UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 6 June 2007 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

*Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most **four** questions from Section I and at most **six** questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in separate bundles labelled **A**, **B**, ..., **H** according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

Suppose that S, T are endomorphisms of the 3-dimensional complex vector space $\mathbb{C}^3$ and that the eigenvalues of each of them are 1, 2, 3. What are their characteristic and minimal polynomials? Are they conjugate?

2G Groups, Rings and Modules

Define the term *Euclidean domain*.

Show that the ring of integers $\mathbb{Z}$ is a Euclidean domain.

3H Analysis II

For integers a and b, define $d(a,b)$ to be 0 if $a=b$, or $\frac{1}{2^n}$ if $a \neq b$ and n is the largest non-negative integer such that $a-b$ is a multiple of 2^n. Show that d is a metric on the integers $\mathbb{Z}$.

Does the sequence $x_n = 2^n - 1$ converge in this metric?

4A Metric and Topological Spaces

Are the following statements true or false? Give a proof or a counterexample as appropriate.

(i) If $f : X \to Y$ is a continuous map of topological spaces and $S \subseteq X$ is compact then $f(S)$ is compact.

(ii) If $f : X \to Y$ is a continuous map of topological spaces and $K \subseteq Y$ is compact then $f^{-1}(K) = \{x \in X : f(x) \in K\}\}$ is compact.

(iii) If a metric space M is complete and a metric space T is homeomorphic to M then T is complete.

5D Methods

Show that a smooth function $y(x)$ that satisfies $y(0) = y'(1) = 0$ can be written as a Fourier series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \sin \lambda_n x, \quad 0 \leqslant x \leqslant 1\,,$$

where the λ_n should be specified. Write down an integral expression for a_n.

Hence solve the following differential equation

$$y'' - \alpha^2 y = x \cos \pi x\,,$$

with boundary conditions $y(0) = y'(1) = 0$, in the form of an infinite series.

6E Electromagnetism

A metal has uniform conductivity σ. A cylindrical wire with radius a and length l is manufactured from the metal. Show, using Maxwell's equations, that when a steady current I flows along the wire the current density within the wire is uniform.

Deduce the electrical resistance of the wire and the rate of Ohmic dissipation within it.

Indicate briefly, and without detailed calculation, whether your results would be affected if the wire was not straight.

7B Special Relativity

A particle in inertial frame S has coordinates (t, x), whilst the coordinates are (t', x') in frame S', which moves with relative velocity v in the x direction. What is the relationship between the coordinates of S and S'?

Frame S'', with cooordinates (t'', x''), moves with velocity u with respect to S' and velocity V with respect to S. Derive the relativistic formula for V in terms of u and v. Show how the Newtonian limit is recovered.

8D Fluid Dynamics

An incompressible, inviscid fluid occupies the region beneath the free surface $y = \eta(x,t)$ and moves with a velocity field given by the velocity potential $\phi(x,y,t)$; gravity acts in the $-y$ direction. Derive the kinematic and dynamic boundary conditions that must be satisfied by ϕ on $y = \eta(x,t)$.

[*You may assume Bernoulli's integral of the equation of motion:*

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + gy \;=\; F(t)\,.]$$

In the absence of waves, the fluid has uniform velocity U in the x direction. Derive the linearised form of the above boundary conditions for small amplitude waves, and verify that they and Laplace's equation are satisfied by the velocity potential

$$\phi \;=\; Ux + Re\{be^{ky}e^{i(kx-\omega t)}\}\,,$$

where $|kb| \ll U$, with a corresponding expression for η, as long as

$$(\omega - kU)^2 \;=\; gk\,.$$

What are the propagation speeds of waves with a given wave-number k?

9C Optimization

Consider the game with payoff matrix

$$\begin{pmatrix} 2 & 5 & 4 \\ 3 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix}\,,$$

where the (i,j) entry is the payoff to the row player if the row player chooses row i and the column player chooses column j.

Find the value of the game and the optimal strategies for each player.

SECTION II

10G Linear Algebra

Suppose that P is the complex vector space of complex polynomials in one variable, z.

(i) Show that the form $\langle\ ,\ \rangle$ defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\theta}\right)\cdot\overline{g\left(e^{i\theta}\right)}\,d\theta$$

is a positive definite Hermitian form on P.

(ii) Find an orthonormal basis of P for this form, in terms of the powers of z.

(iii) Generalize this construction to complex vector spaces of complex polynomials in any finite number of variables.

11G Groups, Rings and Modules

(i) Give an example of a Noetherian ring and of a ring that is not Noetherian. Justify your answers.

(ii) State and prove Hilbert's basis theorem.

12A Geometry

(i) The spherical circle with centre $P \in S^2$ and radius r, $0 < r < \pi$, is the set of all points on the unit sphere S^2 at spherical distance r from P. Find the circumference of a spherical circle with spherical radius r. Compare, for small r, with the formula for a Euclidean circle and comment on the result.

(ii) The cross ratio of four distinct points z_i in $\mathbf{C}$ is

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)}\,.$$

Show that the cross-ratio is a real number if and only if z_1, z_2, z_3, z_4 lie on a circle or a line.

[*You may assume that Möbius transformations preserve the cross-ratio.*]

13H Analysis II

Show that the limit of a uniformly convergent sequence of real valued continuous functions on $[0,1]$ is continuous on $[0,1]$.

Let f_n be a sequence of continuous functions on $[0,1]$ which converge point-wise to a continuous function. Suppose also that the integrals $\int_0^1 f_n(x)\,dx$ converge to $\int_0^1 f(x)\,dx$. Must the functions f_n converge uniformly to f? Prove or give a counterexample.

Let f_n be a sequence of continuous functions on $[0,1]$ which converge point-wise to a function f. Suppose that f is integrable and that the integrals $\int_0^1 f_n(x)\,dx$ converge to $\int_0^1 f(x)\,dx$. Is the limit f necessarily continuous? Prove or give a counterexample.

14F Complex Analysis or Complex Methods

Let Ω be the half-strip in the complex plane,

$$\Omega = \{z = x + iy \in \mathbb{C} \ : \ -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y > 0\}\,.$$

Find a conformal mapping that maps Ω onto the unit disc.

15D Methods

Let $y_0(x)$ be a non-zero solution of the Sturm-Liouville equation

$$L(y_0;\lambda_0) \equiv \frac{d}{dx}\left(p(x)\frac{dy_0}{dx}\right) + (q(x) + \lambda_0 w(x))\, y_0 = 0$$

with boundary conditions $y_0(0) = y_0(1) = 0$. Show that, if $y(x)$ and $f(x)$ are related by

$$L(y;\lambda_0) = f\,,$$

with $y(x)$ satisfying the same boundary conditions as $y_0(x)$, then

$$\int_0^1 y_0 f dx = 0\,. \qquad (*)$$

Suppose that y_0 is normalised so that

$$\int_0^1 w y_0^2 dx = 1\,,$$

and consider the problem

$$L(y;\lambda) = y^3\,; \quad y(0) = y(1) = 0\,.$$

By choosing f appropriately in $(*)$ deduce that, if

$$\lambda - \lambda_0 = \epsilon^2 \mu \;\; [\mu = O(1), \epsilon \ll 1\,]\,, \quad \text{and} \quad y(x) = \epsilon y_0(x) + \epsilon^2 y_1(x)$$

then

$$\mu = \int_0^1 y_0^4 dx + O(\epsilon)\,.$$

16B Quantum Mechanics

Write down the angular momentum operators L_1, L_2, L_3 in terms of the position and momentum operators, $\mathbf{x}$ and $\mathbf{p}$, and the commutation relations satisfied by $\mathbf{x}$ and $\mathbf{p}$.

Verify the commutation relations

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k \,.$$

Further, show that

$$[L_i, p_j] = i\hbar\epsilon_{ijk}p_k \,.$$

A wave-function $\Psi_0(r)$ is spherically symmetric. Verify that

$$\mathbf{L}\Psi_0(r) = 0 \,.$$

Consider the vector function $\mathbf{\Phi} = \nabla\Psi_0(r)$. Show that Φ_3 and $\Phi_1 \pm i\Phi_2$ are eigenfunctions of L_3 with eigenvalues $0, \pm\hbar$ respectively.

17E Electromagnetism

If S is a fixed surface enclosing a volume V, use Maxwell's equations to show that

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right) dV + \int_S \mathbf{P}\cdot\mathbf{n}\, dS = -\int_V \mathbf{j}\cdot\mathbf{E}\, dV \,,$$

where $\mathbf{P} = (\mathbf{E}\times\mathbf{B})/\mu_0$. Give a physical interpretation of each term in this equation.

Show that Maxwell's equations for a vacuum permit plane wave solutions with $\mathbf{E} = E_0(0,1,0)\cos(kx - \omega t)$ with E_0, k and ω constants, and determine the relationship between k and ω.

Find also the corresponding $\mathbf{B}(\mathbf{x}, t)$ and hence the time average $< \mathbf{P} >$. What does $< \mathbf{P} >$ represent in this case?

18F Numerical Analysis

For a symmetric, positive definite matrix A with the spectral radius $\rho(A)$, the linear system $Ax = b$ is solved by the iterative procedure

$$x^{(k+1)} = x^{(k)} - \tau(Ax^{(k)} - b), \qquad k \geq 0 \,,$$

where τ is a real parameter. Find the range of τ that guarantees convergence of $x^{(k)}$ to the exact solution for any choice of $x^{(0)}$.

19C Statistics

State and prove the Neyman–Pearson lemma.

Suppose that X is a random variable drawn from the probability density function

$$f(x \mid \theta) = \tfrac{1}{2}|\,x\,|^{\theta-1}e^{-|x|}/\Gamma(\theta), \quad -\infty < x < \infty\,,$$

where $\Gamma(\theta) = \int\limits_0^\infty y^{\theta-1}e^{-y}dy$ and $\theta \geqslant 1$ is unknown. Find the most powerful test of size α, $0 < \alpha < 1$, of the hypothesis $H_0 : \theta = 1$ against the alternative $H_1 : \theta = 2$. Express the power of the test as a function of α.

Is your test uniformly most powerful for testing $H_0 : \theta = 1$ against $H_1 : \theta > 1$? Explain your answer carefully.

20C Markov Chains

Consider a Markov chain with state space $S = \{0, 1, 2, \ldots\}$ and transition matrix given by

$$P_{i,j} = \begin{cases} qp^{j-i+1} & \text{for } i \geqslant 1 \text{ and } j \geqslant i-1, \\ qp^{j} & \text{for } i = 0 \text{ and } j \geqslant 0, \end{cases}$$

and $P_{i,j} = 0$ otherwise, where $0 < p = 1 - q < 1$.

For each value of p, $0 < p < 1$, determine whether the chain is transient, null recurrent or positive recurrent, and in the last case find the invariant distribution.

END OF PAPER