UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Wednesday 6 June 2007 9 to 12

PAPER 2

Before you begin read these instructions carefully.

*The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt **at most six** questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in bundles, marked **A**, **B**, **C**,...,**J** according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Number Theory

Let p be an odd prime number. Prove that 2 is a quadratic residue modulo p when $p \equiv 7 \pmod 8$. Deduce that, if q is a prime number strictly greater than 3 with $q \equiv 3 \pmod 4$ such that $2q+1$ is also a prime number, then $2^q - 1$ is necessarily composite. Why does the argument break down for $q = 3$?

2F Topics in Analysis

Write

$$P^+ = \{(x, y) \in \mathbb{R}^2 : x,\, y > 0\}.$$

Suppose that K is a convex, compact subset of $\mathbb{R}^2$ with $K \cap P^+ \neq \emptyset$. Show that there is a unique point $(x_0, y_0) \in K \cap P^+$ such that

$$xy \leqslant x_0 y_0$$

for all $(x, y) \in K \cap P^+$.

3G Geometry of Group Actions

Explain what is meant by a lattice in the Euclidean plane $\mathbb{R}^2$. Prove that such a lattice is either $\mathbb{Z}\boldsymbol{w}$ for some vector $\boldsymbol{w} \in \mathbb{R}^2$ or else $\mathbb{Z}\boldsymbol{w}_1 + \mathbb{Z}\boldsymbol{w}_2$ for two linearly independent vectors $\boldsymbol{w}_1, \boldsymbol{w}_2$ in $\mathbb{R}^2$.

4G Coding and Cryptography

Briefly explain how and why a signature scheme is used. Describe the El Gamal scheme.

5I Statistical Modelling

Consider the linear regression setting where the responses Y_i, $i = 1, \ldots, n$ are assumed independent with means $\mu_i = x_i^{\mathrm{T}} \beta$. Here x_i is a vector of known explanatory variables and β is a vector of unknown regression coefficients.

Show that if the response distribution is Laplace, i.e.,

$$Y_i \sim f(y_i; \mu_i, \sigma) = (2\sigma)^{-1} \exp\left\{ -\frac{|y_i - \mu_i|}{\sigma} \right\}, \qquad i = 1, \ldots, n; \quad y_i, \mu_i \in \mathbb{R};\ \sigma \in (0, \infty);$$

then the maximum likelihood estimate $\hat{\beta}$ of β is obtained by minimising

$$S_1(\beta) = \sum_{i=1}^{n} |Y_i - x_i^{\mathrm{T}} \beta|.$$

Obtain the maximum likelihood estimate for σ in terms of $S_1(\hat{\beta})$.

Briefly comment on why the Laplace distribution cannot be written in exponential dispersion family form.

6B Mathematical Biology

A field contains X_n seed-producing poppies in August of year n. On average each poppy produces γ seeds, a number that is assumed not to vary from year to year. A fraction σ of seeds survive the winter and a fraction α of those germinate in May of year $n+1$. A fraction β of those that survive the next winter germinate in year $n+2$. Show that X_n satisfies the following difference equation:

$$X_{n+1} = \alpha\sigma\gamma X_n + \beta\sigma^2(1-\alpha)\gamma X_{n-1}\,.$$

Write down the general solution of this equation, and show that the poppies in the field will eventually die out if

$$\sigma\gamma[(1-\alpha)\beta\sigma + \alpha] < 1\,.$$

7E Dynamical Systems

Find and classify the fixed points of the system

$$\begin{aligned} \dot{x} &= (1-x^2)y\ , \\ \dot{y} &= x(1-y^2)\ . \end{aligned}$$

What are the values of their Poincaré indices? Prove that there are no periodic orbits. Sketch the phase plane.

8B Further Complex Methods

The function $I(z)$ is defined by

$$I(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t+1} dt\ .$$

For what values of z is $I(z)$ analytic?

By considering $I(z) - \zeta(z)$, where $\zeta(z)$ is the Riemann zeta function which you may assume is given by

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t-1} dt \qquad (\operatorname{Re} z > 1)\,,$$

show that $I(z) = \big(1 - 2^{1-z}\big)\zeta(z)$. Deduce from this result that the analytic continuation of $I(z)$ is an entire function. [*You may use properties of* $\zeta(z)$ *without proof.*]

9C Classical Dynamics

The Lagrangian for a particle of mass m and charge e moving in a magnetic field with position vector $\mathbf{r} = (x, y, z)$ is given by

$$L \;=\; \tfrac{1}{2} m\dot{\mathbf{r}}^2 \;+\; e\,\frac{\dot{\mathbf{r}} \cdot \mathbf{A}}{c}\ ,$$

where the vector potential $\mathbf{A}(\mathbf{r})$, which does not depend on time explicitly, is related to the magnetic field $\mathbf{B}$ through

$$\mathbf{B} = \nabla \times \mathbf{A}\ .$$

Write down Lagrange's equations and use them to show that the equation of motion of the particle can be written in the form

$$m\ddot{\mathbf{r}} \;=\; e\,\frac{\dot{\mathbf{r}}\times\mathbf{B}}{c}\ .$$

Deduce that the kinetic energy, T, is constant.

When the magnetic field is of the form $\mathbf{B} = (0, 0, dF/dx)$ for some specified function $F(x)$, show further that

$$\dot{x}^2 \;=\; \frac{2T}{m} \;-\; \frac{\big(eF(x) + C\big)^2}{m^2c^2} \;+\; D\ ,$$

where C and D are constants.

10A Cosmology

The number density of photons in thermal equilibrium at temperature T takes the form

$$n = \frac{8\pi}{c^3} \int \frac{\nu^2 d\nu}{\exp(h\nu/kT) - 1} \,.$$

At time $t = t_{\rm dec}$ and temperature $T = T_{\rm dec}$, photons decouple from thermal equilibrium. By considering how the photon frequency redshifts as the universe expands, show that the form of the equilibrium frequency distribution is preserved, with the temperature for $t > t_{\rm dec}$ defined by

$$T \equiv \frac{a(t_{\rm dec})}{a(t)} T_{\rm dec} \,.$$

Show that the photon number density n and energy density ϵ can be expressed in the form

$$n = \alpha T^3 \,, \qquad\qquad \epsilon = \xi T^4 \,,$$

where the constants α and ξ need not be evaluated explicitly.

SECTION II

11G Coding and Cryptography

Define the capacity of a discrete memoryless channel. State Shannon's second coding theorem and use it to show that the discrete memoryless channel with channel matrix

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

has capacity $\log 5 - 2$.

12F Topics in Analysis

(i) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Prove the theorem of Bernstein which states that, if we write

$$f_m(t) \;=\; \sum_{r=0}^{m} \binom{m}{r} f(r/m) t^r (1-t)^{m-r},$$

for $0 \leqslant t \leqslant 1$, then $f_m \to f$ uniformly as $m \to \infty$.

(ii) Let $n \geqslant 1$, $a_{1,n}, a_{2,n}, \ldots, a_{n,n} \in \mathbb{R}$ and let $x_{1,n}, x_{2,n}, \ldots, x_{n,n}$ be distinct points in $[0,1]$. We write

$$I_n(g) \;=\; \sum_{j=1}^{n} a_{j,n} g(x_{j,n})$$

for every continuous function $g : [0,1] \to \mathbb{R}$. Show that, if

$$I_n(P) \;=\; \int_0^1 P(t)\, dt\,,$$

for all polynomials P of degree $2n-1$ or less, then $a_{j,n} \geqslant 0$ for all $1 \leqslant j \leqslant n$ and $\sum_{j=1}^{n} a_{j,n} \;=\; 1$.

(iii) If I_n satisfies the conditions set out in (ii), show that

$$I_n(f) \to \int_0^1 f(t)\, dt$$

as $n \to \infty$ whenever $f : [0,1] \to \mathbb{R}$ is continuous.

13B Mathematical Biology

Show that the concentration $C(\mathbf{x}, t)$ of a diffusible chemical substance in a stationary medium satisfies the partial differential equation

$$\frac{\partial C}{\partial t} = \nabla \cdot (D \nabla C) + F \,,$$

where D is the diffusivity and $F(\mathbf{x}, t)$ is the rate of supply of the chemical.

A finite amount of the chemical, $4\pi M$, is supplied at the origin at time $t = 0$, and spreads out in a spherically symmetric manner, so that $C = C(r, t)$ for $r > 0, t > 0$, where r is the radial coordinate. The diffusivity is given by $D = kC$, for constant k. Show, by dimensional analysis or otherwise, that it is appropriate to seek a similarity solution in which

$$C = \frac{M^{\alpha}}{(k\,t)^{\beta}}\, f(\xi)\,, \quad \xi = \frac{r}{(Mkt)^{\gamma}} \quad \text{and} \quad \int_0^{\infty} \xi^2 f(\xi)\, d\xi = 1\,,$$

where α, β, γ are constants to be determined, and derive the ordinary differential equation satisfied by $f(\xi)$.

Solve this ordinary differential equation, subject to appropriate boundary conditions, and deduce that the chemical occupies a finite spherical region of radius

$$r_0(t) = (75Mkt)^{1/5}\,.$$

[*Note: in spherical polar coordinates*

$$\nabla C \equiv \left(\frac{\partial C}{\partial r}\,, 0, 0 \right) \quad \textit{and} \quad \nabla \cdot (V(r,t), 0, 0) \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 V \right).\Big]$$

14B Further Complex Methods

Show that the equation

$$zw'' - (1 + z)w' + 2(1 - z)w = 0$$

has solutions of the form $w(z) = \int_\gamma e^{zt} f(t)\, dt$, where

$$f(t) = \frac{1}{(t-2)(t+1)^2}\,,$$

provided that γ is suitably chosen.

Hence find the general solution, evaluating the integrals explicitly. Show that the general solution is entire, but that there is no solution that satisfies $w(0) = 0$ and $w'(0) \neq 0$.

15C Classical Dynamics

(a) A Hamiltonian system with n degrees of freedom is described by the phase space coordinates $(q_1, q_2, ..., q_n)$ and momenta $(p_1, p_2, ..., p_n)$. Show that the phase-space volume element

$$d\tau \;=\; dq_1 dq_2 .....dq_n dp_1 dp_2 .....dp_n$$

is conserved under time evolution.

(b) The Hamiltonian, H , for the system in part (a) is independent of time. Show that if $F(q_1, ..., q_n, p_1, ..., p_n)$ is a constant of the motion, then the Poisson bracket $[F, H]$ vanishes. Evaluate $[F, H]$ when

$$F \;=\; \sum_{k=1}^{n} p_k$$

and

$$H \;=\; \sum_{k=1}^{n} p_k^2 + V(q_1, q_2, ..., q_n) \;,$$

where the potential V depends on the q_k $(k = 1, 2, ..., n)$ only through quantities of the form $q_i - q_j$ for $i \neq j$.

(c) For a system with one degree of freedom, state what is meant by the transformation

$$(q, p) \rightarrow \big(Q(q, p),\ P(q, p)\,\big)$$

being canonical. Show that the transformation is canonical if and only if the Poisson bracket $[Q, P] = 1$.

16G Set Theory and Logic

Explain carefully what is meant by a *deduction* in the propositional calculus. State the completeness theorem for the propositional calculus, and deduce the compactness theorem.

Let P, Q, R be three pairwise-disjoint sets of primitive propositions, and suppose given compound propositions $s \in \mathcal{L}(P \cup Q)$ and $t \in \mathcal{L}(Q \cup R)$ such that $(s \vdash t)$ holds. Let U denote the set

$$\{u \in \mathcal{L}(Q) \mid \ (s \vdash u)\} \;.$$

If $v \colon Q \rightarrow 2$ is any valuation making all the propositions in U true, show that the set

$$\{s\} \cup \{q \mid q \in Q, v(q) = 1\} \cup \{\neg q \mid q \in Q, v(q) = 0\}$$

is consistent. Deduce that $U \cup \{\neg t\}$ is inconsistent, and hence show that there exists $u \in \mathcal{L}(Q)$ such that $(s \vdash u)$ and $(u \vdash t)$ both hold.

17H Graph Theory

The Ramsey number $R(G)$ of a graph G is the smallest n such that in any red/blue colouring of the edges of K_n there is a monochromatic copy of G.

Show that $R(K_t) \leqslant \binom{2t-2}{t-1}$ for every $t \geqslant 3$.

Let H be the graph on four vertices obtained by adding an edge to a triangle. Show that $R(H) = 7$.

18F Galois Theory

Let $L = K(\xi_n)$, where ξ_n is a primitive nth root of unity and $G = \mathrm{Aut}(L/K)$. Prove that there is an injective group homomorphism $\chi : G \to (\mathbb{Z}/n\mathbb{Z})^*$.

Show that, if M is an intermediate subfield of $K(\xi_n)/K$, then M/K is Galois. State carefully any results that you use.

Give an example where G is non-trivial but χ is not surjective. Show that χ is surjective when $K = \mathbb{Q}$ and n is a prime.

Determine all the intermediate subfields M of $\mathbb{Q}(\xi_7)$ and the automorphism groups $\mathrm{Aut}(\mathbb{Q}\,(\xi_7)/M)$. Write the quadratic subfield in the form $\mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Q}$.

19H Representation Theory

Let G be a finite group and let Z be its centre. Show that if ρ is a complex irreducible representation of G, assumed to be faithful (that is, the kernel of ρ is trivial), then Z is cyclic.

Now assume that G is a p-group (that is, the order of G is a power of the prime p), and assume that Z is cyclic. If ρ is a faithful representation of G, show that some irreducible component of ρ is faithful.

[*You may use without proof the fact that, since G is a p-group, Z is non-trivial and any non-trivial normal subgroup of G intersects Z non-trivially.*]

Deduce that a finite p-group has a faithful irreducible representation if and only if its centre is cyclic.

20H Number Fields

Let $K = \mathbb{Q}(\sqrt{10})$ and put $\varepsilon = 3 + \sqrt{10}$.

(a) Show that 2, 3 and $\varepsilon + 1$ are irreducible elements in $\mathcal{O}_K$. Deduce from the equation

$$6 = 2 \cdot 3 = (\varepsilon + 1)(\bar{\varepsilon} + 1)$$

that $\mathcal{O}_K$ is not a principal ideal domain.

(b) Put $\mathfrak{p}_2 = [2, \varepsilon + 1]$ and $\mathfrak{p}_3 = [3, \varepsilon + 1]$. Show that

$$[2] = \mathfrak{p}_2^2\,, \quad [3] = \mathfrak{p}_3 \bar{\mathfrak{p}}_3\,, \quad \mathfrak{p}_2 \mathfrak{p}_3 = [\varepsilon + 1]\,, \quad \mathfrak{p}_2 \bar{\mathfrak{p}}_3 = [\varepsilon - 1]\,.$$

Deduce that K has class number 2.

(c) Show that ε is the fundamental unit of K. Hence prove that all solutions in integers x, y of the equation $x^2 - 10y^2 = 6$ are given by

$$x + \sqrt{10}y = \pm \varepsilon^n (\varepsilon + (-1)^n)\,, \;\; n = 0, 1, 2, \ldots$$

21H Algebraic Topology

State the Mayer–Vietoris sequence for a simplicial complex X which is a union of two subcomplexes A and B. Define the homomorphisms in the sequence (but do *not* check that they are well-defined). Prove exactness of the sequence at the term $H_i(A \cap B)$.

22G Linear Analysis

Let X be a Banach space, Y a normed vector space, and $T : X \to Y$ a bounded linear map. Assume that $T(X)$ is of second category in Y. Show that T is surjective and $T(\mathcal{U})$ is open whenever $\mathcal{U}$ is open. Show that, if T is also injective, then T^{-1} exists and is bounded.

Give an example of a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R})$ is of second category in $\mathbb{R}$ but f is not surjective. Give an example of a continuous surjective map $f : \mathbb{R} \to \mathbb{R}$ which does not take open sets to open sets.

23F Riemann Surfaces

A function ψ is defined for $z \in \mathbb{C}$ by

$$\psi(z) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i \left(n+\tfrac{1}{2}\right)^2 \tau + 2\pi i \left(n+\tfrac{1}{2}\right)\left(z+\tfrac{1}{2}\right)\right)$$

where τ is a complex parameter with $\mathrm{Im}(\tau) > 0$. Prove that this series converges uniformly on the subsets $\{|\mathrm{Im}(z)| \leqslant R\}$ for $R > 0$ and deduce that ψ is holomorphic on $\mathbb{C}$.

You may assume without proof that

$$\psi(z+1) = -\psi(z) \qquad \text{and} \qquad \psi(z+\tau) = -\exp(-\pi i\tau - 2\pi i z)\,\psi(z)$$

for all $z \in \mathbb{C}$. Let $\ell(z)$ be the logarithmic derivative $\ell(z) = \dfrac{\psi'(z)}{\psi(z)}$. Show that

$$\ell(z+1) = \ell(z) \qquad \text{and} \qquad \ell(z+\tau) = -2\pi i + \ell(z)$$

for all $z \in \mathbb{C}$. Deduce that ψ has only one zero in the parallelogram P with vertices $\frac{1}{2}(\pm 1 \pm \tau)$. Find all of the zeros of ψ.

Let Λ be the lattice in $\mathbb{C}$ generated by 1 and τ. Show that, for $\lambda_j, a_j \in \mathbb{C}$ $(j = 1, \ldots, n)$, the formula

$$f(z) = \lambda_1 \frac{\psi'(z-a_1)}{\psi(z-a_1)} + \ldots + \lambda_n \frac{\psi'(z-a_n)}{\psi(z-a_n)}$$

gives a Λ-periodic meromorphic function f if and only if $\lambda_1 + \ldots + \lambda_n = 0$. Deduce that $\dfrac{d}{dz}\left(\dfrac{\psi'(z-a)}{\psi(z-a)}\right)$ is Λ-periodic.

24H Differential Geometry

(i) What is a minimal surface? Explain why minimal surfaces always have non-positive Gaussian curvature.

(ii) A smooth map $f : S_1 \to S_2$ between two surfaces in 3-space is said to be *conformal* if

$$\langle df_p(v_1), df_p(v_2)\rangle = \lambda(p)\langle v_1, v_2\rangle$$

for all $p \in S_1$ and all $v_1, v_2 \in T_pS_1$, where $\lambda(p) \neq 0$ is a number which depends only on p.

Let S be a surface without umbilical points. Prove that S is a minimal surface if and only if the Gauss map $N : S \to S^2$ is conformal.

(iii) Show that isothermal coordinates exist around a non-planar point in a minimal surface.

25J Probability and Measure

(a) State and prove the first Borel–Cantelli lemma. State the second Borel–Cantelli lemma.

(b) Let $X_1, X_2, \dots$ be a sequence of independent random variables that converges in probability to the limit X. Show that X is almost surely constant.

A sequence $X_1, X_2, \dots$ of random variables is said to be *completely convergent* to X if

$$\sum_{n\in\mathbb{N}} \mathbb{P}\big(A_n(\epsilon)\big) < \infty \quad \text{for all } \epsilon > 0 \ , \quad \text{where } A_n(\epsilon) = \big\{|X_n - X| > \epsilon\big\} \ .$$

(c) Show that complete convergence implies almost sure convergence.

(d) Show that, for sequences of independent random variables, almost sure convergence also implies complete convergence.

(e) Find a sequence of (dependent) random variables that converges almost surely but does not converge completely.

26J Applied Probability

In this question we work with a continuous-time Markov chain where the rate of jump $i \to j$ may depend on j but not on i. A virus can be in one of s strains $1, \dots, s$, and it mutates to strain j with rate $r_j \geqslant 0$ from each strain $i \neq j$. (Mutations are caused by the chemical environment.) Set $R = r_1 + \ldots + r_s$.

(a) Write down the Q-matrix (the generator) of the chain (X_t) in terms of r_j and R.

(b) If $R = 0$, that is, $r_1 = \ldots = r_s = 0$, what are the communicating classes of the chain (X_t)?

(c) From now on assume that $R > 0$. State and prove a necessary and sufficient condition, in terms of the numbers r_j, for the chain (X_t) to have a single communicating class (which therefore should be closed).

(d) In general, what is the number of closed communicating classes in the chain (X_t)? Describe all open communicating classes of (X_t).

(e) Find the equilibrium distribution of (X_t). Is the chain (X_t) reversible? Justify your answer.

(f) Write down the transition matrix $\widehat{P} = (\widehat{p}_{ij})$ of the discrete-time jump chain for (X_t) and identify its equilibrium distribution. Is the jump chain reversible? Justify your answer.

27I Principles of Statistics

(i) State Wilks' likelihood ratio test of the null hypothesis $H_0 : \theta \in \Theta_0$ against the alternative $H_1 : \theta \in \Theta_1$, where $\Theta_0 \subset \Theta_1$. Explain when this test may be used.

(ii) Independent identically-distributed observations $X_1, \ldots, X_n$ take values in the set $S = \{1, \ldots, K\}$, with common distribution which under the null hypothesis is of the form

$$P(X_1 = k|\theta) = f(k|\theta) \qquad (k \in S)$$

for some $\theta \in \Theta_0$, where Θ_0 is an open subset of some Euclidean space $\mathbb{R}^d$, $d < K - 1$. Under the alternative hypothesis, the probability mass function of the X_i is unrestricted.

Assuming sufficient regularity conditions on f to guarantee the existence and uniqueness of a maximum-likelihood estimator $\hat{\theta}_n(X_1, \ldots, X_n)$ of θ for each n, show that for large n the Wilks' likelihood ratio test statistic is approximately of the form

$$\sum_{j=1}^{K} (N_j - n\hat{\pi}_j)^2 / N_j,$$

where $N_j = \sum_{i=1}^{n} I_{\{X_i = j\}}$, and $\hat{\pi}_j = f(j|\hat{\theta}_n)$. What is the asymptotic distribution of this statistic?

28J Stochastic Financial Models

In the context of a single-period financial market with n traded assets, what is an arbitrage? What is an equivalent martingale measure?

Fix $\epsilon \in (0, 1)$ and consider the following single-period market with 3 assets:

Asset 1 is a riskless bond and pays no interest.

Asset 2 is a stock with initial price £1 per share; its possible final prices are $u = 1+\epsilon$ with probability $3/5$ and $d = 1 - \epsilon$ with probability $2/5$.

Asset 3 is another stock that behaves like an independent copy of asset 2.

Find all equivalent martingale measures for the problem and characterise all contingent claims that can be replicated.

Consider a contingent claim Y that pays 1 if both risky assets move in the same direction and zero otherwise. Show that the lower arbitrage bound, simply obtained by calculating all possible prices as the pricing measure ranges over all equivalent martingale measures, is zero. Why might someone pay for such a contract?

29I Optimization and Control

State Pontryagin's maximum principle in the case where both the terminal time and the terminal state are given.

Show that π is the minimum value taken by the integral

$$\tfrac{1}{2}\int_0^1 (u_t^2 + v_t^2)\,dt$$

subject to the constraints $x_0 = y_0 = z_0 = x_1 = y_1 = 0$ and $z_1 = 1$, where

$$\dot{x}_t = u_t, \quad \dot{y}_t = v_t, \quad \dot{z}_t = u_t y_t - v_t x_t, \quad 0 \leqslant t \leqslant 1.$$

[*You may find it useful to note the fact that the problem is rotationally symmetric about the z-axis, so that the angle made by the initial velocity $(\dot{x}_0, \dot{y}_0)$ with the positive x-axis may be chosen arbitrarily.*]

30A Partial Differential Equations

Define (i) the Fourier transform of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^3)$, and (ii) the convolution $T * g$ of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^3)$ and a Schwartz function $g \in \mathcal{S}(\mathbb{R}^3)$. Give a formula for the Fourier transform of $T * g$ ("convolution theorem").

Let $t > 0$. Compute the Fourier transform of the tempered distribution $A_t \in \mathcal{S}'(\mathbb{R}^3)$ defined by

$$\langle A_t, \phi \rangle = \int_{\|y\|=t} \phi(y) d\Sigma(y), \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R}^3),$$

and deduce the Kirchhoff formula for the solution $u(t,x)$ of

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0,$$

$$u(0,x) = 0, \qquad \frac{\partial u}{\partial t}(0,x) = g(x), \quad g \in \mathcal{S}(\mathbb{R}^3)\ .$$

Prove, by consideration of the quantities $e = \frac{1}{2}(u_t^2 + |\nabla u|^2)$ and $p = -u_t \nabla u$, that any C^2 solution is also given by the Kirchhoff formula (uniqueness).

Prove a corresponding uniqueness statement for the initial value problem

$$\frac{\partial^2 w}{\partial t^2} - \Delta w + V(x) w = 0,$$

$$w(0,x) = 0, \qquad \frac{\partial w}{\partial t}(0,x) = g(x), \quad g \in \mathcal{S}(\mathbb{R}^3)$$

where V is a smooth positive real-valued function of $x \in \mathbb{R}^3$ only.

31E Integrable Systems

Solve the following linear singular equation

$$\left(t+t^{-1}\right)\phi(t)+\frac{\left(t-t^{-1}\right)}{\pi i}\fint_C \frac{\phi(\tau)}{\tau-t}\,d\tau-\frac{\left(t+t^{-1}\right)}{2\pi i}\oint_C\left(\tau+2\tau^{-1}\right)\phi(\tau)\,d\tau = 2t^{-1},$$

where C denotes the unit circle, $t \in C$ and $\fint_C$ denotes the principal value integral.

32D Principles of Quantum Mechanics

Let $|s\,m\rangle$ denote the combined spin eigenstates for a system of two particles, each with spin 1. Derive expressions for all states with $m = s$ in terms of product states.

Given that the particles are identical, and that the spatial wavefunction describing their relative position has definite orbital angular momentum ℓ, show that $\ell + s$ must be even. Suppose that this two-particle state is known to arise from the decay of a single particle, X, also of spin 1. Assuming that total angular momentum and parity are conserved in this process, find the values of ℓ and s that are allowed, depending on whether the intrinsic parity of X is even or odd.

[*You may set* $\hbar = 1$ *and use* $J_\pm|\,j\,m\,\rangle = \sqrt{(j\mp m)(j\pm m+1)}\;|\,j\;m{\pm}1\,\rangle$.]

33A Applications of Quantum Mechanics

Describe the variational method for estimating the ground state energy of a quantum system. Prove that an error of order ϵ in the wavefunction leads to an error of order ϵ^2 in the energy.

Explain how the variational method can be generalized to give an estimate of the energy of the first excited state of a quantum system.

Using the variational method, estimate the energy of the first excited state of the anharmonic oscillator with Hamiltonian

$$H = -\frac{d^2}{dx^2} + x^2 + x^4 \,.$$

How might you improve your estimate?

[*Hint: If* $I_{2n} = \int_{-\infty}^{\infty} x^{2n}\, e^{-ax^2} dx$ *then*

$$I_0 = \sqrt{\frac{\pi}{a}}, \quad I_2 = \sqrt{\frac{\pi}{a}}\,\frac{1}{2a}, \quad I_4 = \sqrt{\frac{\pi}{a}}\,\frac{3}{4a^2}, \quad I_6 = \sqrt{\frac{\pi}{a}}\,\frac{15}{8a^3}. \;\Big]$$

34D Statistical Physics

Derive the Maxwell relation

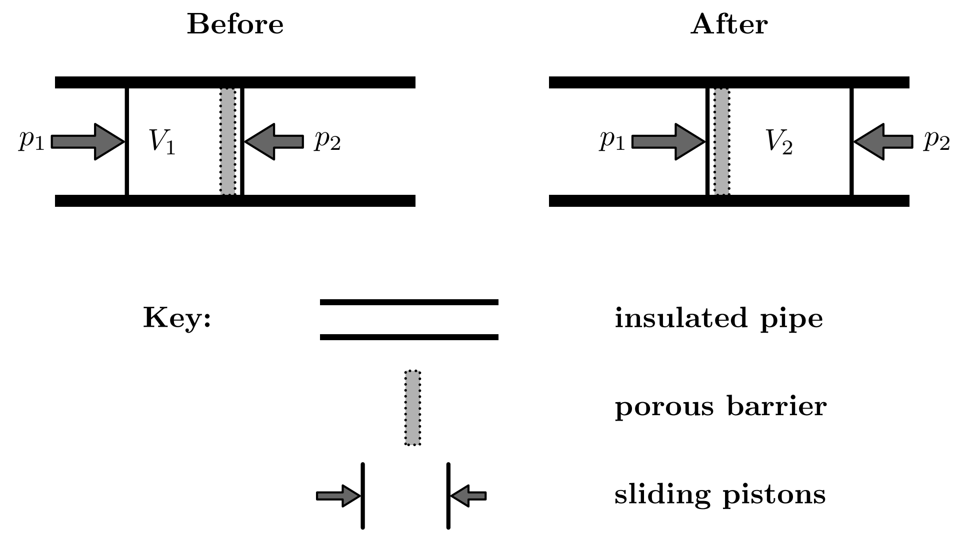
$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p .$$

The diagram below illustrates the Joule–Thomson throttling process for a porous barrier. A gas of volume V_1, initially on the left-hand side of a thermally insulated pipe, is forced by a piston to go through the barrier using constant pressure p_1. As a result the gas flows to the right-hand side, resisted by a piston which applies a constant pressure p_2 (with $p_2 < p_1$). Eventually all of the gas occupies a volume V_2 on the right-hand side. Show that this process conserves enthalpy.

The Joule–Thomson coefficient μ_{JT} is the change in temperature with respect to a change in pressure during a process that conserves enthalpy H. Express the Joule–Thomson coefficient, $\mu_{\mathrm{JT}} \equiv \left(\frac{\partial T}{\partial p}\right)_H$, in terms of T, V, the heat capacity at constant pressure C_p, and the volume coefficient of expansion $\alpha \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$.

What is μ_{JT} for an ideal gas?

If one wishes to use the Joule–Thomson process to cool a real (non-ideal) gas, what must the sign of μ_{JT} be?



35A General Relativity

The symbol ∇_a denotes the covariant derivative defined by the Christoffel connection $\Gamma^a{}_{bc}$ for a metric g_{ab}. Explain briefly why

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\phi = 0,$$
$$(\nabla_a \nabla_b - \nabla_b \nabla_a)v_c \neq 0,$$

in general, where ϕ is a scalar field and v_c is a covariant vector field.

A Killing vector field v_a satisfies the equation

$$S_{ab} \;\equiv\; \nabla_a v_b + \nabla_b v_a \;=\; 0\,.$$

By considering the quantity $\nabla_a S_{bc} + \nabla_b S_{ac} - \nabla_c S_{ab}$, show that

$$\nabla_a \nabla_b v_c \;=\; -R^d{}_{abc} v_d\,.$$

Find all Killing vector fields v_a in the case of flat Minkowski space-time.

For a metric of the form

$$\mathrm{d}s^2 \;=\; -f(\mathbf{x})\,\mathrm{d}t^2 \,+\, g_{ij}(\mathbf{x})\,\mathrm{d}x^i \mathrm{d}x^j\,, \qquad i,j = 1,2,3\,,$$

where $\mathbf{x}$ denotes the coordinates x^i, show that $\Gamma^0{}_{00} = \Gamma^0{}_{ij} = 0$ and that $\Gamma^0{}_{0i} = \Gamma^0{}_{i0} = \frac{1}{2}\,(\partial_i f)\,/f$. Deduce that the vector field $v^a = (1,0,0,0)$ is a Killing vector field.

[*You may assume the standard symmetries of the Riemann tensor.*]

36B Fluid Dynamics II

Viscous fluid is extracted through a small hole in the tip of the cone given by $\theta = \alpha$ in spherical polar coordinates (R, θ, ϕ). The total volume flux through the hole takes the constant value Q. It is given that there is a steady solution of the Navier–Stokes equations for the fluid velocity $\mathbf{u}$. For small enough R, the velocity $\mathbf{u}$ is well approximated by $\mathbf{u} \sim (-A/R^2, 0, 0)$, where $A = Q/[2\pi(1 - \cos\alpha)]$ except in thin boundary layers near $\theta = \alpha$.

(i) Verify that the volume flux through the hole is approximately Q.

(ii) Construct a Reynolds number (depending on R) in terms of Q and the kinematic viscosity ν, and thus give an estimate of the value of R below which solutions of this type will appear.

(iii) Assuming that there is a boundary layer near $\theta = \alpha$, write down the boundary layer equations in the usual form, using local Cartesian coordinates x and y parallel and perpendicular to the boundary. Show that the boundary layer thickness $\delta(x)$ is proportional to $x^{\frac{3}{2}}$, and show that the x component of the velocity u_x may be written in the form

$$u_x = -\frac{A}{x^2}F'(\eta), \quad \text{where} \quad \eta = \frac{y}{\delta(x)}.$$

Derive the equation and boundary conditions satisfied by F. Give an expression, in terms of F, for the volume flux through the boundary layer, and use this to derive the R-dependence of the first correction to the flow outside the boundary layer.

37C Waves

Show that for a one-dimensional flow of a perfect gas at constant entropy the Riemann invariants $u \pm 2(c-c_0)/(\gamma-1)$ are constant along characteristics $dx/dt = u \pm c$.

Define a simple wave. Show that in a right-propagating simple wave

$$\frac{\partial u}{\partial t} + \left(c_0 + \frac{\gamma+1}{2}u\right)\frac{\partial u}{\partial x} = 0 \, .$$

Now suppose instead that, owing to dissipative effects,

$$\frac{\partial u}{\partial t} + \left(c_0 + \frac{\gamma+1}{2}u\right)\frac{\partial u}{\partial x} = -\alpha u$$

where α is a positive constant. Suppose also that u is prescribed at $t=0$ for all x, say $u(x,0) = v(x)$. Demonstrate that, unless a shock forms,

$$u(x,t) = v(x_0)\, e^{-\alpha t}$$

where, for each x and t, x_0 is determined implicitly as the solution of the equation

$$x - c_0 t = x_0 + \frac{\gamma+1}{2}\left(\frac{1-e^{-\alpha t}}{\alpha}\right) v(x_0) \, .$$

Deduce that a shock will not form at any (x,t) if

$$\alpha > \frac{\gamma+1}{2} \max_{v'<0} |v'(x_0)| \, .$$

38C Numerical Analysis

(a) State the Householder–John theorem and explain how it can be used to design iterative methods for solving a system of linear equations $Ax = b$.

(b) Let $A = L+D+U$ where D is the diagonal part of A, and L and U are, respectively, the strictly lower and strictly upper triangular parts of A. Given a vector b, consider the following iterative scheme:

$$(D+\omega L)x^{(k+1)} = (1-\omega)Dx^{(k)} - \omega U x^{(k)} + \omega b \, .$$

Prove that if A is a symmetric positive definite matrix, and $\omega \in (0,2)$, then the above iteration converges to the solution of the system $Ax = b$.

END OF PAPER