UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 7 June 2007 9 to 12

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, ..., H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet <u>must</u> bear your examination number and desk number.

STATIONERY REQUIREMENTS	***SPECIAL REQUIREMENTS***
Gold cover sheet	*None*
Green master cover sheet	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Groups, Rings and Modules

What are the orders of the groups $GL_2(\mathbb{F}_p)$ and $SL_2(\mathbb{F}_p)$ where $\mathbb{F}_p$ is the field of p elements?

2A Geometry

Let l be a line in the Euclidean plane $\mathbf{R}^2$ and P a point on l. Denote by ρ the reflection in l and by τ the rotation through an angle α about P. Describe, in terms of l, P, and α, a line fixed by the composition $\tau\rho$ and show that $\tau\rho$ is a reflection.

3H Analysis II

Define uniform continuity for a real-valued function on an interval in the real line. Is a uniformly continuous function on the real line necessarily bounded?

Which of the following functions are uniformly continuous on the real line?

(i) $f(x) = x \sin x$,

(ii) $f(x) = e^{-x^4}$.

Justify your answers.

4A Metric and Topological Spaces

(a) Let X be a connected topological space such that each point x of X has a neighbourhood homeomorphic to $\mathbb{R}^n$. Prove that X is path-connected.

(b) Let τ denote the topology on $\mathbb{N} = \{1, 2, \ldots\}$, such that the open sets are $\mathbb{N}$, the empty set, and all the sets $\{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$. Prove that any continuous map from the topological space $(\mathbb{N}, \tau)$ to the Euclidean $\mathbb{R}$ is constant.

5F Complex Methods

Show that the function $\phi(x, y) = \tan^{-1} \dfrac{y}{x}$ is harmonic. Find its harmonic conjugate $\psi(x, y)$ and the analytic function $f(z)$ whose real part is $\phi(x, y)$. Sketch the curves $\phi(x, y) = C$ and $\psi(x, y) = K$.

6E Methods

Describe the method of Lagrange multipliers for finding extrema of a function $f(x,y,z)$ subject to the constraint that $g(x,y,z)=c$.

Illustrate the method by finding the maximum and minimum values of xy for points (x,y,z) lying on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\,,$$

with a, b and c all positive.

7B Quantum Mechanics

The quantum mechanical harmonic oscillator has Hamiltonian

$$H=\frac{1}{2m}\,p^2+\frac{1}{2}\,m\omega^2x^2\,,$$

and is in a stationary state of energy $< H >= E$. Show that

$$E \geqslant \frac{1}{2m}\,(\Delta p)^2+\frac{1}{2}\,m\omega^2(\Delta x)^2\,,$$

where $(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2$ and $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$. Use the Heisenberg Uncertainty Principle to show that

$$E \geqslant \frac{1}{2}\hbar\omega\,.$$

8C Statistics

Light bulbs are sold in packets of 3 but some of the bulbs are defective. A sample of 256 packets yields the following figures for the number of defectives in a packet:

No. of defectives	0	1	2	3
No. of packets	116	94	40	6

Test the hypothesis that each bulb has a constant (but unknown) probability θ of being defective independently of all other bulbs.

[*Hint: You may wish to use some of the following percentage points:*

Distribution	χ_1^2	χ_2^2	χ_3^2	χ_4^2	t_1	t_2	t_3	t_4
90% *percentile*	2·71	4·61	6·25	7·78	3·08	1·89	1·64	1·53
95% *percentile*	3·84	5·99	7·81	9·49	6·31	2·92	2·35	2·13

]

9C Markov Chains

Consider a Markov chain $(X_n)_{n \geqslant 0}$ with state space $S = \{0, 1\}$ and transition matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix},$$

where $0 < \alpha < 1$ and $0 < \beta < 1$.

Calculate $\mathbb{P}(X_n = 0 \mid X_0 = 0)$ for each $n \geqslant 0$.

SECTION II

10G Linear Algebra

(i) Define the terms *row-rank*, *column-rank* and *rank* of a matrix, and state a relation between them.

(ii) Fix positive integers m, n, p with $m, n \geqslant p$. Suppose that A is an $m \times p$ matrix and B a $p \times n$ matrix. State and prove the best possible upper bound on the rank of the product AB.

11G Groups, Rings and Modules

(i) State the Sylow theorems for Sylow p-subgroups of a finite group.

(ii) Write down one Sylow 3-subgroup of the symmetric group S_5 on 5 letters. Calculate the number of Sylow 3-subgroups of S_5.

12A Geometry

For a parameterized smooth embedded surface $\sigma : V \to U \subset \mathbf{R}^3$, where V is an open domain in $\mathbf{R}^2$, define the *first fundamental form*, the *second fundamental form*, and the *Gaussian curvature* K. State the Gauss–Bonnet formula for a compact embedded surface $S \subset \mathbf{R}^3$ having Euler number $e(S)$.

Let S denote a surface defined by rotating a curve

$$\eta(u) = (r + a\sin u, 0, b\cos u) \qquad 0 \leq u \leq 2\pi\,,$$

about the z-axis. Here a, b, r are positive constants, such that $a^2 + b^2 = 1$ and $a < r$. By considering a smooth parameterization, find the first fundamental form and the second fundamental form of S.

13H Analysis II

Let V be the real vector space of continuous functions $f : [0,1] \to \mathbb{R}$. Show that defining

$$||f|| = \int_0^1 |f(x)| dx$$

makes V a normed vector space.

Define $f_n(x) = \sin nx$ for positive integers n. Is the sequence (f_n) convergent to some element of V? Is (f_n) a Cauchy sequence in V? Justify your answers.

14H Complex Analysis

Say that a function on the complex plane $\mathbb{C}$ is *periodic* if $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all z. If f is a periodic analytic function, show that f is constant.

If f is a meromorphic periodic function, show that the number of zeros of f in the square $[0,1) \times [0,1)$ is equal to the number of poles, both counted with multiplicities.

Define

$$f(z) = \frac{1}{z^2} + \sum_w \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right],$$

where the sum runs over all $w = a + bi$ with a and b integers, not both 0. Show that this series converges to a meromorphic periodic function on the complex plane.

15E Methods

Legendre's equation may be written

$$\left(1 - x^2\right) y'' - 2xy' + n(n+1)y = 0 \quad \text{with} \quad y(1) = 1\,.$$

Show that if n is a positive integer, this equation has a solution $y = P_n(x)$ that is a polynomial of degree n. Find P_0, P_1 and P_2 explicitly.

Write down a general separable solution of Laplace's equation, $\nabla^2 \phi = 0$, in spherical polar coordinates (r, θ). (A derivation of this result is *not* required.)

Hence or otherwise find ϕ when

$$\nabla^2 \phi = 0, \quad a < r < b\,,$$

with $\phi = \sin^2 \theta$ both when $r = a$ and when $r = b$.

16B Quantum Mechanics

A quantum system has a complete set of orthonormal eigenstates, $\psi_n(x)$, with non-degenerate energy eigenvalues, E_n, where $n = 1, 2, 3 \ldots$. Write down the wave-function, $\Psi(x,t), t \geqslant 0$ in terms of the eigenstates.

A linear operator acts on the system such that

$$\begin{aligned} A\psi_1 &= 2\psi_1 - \psi_2 \\ A\psi_2 &= 2\psi_2 - \psi_1 \ . \\ A\psi_n &= 0, \ n \geqslant 3 \end{aligned}$$

Find the eigenvalues of A and obtain a complete set of normalised eigenfunctions, ϕ_n, of A in terms of the ψ_n.

At time $t = 0$ a measurement is made and it is found that the observable corresponding to A has value 3. After time t, A is measured again. What is the probability that the value is found to be 1?

17E Electromagnetism

A capacitor consists of three long concentric cylinders of radii a, λa and $2a$ respectively, where $1 < \lambda < 2$. The inner and outer cylinders are earthed (i.e. held at zero potential); the cylinder of radius λa is raised to a potential V. Find the electrostatic potential in the regions between the cylinders and deduce the capacitance, $C(\lambda)$ per unit length, of the system.

For $\lambda = 1 + \delta$ with $0 < \delta \ll 1$ find $C(\lambda)$ correct to leading order in δ and comment on your result.

Find also the value of λ at which $C(\lambda)$ has an extremum. Is the extremum a maximum or a minimum? Justify your answer.

18D Fluid Dynamics

Given that the circulation round every closed material curve in an inviscid, incompressible fluid remains constant in time, show that the velocity field of such a fluid started from rest can be written as the gradient of a potential, ϕ, that satisfies Laplace's equation.

A rigid sphere of radius a moves in a straight line at speed U in a fluid that is at rest at infinity. Using axisymmetric spherical polar coordinates (r, θ), with $\theta = 0$ in the direction of motion, write down the boundary conditions on ϕ and, by looking for a solution of the form $\phi = f(r)\cos\theta$, show that the velocity potential is given by

$$\phi = \frac{-Ua^3\cos\theta}{2r^2}\,.$$

Calculate the kinetic energy of the fluid.

A rigid sphere of radius a and uniform density ρ_b is submerged in an infinite fluid of density ρ, under the action of gravity. Show that, when the sphere is released from rest, its initial upwards acceleration is

$$\frac{2(\rho-\rho_b)g}{\rho+2\rho_b}\,.$$

[*Laplace's equation for an axisymmetric scalar field in spherical polars is:*

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\phi}{\partial\theta}\right)=0\,.]$$

19F Numerical Analysis

Prove that the monic polynomials Q_n, $n \geq 0$, orthogonal with respect to a given weight function $w(x) > 0$ on $[a, b]$, satisfy the three-term recurrence relation

$$Q_{n+1}(x) = (x-a_n)Q_n(x) - b_nQ_{n-1}(x), \quad n \geq 0\,.$$

where $Q_{-1}(x) \equiv 0$, $Q_0(x) \equiv 1$. Express the values a_n and b_n in terms of Q_n and Q_{n-1} and show that $b_n > 0$.

20C Optimization

State and prove the Lagrangian sufficiency theorem.

Solve the problem

$$\begin{array}{ll} \text{maximize} & x_1 + 3\ln(1+x_2) \\ \text{subject to} & 2x_1 + 3x_2 \leqslant c_1, \\ & \ln(1+x_1) \geqslant c_2, \quad x_1 \geqslant 0,\ x_2 \geqslant 0, \end{array}$$

where c_1 and c_2 are non-negative constants satisfying $c_1 + 2 \geqslant 2e^{c_2}$.

END OF PAPER