1}{R} \frac{\partial}{\partial R}(RF),\ 0\right)$$

in spherical polar coordinates (R, θ, ϕ).]

1/II/37C **Waves**

A uniform elastic solid with density ρ and Lamé moduli λ and μ occupies the region between rigid plane boundaries $y = 0$ and $y = h$. Show that SH waves can propagate in the x direction within this layer, and find the dispersion relation for such waves.

Deduce for each mode (a) the cutoff frequency, (b) the phase velocity, and (c) the group velocity.

Show also that for each mode the kinetic energy and elastic energy are equal in an average sense to be made precise.

[*You may assume that the elastic energy per unit volume* $W = \frac{1}{2}(\lambda e_{kk}^2 + 2\mu e_{ij}e_{ij})$.]

2/II/37C **Waves**

Show that for a one-dimensional flow of a perfect gas at constant entropy the Riemann invariants $u \pm 2(c - c_0)/(\gamma - 1)$ are constant along characteristics $dx/dt = u \pm c$.

Define a simple wave. Show that in a right-propagating simple wave

$$\frac{\partial u}{\partial t} + \left(c_0 + \frac{\gamma+1}{2}u\right)\frac{\partial u}{\partial x} = 0 .$$

Now suppose instead that, owing to dissipative effects,

$$\frac{\partial u}{\partial t} + \left(c_0 + \frac{\gamma+1}{2}u\right)\frac{\partial u}{\partial x} = -\alpha u$$

where α is a positive constant. Suppose also that u is prescribed at $t = 0$ for all x, say $u(x, 0) = v(x)$. Demonstrate that, unless a shock forms,

$$u(x,t) = v(x_0)\, e^{-\alpha t}$$

where, for each x and t, x_0 is determined implicitly as the solution of the equation

$$x - c_0 t = x_0 + \frac{\gamma+1}{2}\left(\frac{1 - e^{-\alpha t}}{\alpha}\right) v(x_0) .$$

Deduce that a shock will not form at any (x, t) if

$$\alpha > \frac{\gamma+1}{2} \max_{v'<0} |v'(x_0)| .$$

3/II/37C **Waves**

Waves propagating in a slowly-varying medium satisfy the local dispersion relation

$$\omega \;=\; \Omega(\mathbf{k}, \mathbf{x}, t)$$

in the standard notation. Give a brief derivation of the ray-tracing equations for such waves; a formal justification is *not* required.

An ocean occupies the region $x > 0\,, \quad -\infty < y < \infty\,.$ Water waves are incident on a beach near $x = 0$. The undisturbed water depth is

$$h(x) \;=\; \alpha\, x^p$$

with α a small positive constant and p positive. The local dispersion relation is

$$\Omega^2 \;=\; g\kappa\, \tanh(\kappa h) \qquad \text{where} \qquad \kappa^2 \;=\; k_1^2 \;+\; k_2^2$$

and where k_1, k_2 are the wavenumber components in the x, y directions. Far from the beach, the waves are planar with frequency ω_∞ and crests making an acute angle θ_∞ with the shoreline $x = 0\,.$ Obtain a differential equation (in implicit form) for a ray $y = y(x)\,,$ and show that near the shore the ray satisfies

$$y - y_0 \;\sim\; A\, x^q$$

where A and q should be found. Sketch the appearance of the wavecrests near the shoreline.

4/II/38C **Waves**

Show that, for a plane acoustic wave, the acoustic intensity $\tilde{p}\,\mathbf{u}$ may be written as $\rho_0 c_0 |\mathbf{u}|^2 \hat{\mathbf{k}}$ in the standard notation.

Derive the general spherically-symmetric solution of the wave equation. Use it to find the velocity potential $\phi(r,t)$ for waves radiated into an unbounded fluid by a pulsating sphere of radius

$$a\,(1 \;+\; \varepsilon\, e^{i\omega t}) \qquad (\varepsilon \ll 1)\;.$$

By considering the far field, or otherwise, find the time-average rate at which energy is radiated by the sphere.

$$\left[\; \textit{You may assume that}\;\; \nabla^2\phi \;=\; \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right). \;\right]$$

1/II/38C Numerical Analysis

(a) For a numerical method to solve $y' = f(t, y)$, define the linear stability domain and state when such a method is A-stable.

(b) Determine all values of the real parameter a for which the Runge–Kutta method

$$\begin{aligned} k_1 &= f\Big(t_n + (\tfrac{1}{2} - a)h,\ y_n + h\big[\tfrac{1}{4}k_1 + (\tfrac{1}{4} - a)k_2\big]\Big), \\ k_2 &= f\Big(t_n + (\tfrac{1}{2} + a)h,\ y_n + h\big[(\tfrac{1}{4} + a)k_1 + \tfrac{1}{4}k_2\big]\Big), \\ y_{n+1} &= y_n + \tfrac{1}{2}h(k_1 + k_2) \end{aligned}$$

is A-stable.

2/II/38C Numerical Analysis

(a) State the Householder–John theorem and explain how it can be used to design iterative methods for solving a system of linear equations $Ax = b$.

(b) Let $A = L+D+U$ where D is the diagonal part of A, and L and U are, respectively, the strictly lower and strictly upper triangular parts of A. Given a vector b, consider the following iterative scheme:

$$(D + \omega L)x^{(k+1)} = (1 - \omega)Dx^{(k)} - \omega Ux^{(k)} + \omega b\,.$$

Prove that if A is a symmetric positive definite matrix, and $\omega \in (0, 2)$, then the above iteration converges to the solution of the system $Ax = b$.

3/II/38C **Numerical Analysis**

(a) Prove that all Toeplitz symmetric tridiagonal $M \times M$ matrices

$$A = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b & a & b \\ 0 & \cdots & 0 & b & a \end{bmatrix}$$

share the same eigenvectors $(v^{(k)})_{k=1}^{M}$ with components $v_i^{(k)} = \sin \dfrac{ki\pi}{M+1}$, $i = 1, \ldots, M$, and eigenvalues to be determined.

(b) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant T,$$

is approximated by the Crank–Nicolson scheme

$$u_m^{n+1} - \tfrac{1}{2}\mu \left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right) = u_m^n + \tfrac{1}{2}\mu \left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right) ,$$
$$\text{for} \qquad m = 1, \ldots, M,$$

where $\mu = \Delta t/(\Delta x)^2$, $\Delta x = 1/(M+1)$, and u_m^n is an approximation to $u(m\Delta x, n\Delta t)$. Assuming that $u(0,t) = u(1,t) = 0$, $\forall t$, show that the above scheme can be written in the form

$$Bu^{n+1} = Cu^n, \qquad 0 \leqslant n \leqslant (T/\Delta t) - 1$$

where $u^n = [u_1^n, \ldots, u_M^n]^{\mathsf{T}}$ and the real matrices B and C should be found. Using matrix analysis, find the range of μ for which the scheme is stable. [*Do not use Fourier analysis.*]

4/II/39C **Numerical Analysis**

(a) Suppose that A is a real $n \times n$ matrix, and that $w \in \mathbb{R}^n$ and $\lambda_1 \in \mathbb{R}$ are given so that $Aw = \lambda_1 w$. Further, let S be a non-singular matrix such that $Sw = ce^{(1)}$, where $e^{(1)}$ is the first coordinate vector and $c \neq 0$. Let $\widehat{A} = SAS^{-1}$. Prove that the eigenvalues of A are λ_1 together with the eigenvalues of the bottom right $(n-1) \times (n-1)$ submatrix of $\widehat{A}$.

(b) Suppose again that A is a real $n \times n$ matrix, and that two linearly independent vectors $v, w \in \mathbb{R}^n$ are given such that the linear subspace $L\{v, w\}$ spanned by v and w is invariant under the action of A, i.e.,

$$x \in L\{v, w\} \quad \Rightarrow \quad Ax \in L\{v, w\}.$$

Denote by V an $n \times 2$ matrix whose two columns are the vectors v and w, and let S be a non-singular matrix such that $R = SV$ is upper triangular, that is,

$$R = SV = S \times \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ \vdots & \vdots \\ v_n & w_n \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

Again let $\widehat{A} = SAS^{-1}$. Prove that the eigenvalues of A are the eigenvalues of the top left 2×2 submatrix of $\widehat{A}$ together with the eigenvalues of the bottom right $(n-2) \times (n-2)$ submatrix of $\widehat{A}$.