UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2006

List of Courses

1/I/1H Linear Algebra

Define what is meant by the *minimal polynomial* of a complex $n \times n$ matrix, and show that it is unique. Deduce that the minimal polynomial of a real $n \times n$ matrix has real coefficients.

For $n > 2$, find an $n \times n$ matrix with minimal polynomial $(t-1)^2(t+1)$.

1/II/9H Linear Algebra

Let U, V be finite-dimensional vector spaces, and let θ be a linear map of U into V. Define the *rank* $r(\theta)$ and the *nullity* $n(\theta)$ of θ, and prove that

$$r(\theta) + n(\theta) = \dim U.$$

Now let θ, ϕ be endomorphisms of a vector space U. Define the endomorphisms $\theta + \phi$ and $\theta\phi$, and prove that

$$\begin{aligned} r(\theta+\phi) &\leqslant r(\theta) + r(\phi) \\ n(\theta\phi) &\leqslant n(\theta) + n(\phi). \end{aligned}$$

Prove that equality holds in **both** inequalities if and only if $\theta + \phi$ is an isomorphism and $\theta\phi$ is zero.

2/I/1E Linear Algebra

State Sylvester's law of inertia.

Find the rank and signature of the quadratic form q on $\mathbf{R}^n$ given by

$$q(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2.$$

2/II/10E Linear Algebra

Suppose that V is the set of complex polynomials of degree at most n in the variable x. Find the dimension of V as a complex vector space.

Define

$$e_k : V \to \mathbf{C} \quad \text{by} \quad e_k(\phi) = \frac{d^k \phi}{dx^k}(0).$$

Find a subset of $\{e_k \,|\, k \in \mathbf{N}\}$ that is a basis of the dual vector space V^*. Find the corresponding dual basis of V.

Define

$$D : V \to V \quad \text{by} \quad D(\phi) = \frac{d\phi}{dx}.$$

Write down the matrix of D with respect to the basis of V that you have just found, and the matrix of the map dual to D with respect to the dual basis.

3/II/10H Linear Algebra

(a) Define what is meant by the *trace* of a complex $n \times n$ matrix A. If T denotes an $n \times n$ invertible matrix, show that A and TAT^{-1} have the same trace.

(b) If $\lambda_1, \ldots, \lambda_r$ are distinct non-zero complex numbers, show that the endomorphism of $\mathbf{C}^r$ defined by the matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & \ldots & \lambda_1^r \\ \vdots & \ldots & \vdots \\ \lambda_r & \ldots & \lambda_r^r \end{pmatrix}$$

has trivial kernel, and hence that the same is true for the transposed matrix Λ^t.

For arbitrary complex numbers $\lambda_1, \ldots, \lambda_n$, show that the vector $(1, \ldots, 1)^t$ is not in the kernel of the endomorphism of $\mathbf{C}^n$ defined by the matrix

$$\begin{pmatrix} \lambda_1 & \ldots & \lambda_n \\ \vdots & \ldots & \vdots \\ \lambda_1^n & \ldots & \lambda_n^n \end{pmatrix},$$

unless all the λ_i are zero.

[*Hint: reduce to the case when $\lambda_1, \ldots, \lambda_r$ are distinct non-zero complex numbers, with $r \leqslant n$, and each λ_j for $j > r$ is either zero or equal to some λ_i with $i \leqslant r$. If the kernel of the endomorphism contains $(1, \ldots, 1)^t$, show that it also contains a vector of the form $(m_1, \ldots, m_r, 0, \ldots, 0)^t$ with the m_i strictly positive integers.*]

(c) Assuming the fact that any complex $n \times n$ matrix is conjugate to an upper-triangular one, prove that if A is an $n \times n$ matrix such that A^k has zero trace for all $1 \leqslant k \leqslant n$, then $A^n = 0$.

4/I/1H **Linear Algebra**

Suppose V is a vector space over a field k. A finite set of vectors is said to be a *basis* for V if it is both linearly independent and spanning. Prove that any two finite bases for V have the same number of elements.

4/II/10E **Linear Algebra**

Suppose that α is an orthogonal endomorphism of the finite-dimensional real inner product space V. Suppose that V is decomposed as a direct sum of mutually orthogonal α-invariant subspaces. How small can these subspaces be made, and how does α act on them? Justify your answer.

Describe the possible matrices for α with respect to a suitably chosen orthonormal basis of V when $\dim V = 3$.

1/II/10E Groups, Rings and Modules

Find all subgroups of indices 2, 3, 4 and 5 in the alternating group A_5 on 5 letters. You may use any general result that you choose, provided that you state it clearly, but you must justify your answers.

[*You may take for granted the fact that A_4 has no subgroup of index 2.*]

2/I/2E Groups, Rings and Modules

(i) Give the definition of a Euclidean domain and, with justification, an example of a Euclidean domain that is not a field.

(ii) State the structure theorem for finitely generated modules over a Euclidean domain.

(iii) In terms of your answer to (ii), describe the structure of the $\mathbb{Z}$-module M with generators $\{m_1, m_2, m_3\}$ and relations $2m_3 = 2m_2$, $4m_2 = 0$.

2/II/11E Groups, Rings and Modules

(i) Prove the first Sylow theorem, that a finite group of order $p^n r$ with p prime and p not dividing the integer r has a subgroup of order p^n.

(ii) State the remaining Sylow theorems.

(iii) Show that if p and q are distinct primes then no group of order pq is simple.

3/I/1E Groups, Rings and Modules

(i) Give an example of an integral domain that is not a unique factorization domain.

(ii) For which integers n is $\mathbb{Z}/n\mathbb{Z}$ an integral domain?

3/II/11E Groups, Rings and Modules

Suppose that R is a ring. Prove that $R[X]$ is Noetherian if and only if R is Noetherian.

4/I/2E **Groups, Rings and Modules**

How many elements does the ring $\mathbb{Z}[X]/(3, X^2 + X + 1)$ have?

Is this ring an integral domain?

Briefly justify your answers.

4/II/11E **Groups, Rings and Modules**

(a) Suppose that R is a commutative ring, M an R-module generated by $m_1, \ldots, m_n$ and $\phi \in End_R(M)$. Show that, if $A = (a_{ij})$ is an $n \times n$ matrix with entries in R that represents ϕ with respect to this generating set, then in the sub-ring $R[\phi]$ of $End_R(M)$ we have $\det(a_{ij} - \phi\delta_{ij}) = 0$.

[*Hint: A is a matrix such that $\phi(m_i) = \sum a_{ij} m_j$ with $a_{ij} \in R$. Consider the matrix $C = (a_{ij} - \phi\delta_{ij})$ with entries in $R[\phi]$ and use the fact that for any $n \times n$ matrix N over any commutative ring, there is a matrix N' such that $N'N = (\det N)1_n$.*]

(b) Suppose that k is a field, V a finite-dimensional k-vector space and that $\phi \in End_k(V)$. Show that if A is the matrix of ϕ with respect to some basis of V then ϕ satisfies the characteristic equation $\det(A - \lambda 1) = 0$ of A.

1/I/2H **Geometry**

Define the hyperbolic metric in the upper half-plane model H of the hyperbolic plane. How does one define the hyperbolic area of a region in H? State the Gauss–Bonnet theorem for hyperbolic triangles.

Let R be the region in H defined by

$$0 < x < \frac{1}{2}, \quad \sqrt{1-x^2} < y < 1.$$

Calculate the hyperbolic area of R.

2/II/12H **Geometry**

Let $\sigma : V \to U \subset \mathbf{R}^3$ denote a parametrized smooth embedded surface, where V is an open ball in $\mathbf{R}^2$ with coordinates (u, v). Explain briefly the geometric meaning of the *second fundamental form*

$$L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

where $L = \sigma_{uu} \cdot \mathbf{N},\ M = \sigma_{uv} \cdot \mathbf{N},\ N = \sigma_{vv} \cdot \mathbf{N}$, with $\mathbf{N}$ denoting the unit normal vector to the surface U.

Prove that if the second fundamental form is identically zero, then $\mathbf{N}_u = \mathbf{0} = \mathbf{N}_v$ as vector-valued functions on V, and hence that $\mathbf{N}$ is a constant vector. Deduce that U is then contained in a plane given by $\mathbf{x} \cdot \mathbf{N} =$ constant.

3/I/2H **Geometry**

Show that the Gaussian curvature K at an arbitrary point (x, y, z) of the hyperboloid $z = xy$, as an embedded surface in $\mathbf{R}^3$, is given by the formula

$$K = -1/(1 + x^2 + y^2)^2.$$

3/II/12H **Geometry**

Describe the stereographic projection map from the sphere S^2 to the extended complex plane $\mathbf{C}_\infty$, positioned equatorially. Prove that $w, z \in \mathbf{C}_\infty$ correspond to antipodal points on S^2 if and only if $w = -1/\bar{z}$. State, without proof, a result which relates the rotations of S^2 to a certain group of Möbius transformations on $\mathbf{C}_\infty$.

Show that any circle in the complex plane corresponds, under stereographic projection, to a circle on S^2. Let C denote any circle in the complex plane other than the unit circle; show that C corresponds to a great circle on S^2 if and only if its intersection with the unit circle consists of two points, one of which is the negative of the other.

[*You may assume the result that a Möbius transformation on the complex plane sends circles and straight lines to circles and straight lines.*]

4/II/12H **Geometry**

Describe the hyperbolic lines in both the disc and upper half-plane models of the hyperbolic plane. Given a hyperbolic line l and a point $P \notin l$, we define

$$d(P, l) := \inf_{Q \in l} \rho(P, Q),$$

where ρ denotes the hyperbolic distance. Show that $d(P, l) = \rho(P, Q')$, where Q' is the unique point of l for which the hyperbolic line segment PQ' is perpendicular to l.

Suppose now that L_1 is the positive imaginary axis in the upper half-plane model of the hyperbolic plane, and L_2 is the semicircle with centre $a > 0$ on the real line, and radius r, where $0 < r < a$. For any $P \in L_2$, show that the hyperbolic line through P which is perpendicular to L_1 is a semicircle centred on the origin of radius $\leqslant a + r$, and prove that

$$d(P, L_1) \geqslant \frac{a - r}{a + r}.$$

For arbitrary hyperbolic lines L_1, L_2 in the hyperbolic plane, we define

$$d(L_1, L_2) := \inf_{P \in L_1, Q \in L_2} \rho(P, Q).$$

If L_1 and L_2 are *ultraparallel* (i.e. hyperbolic lines which do not meet, either inside the hyperbolic plane or at its boundary), prove that $d(L_1, L_2)$ is strictly positive.

[*The equivalence of the disc and upper half-plane models of the hyperbolic plane, and standard facts about the metric and isometries of these models, may be quoted without proof.*]

1/II/11F Analysis II

Let a_n and b_n be sequences of real numbers for $n \geqslant 1$ such that $|a_n| \leqslant c/n^{1+\epsilon}$ and $|b_n| \leqslant c/n^{1+\epsilon}$ for all $n \geqslant 1$, for some constants $c > 0$ and $\epsilon > 0$. Show that the series

$$f(x) = \sum_{n \geqslant 1} a_n \cos nx + \sum_{n \geqslant 1} b_n \sin nx$$

converges uniformly to a continuous function on the real line. Show that f is periodic in the sense that $f(x + 2\pi) = f(x)$.

Now suppose that $|a_n| \leqslant c/n^{2+\epsilon}$ and $|b_n| \leqslant c/n^{2+\epsilon}$ for all $n \geqslant 1$, for some constants $c > 0$ and $\epsilon > 0$. Show that f is differentiable on the real line, with derivative

$$f'(x) = -\sum_{n \geqslant 1} n a_n \sin nx + \sum_{n \geqslant 1} n b_n \cos nx.$$

[*You may assume the convergence of standard series.*]

2/I/3F Analysis II

Define uniform convergence for a sequence $f_1, f_2, \ldots$ of real-valued functions on an interval in $\mathbf{R}$. If (f_n) is a sequence of continuous functions converging uniformly to a (necessarily continuous) function f on a closed interval $[a, b]$, show that

$$\int_a^b f_n(x)\, dx \to \int_a^b f(x)\, dx$$

as $n \to \infty$.

Which of the following sequences of functions $f_1, f_2, \ldots$ converges uniformly on the open interval $(0, 1)$? Justify your answers.

(i) $f_n(x) = 1/(nx)$;

(ii) $f_n(x) = e^{-x/n}$.

2/II/13F **Analysis II**

For a smooth mapping $F : \mathbf{R}^2 \to \mathbf{R}^2$, the Jacobian $J(F)$ at a point (x, y) is defined as the determinant of the derivative DF, viewed as a linear map $\mathbf{R}^2 \to \mathbf{R}^2$. Suppose that F maps into a curve in the plane, in the sense that F is a composition of two smooth mappings $\mathbf{R}^2 \to \mathbf{R} \to \mathbf{R}^2$. Show that the Jacobian of F is identically zero.

Conversely, let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be a smooth mapping whose Jacobian is identically zero. Write $F(x, y) = (f(x, y), g(x, y))$. Suppose that $\partial f/\partial y|_{(0,0)} \neq 0$. Show that $\partial f/\partial y \neq 0$ on some open neighbourhood U of $(0, 0)$ and that on U

$$(\partial g/\partial x,\, \partial g/\partial y) = e(x, y)\,(\partial f/\partial x,\, \partial f/\partial y)$$

for some smooth function e defined on U. Now suppose that $c : \mathbf{R} \to U$ is a smooth curve of the form $t \mapsto (t, \alpha(t))$ such that $F \circ c$ is constant. Write down a differential equation satisfied by α. Apply an existence theorem for differential equations to show that there is a neighbourhood V of $(0, 0)$ such that every point in V lies on a curve $t \mapsto (t, \alpha(t))$ on which F is constant.

[*A function is said to be smooth when it is infinitely differentiable. Detailed justification of the smoothness of the functions in question is not expected.*]

3/I/3F **Analysis II**

Define what it means for a function $f : \mathbf{R}^2 \to \mathbf{R}$ to be *differentiable* at a point (a, b). If the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are defined and continuous on a neighbourhood of (a, b), show that f is differentiable at (a, b).

3/II/13F **Analysis II**

State precisely the inverse function theorem for a smooth map F from an open subset of $\mathbf{R}^2$ to $\mathbf{R}^2$.

Define $F : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$F(x, y) = (x^3 - x - y^2, y).$$

Determine the open subset of $\mathbf{R}^2$ on which F is locally invertible.

Let C be the curve $\{(x, y) \in \mathbf{R}^2 : x^3 - x - y^2 = 0\}$. Show that C is the union of the two subsets $C_1 = \{(x, y) \in C : x \in [-1, 0]\}$ and $C_2 = \{(x, y) \in C : x \geqslant 1\}$. Show that for each $y \in \mathbf{R}$ there is a unique $x = p(y)$ such that $(x, y) \in C_2$. Show that F is locally invertible at all points of C_2, and deduce that $p(y)$ is a smooth function of y.

[*A function is said to be smooth when it is infinitely differentiable.*]

4/I/3F Analysis II

Let V be the vector space of all sequences $(x_1, x_2, \ldots)$ of real numbers such that x_i converges to zero. Show that the function

$$|(x_1, x_2, \ldots)| = \max_{i \geqslant 1} |x_i|$$

defines a norm on V.

Is the sequence

$$(1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots), \ldots$$

convergent in V? Justify your answer.

4/II/13F Analysis II

State precisely the contraction mapping theorem.

An ancient way to approximate the square root of a positive number a is to start with a guess $x > 0$ and then hope that the average of x and a/x gives a better guess. We can then repeat the procedure using the new guess. Justify this procedure as follows. First, show that all the guesses after the first one are greater than or equal to $\sqrt{a}$. Then apply the properties of contraction mappings to the interval $[\sqrt{a}, \infty)$ to show that the procedure always converges to $\sqrt{a}$.

Once the above procedure is close enough to $\sqrt{a}$, estimate how many more steps of the procedure are needed to get one more decimal digit of accuracy in computing $\sqrt{a}$.

1/II/12F **Metric and Topological Spaces**

(i) Define the product topology on $X \times Y$ for topological spaces X and Y, proving that your definition does define a topology.

(ii) Let X be the logarithmic spiral defined in polar coordinates by $r = e^{\theta}$, where $-\infty < \theta < \infty$. Show that X (with the subspace topology from $\mathbf{R}^2$) is homeomorphic to the real line.

2/I/4F **Metric and Topological Spaces**

Which of the following subspaces of Euclidean space are connected? Justify your answers.

(i) $\{(x, y, z) \in \mathbf{R}^3 : z^2 - x^2 - y^2 = 1\}$;

(ii) $\{(x, y) \in \mathbf{R}^2 : x^2 = y^2\}$;

(iii) $\{(x, y, z) \in \mathbf{R}^3 : z = x^2 + y^2\}$.

3/I/4F **Metric and Topological Spaces**

Which of the following are topological spaces? Justify your answers.

(i) The set $X = \mathbf{Z}$ of the integers, with a subset A of X called "open" when A is either finite or the whole set X;

(ii) The set $X = \mathbf{Z}$ of the integers, with a subset A of X called "open" when, for each element $x \in A$ and every even integer n, $x + n$ is also in A.

4/II/14F **Metric and Topological Spaces**

(a) Show that every compact subset of a Hausdorff topological space is closed.

(b) Let X be a compact metric space. For F a closed subset of X and p any point of X, show that there is a point q in F with

$$d(p, q) = \inf_{q' \in F} d(p, q').$$

Suppose that for every x and y in X there is a point m in X with $d(x, m) = (1/2)d(x, y)$ and $d(y, m) = (1/2)d(x, y)$. Show that X is connected.

1/I/3D Complex Analysis or Complex Methods

Let L be the Laplace operator, i.e., $L(g) = g_{xx} + g_{yy}$. Prove that if $f : \Omega \to \mathbf{C}$ is analytic in a domain Ω, then

$$L\big(|f(z)|^2\big) = 4|f'(z)|^2\,, \quad z \in \Omega.$$

1/II/13D Complex Analysis or Complex Methods

By integrating round the contour involving the real axis and the line $\mathrm{Im}(z) = 2\pi$, or otherwise, evaluate

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx\,, \quad 0 < a < 1.$$

Explain why the given restriction on the value a is necessary.

2/II/14D Complex Analysis or Complex Methods

Let Ω be the region enclosed between the two circles C_1 and C_2, where

$$C_1 = \{z \in \mathbf{C} : |z - i| = 1\}\,, \qquad C_2 = \{z \in \mathbf{C} : |z - 2i| = 2\}.$$

Find a conformal mapping that maps Ω onto the unit disc.

[*Hint: you may find it helpful first to map Ω to a strip in the complex plane.*]

3/II/14H **Complex Analysis**

Assuming the principle of the argument, prove that any polynomial of degree n has precisely n zeros in $\mathbf{C}$, counted with multiplicity.

Consider a polynomial $p(z) = z^4 + az^3 + bz^2 + cz + d$, and let R be a positive real number such that $|a|R^3 + |b|R^2 + |c|R + |d| < R^4$. Define a curve $\Gamma : [0,1] \to \mathbf{C}$ by

$$\Gamma(t) = \begin{cases} p(Re^{\pi i t}) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} , \\ (2-2t)p(iR) + (2t-1)p(R) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 . \end{cases}$$

Show that the winding number $n(\Gamma, 0) = 1$.

Suppose now that $p(z)$ has real coefficients, that $z^4 - bz^2 + d$ has no real zeros, and that the real zeros of $p(z)$ are all strictly negative. Show that precisely one of the zeros of $p(z)$ lies in the quadrant $\{x + iy : x > 0, y > 0\}$.

[*Standard results about winding numbers may be quoted without proof; in particular, you may wish to use the fact that if* $\gamma_i : [0,1] \to \mathbf{C}$, $i = 1, 2$, *are two closed curves with* $|\gamma_2(t) - \gamma_1(t)| < |\gamma_1(t)|$ *for all* t, *then* $n(\gamma_1, 0) = n(\gamma_2, 0)$.]

4/I/4H **Complex Analysis**

State the principle of isolated zeros for an analytic function on a domain in $\mathbf{C}$.

Suppose f is an analytic function on $\mathbf{C} \setminus \{0\}$, which is real-valued at the points $1/n$, for $n = 1, 2, \ldots$, and does not have an essential singularity at the origin. Prove that $f(z) = \overline{f(\bar{z})}$ for all $z \in \mathbf{C} \setminus \{0\}$.

3/I/5D **Complex Methods**

The transformation

$$w = i\left(\frac{1-z}{1+z}\right)$$

maps conformally the interior of the unit disc D onto the upper half-plane H_+, and maps the upper and lower unit semicircles C_+ and C_- onto the positive and negative real axis $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively.

Consider the Dirichlet problem in the upper half-plane:

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0 \quad \text{in} \quad H_+\,; \qquad f(u,v) = \begin{cases} 1 & \text{on } \mathbb{R}_+, \\ 0 & \text{on } \mathbb{R}_-. \end{cases}$$

Its solution is given by the formula

$$f(u,v) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{u}{v}\right).$$

Using this result, determine the solution to the Dirichlet problem in the unit disc:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0 \quad \text{in} \quad D; \qquad F(x,y) = \begin{cases} 1 & \text{on } C_+, \\ 0 & \text{on } C_-. \end{cases}$$

Briefly explain your answer.

4/II/15D **Complex Methods**

Denote by $f * g$ the convolution of two functions, and by $\widehat{f}$ the Fourier transform, i.e.,

$$[f * g](x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt, \qquad \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,dx\,.$$

(a) Show that, for suitable functions f and g, the Fourier transform $\widehat{F}$ of the convolution $F = f * g$ is given by $\widehat{F} = \widehat{f}\cdot\widehat{g}$.

(b) Let

$$f_1(x) = \begin{cases} 1 & |x| \leqslant 1/2\,, \\ 0 & \text{otherwise.} \end{cases}$$

and let $f_2 = f_1 * f_1$ be the convolution of f_1 with itself. Find the Fourier transforms of f_1 and f_2, and, by applying Parseval's theorem, determine the value of the integral

$$\int_{-\infty}^{\infty}\left(\frac{\sin y}{y}\right)^4 dy\,.$$

1/II/14A **Methods**

Define a *second rank tensor*. Show from your definition that if M_{ij} is a second rank tensor then M_{ii} is a scalar.

A rigid body consists of a thin flat plate of material having density $\rho(\mathbf{x})$ per unit area, where $\mathbf{x}$ is the position vector. The body occupies a region D of the (x, y)-plane; its thickness in the z-direction is negligible. The moment of inertia tensor of the body is given as

$$M_{ij} = \int_D (x_k x_k \delta_{ij} - x_i x_j)\rho \, dS.$$

Show that the z-direction is an eigenvector of M_{ij} and write down an integral expression for the corresponding eigenvalue $M_\perp$.

Hence or otherwise show that if the remaining eigenvalues of M_{ij} are M_1 and M_2 then

$$M_\perp = M_1 + M_2.$$

Find M_{ij} for a circular disc of radius a and uniform density having its centre at the origin.

2/I/5A **Methods**

Describe briefly the method of Lagrange multipliers for finding the stationary values of a function $f(x, y)$ subject to a constraint $g(x, y) = 0$.

Use the method to find the smallest possible surface area (including both ends) of a circular cylinder that has volume V.

2/II/15G **Methods**

Verify that $y = e^{-x}$ is a solution of the differential equation

$$(x+2)y'' + (x+1)y' - y = 0,$$

and find a second solution of the form $ax + b$.

Let L be the operator

$$L[y] = y'' + \frac{(x+1)}{(x+2)}y' - \frac{1}{(x+2)}y$$

on functions $y(x)$ satisfying

$$y'(0) = y(0) \quad \text{and} \quad \lim_{x \to \infty} y(x) = 0.$$

The Green's function $G(x, \xi)$ for L satisfies

$$L[G] = \delta(x - \xi),$$

with $\xi > 0$. Show that

$$G(x, \xi) = -\frac{(\xi+1)}{(\xi+2)}e^{\xi - x}$$

for $x > \xi$, and find $G(x, \xi)$ for $x < \xi$.

Hence or otherwise find the solution of

$$L[y] = -(x+2)e^{-x},$$

for $x \geqslant 0$, with $y(x)$ satisfying the boundary conditions above.

3/I/6A Methods

If T_{ij} is a second rank tensor such that $b_i T_{ij} c_j = 0$ for every vector $\mathbf{b}$ and every vector $\mathbf{c}$, show that $T_{ij} = 0$.

Let S be a closed surface with outward normal $\mathbf{n}$ that encloses a three-dimensional region having volume V. The position vector is $\mathbf{x}$. Use the divergence theorem to find

$$\int_S (\mathbf{b}\cdot\mathbf{x})(\mathbf{c}\cdot\mathbf{n})\, dS$$

for constant vectors $\mathbf{b}$ and $\mathbf{c}$. Hence find

$$\int_S x_i n_j \, dS,$$

and deduce the values of

$$\int_S \mathbf{x}\cdot\mathbf{n}\, dS \quad \text{and} \quad \int_S \mathbf{x}\times\mathbf{n}\, dS.$$

3/II/15G Methods

(a) Find the Fourier sine series of the function

$$f(x) = x$$

for $0 \leqslant x \leqslant 1$.

(b) The differential operator L acting on y is given by

$$L[y] = y'' + y'.$$

Show that the eigenvalues λ in the eigenvalue problem

$$L[y] = \lambda y, \quad y(0) = y(1) = 0,$$

are given by $\lambda = -n^2\pi^2 - \frac{1}{4}, \quad n = 1, 2, \ldots$, and find the corresponding eigenfunctions $y_n(x)$.

By expressing the equation $L[y] = \lambda y$ in Sturm-Liouville form or otherwise, write down the orthogonality relation for the y_n. Assuming the completeness of the eigenfunctions and using the result of part (a), find, in the form of a series, a function $y(x)$ which satisfies

$$L[y] = xe^{-x/2}$$

and $y(0) = y(1) = 0$.

4/I/5G **Methods**

A finite-valued function $f(r, \theta, \phi)$, where r, θ, ϕ are spherical polar coordinates, satisfies Laplace's equation in the regions $r < 1$ and $r > 1$, and $f \to 0$ as $r \to \infty$. At $r = 1$, f is continuous and its derivative with respect to r is discontinuous by $A \sin^2 \theta$, where A is a constant. Write down the general axisymmetric solution for f in the two regions and use the boundary conditions to find f.

$$\left[\mathit{Hint:} \quad P_2(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right) . \right]$$

4/II/16B **Methods**

The integral

$$I = \int_a^b F(y(x), y'(x)) dx \, ,$$

where F is some functional, is defined for the class of functions $y(x)$ for which $y(a) = y_0$, with the value $y(b)$ at the upper endpoint unconstrained. Suppose that $y(x)$ extremises the integral among the functions in this class. By considering perturbed paths of the form $y(x) + \epsilon\eta(x)$, with $\epsilon \ll 1$, show that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

and that

$$\left.\frac{\partial F}{\partial y'}\right|_{x=b} = 0 \, .$$

Show further that

$$F - y'\frac{\partial F}{\partial y'} = k$$

for some constant k.

A bead slides along a frictionless wire under gravity. The wire lies in a vertical plane with coordinates (x, y) and connects the point A with coordinates $(0, 0)$ to the point B with coordinates $(x_0, y(x_0))$, where x_0 is given and $y(x_0)$ can take any value less than zero. The bead is released from rest at A and slides to B in a time T. For a prescribed x_0 find both the shape of the wire, and the value of $y(x_0)$, for which T is as small as possible.

1/II/15B Quantum Mechanics

Let $V_1(x)$ and $V_2(x)$ be two real potential functions of one space dimension, and let a be a positive constant. Suppose also that $V_1(x) \leqslant V_2(x) \leqslant 0$ for all x and that $V_1(x) = V_2(x) = 0$ for all x such that $|x| \geqslant a$. Consider an incoming beam of particles described by the plane wave $\exp(ikx)$, for some $k > 0$, scattering off one of the potentials $V_1(x)$ or $V_2(x)$. Let p_i be the probability that a particle in the beam is reflected by the potential $V_i(x)$. Is it necessarily the case that $p_1 \leqslant p_2$? Justify your answer carefully, either by giving a rigorous proof or by presenting a counterexample with explicit calculations of p_1 and p_2.

2/II/16B Quantum Mechanics

The spherically symmetric bound state wavefunctions $\psi(r)$, where $r = |\mathbf{x}|$, for an electron orbiting in the Coulomb potential $V(r) = -e^2/(4\pi\epsilon_0 r)$ of a hydrogen atom nucleus, can be modelled as solutions to the equation

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} + \frac{a}{r}\psi(r) - b^2\psi(r) = 0$$

for $r \geqslant 0$, where $a = e^2 m/(2\pi\epsilon_0\hbar^2)$, $b = \sqrt{-2mE}/\hbar$, and E is the energy of the corresponding state. Show that there are normalisable and continuous wavefunctions $\psi(r)$ satisfying this equation with energies

$$E = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2 N^2}$$

for all integers $N \geqslant 1$.

3/I/7B Quantum Mechanics

Define the quantum mechanical operators for the angular momentum $\hat{\mathbf{L}}$ and the total angular momentum $\hat{L}^2$ in terms of the operators $\hat{\mathbf{x}}$ and $\boldsymbol{\nabla}$. Calculate the commutators $[\hat{L}_i, \hat{L}_j]$ and $[\hat{L}^2, \hat{L}_i]$.

3/II/16B **Quantum Mechanics**

The expression $\Delta_\psi A$ denotes the uncertainty of a quantum mechanical observable A in a state with normalised wavefunction ψ. Prove that the Heisenberg uncertainty principle

$$(\Delta_\psi x)(\Delta_\psi p) \geqslant \frac{\hbar}{2}$$

holds for all normalised wavefunctions $\psi(x)$ of one spatial dimension.

[*You may quote Schwarz's inequality without proof.*]

A Gaussian wavepacket evolves so that at time t its wavefunction is

$$\psi(x,t) = (2\pi)^{-\frac{1}{4}}\left(1+i\hbar t\right)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4(1+i\hbar t)}\right).$$

Calculate the uncertainties $\Delta_\psi x$ and $\Delta_\psi p$ at each time t, and hence verify explicitly that the uncertainty principle holds at each time t.

[*You may quote without proof the results that if* $\mathrm{Re}(a) > 0$ *then*

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{a^*}\right) x^2 \exp\left(-\frac{x^2}{a}\right) dx = \frac{1}{4}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{|a|^3}{(\mathrm{Re}(a))^{\frac{3}{2}}}$$

and

$$\int_{-\infty}^{\infty} \left(\frac{d}{dx}\exp\left(-\frac{x^2}{a^*}\right)\right)\left(\frac{d}{dx}\exp\left(-\frac{x^2}{a}\right)\right) dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{|a|}{(\mathrm{Re}(a))^{\frac{3}{2}}}\,.\ ]$$

4/I/6B **Quantum Mechanics**

(a) Define the probability density $\rho(\mathbf{x},t)$ and the probability current $\mathbf{J}(\mathbf{x},t)$ for a quantum mechanical wave function $\psi(\mathbf{x},t)$, where the three dimensional vector $\mathbf{x}$ defines spatial coordinates.

Given that the potential $V(\mathbf{x})$ is real, show that

$$\boldsymbol{\nabla}\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0\,.$$

(b) Write down the standard integral expressions for the expectation value $\langle A\rangle_\psi$ and the uncertainty $\Delta_\psi A$ of a quantum mechanical observable A in a state with wavefunction $\psi(\mathbf{x})$. Give an expression for $\Delta_\psi A$ in terms of $\langle A^2\rangle_\psi$ and $\langle A\rangle_\psi$, and justify your answer.

1/II/16G **Electromagnetism**

Three concentric conducting spherical shells of radii a, b and c ($a<b<c$) carry charges $q, -2q$ and $3q$ respectively. Find the electric field and electric potential at all points of space.

Calculate the total energy of the electric field.

2/I/6G **Electromagnetism**

Given that the electric field $\mathbf{E}$ and the current density $\mathbf{j}$ within a conducting medium of uniform conductivity σ are related by $\boldsymbol{j}=\sigma\mathbf{E}$, use Maxwell's equations to show that the charge density ρ in the medium obeys the equation

$$\frac{\partial\rho}{\partial t}=-\frac{\sigma}{\epsilon_0}\rho.$$

An infinitely long conducting cylinder of uniform conductivity σ is set up with a uniform electric charge density ρ_0 throughout its interior. The region outside the cylinder is a vacuum. Obtain ρ within the cylinder at subsequent times and hence obtain $\mathbf{E}$ and $\mathbf{j}$ within the cylinder as functions of time and radius. Calculate the value of $\mathbf{E}$ outside the cylinder.

2/II/17G **Electromagnetism**

Derive from Maxwell's equations the Biot-Savart law

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dV'$$

giving the magnetic field $\mathbf{B}(\mathbf{r})$ produced by a steady current density $\mathbf{j}(\mathbf{r})$ that vanishes outside a bounded region V.

[*You may assume that the divergence of the magnetic vector potential is zero.*]

A steady current density $\mathbf{j}(\mathbf{r})$ has the form $\mathbf{j}=(0, j_\phi(\mathbf{r}), 0)$ in cylindrical polar coordinates (r,ϕ,z) where

$$j_\phi(\mathbf{r})=\begin{cases} kr & 0\leqslant r\leqslant b, \quad -h\leqslant z\leqslant h,\\ 0 & \text{otherwise,}\end{cases}$$

and k is a constant. Find the magnitude and direction of the magnetic field at the origin.

$$\left[\textit{Hint}:\quad \int_{-h}^{h}\frac{dz}{(r^2+z^2)^{3/2}}=\frac{2h}{r^2(h^2+r^2)^{1/2}}\ .\right]$$

3/II/17G **Electromagnetism**

Write down Maxwell's equations in vacuo and show that they admit plane wave solutions in which

$$\mathbf{E}(\mathbf{x}, t) = Re\left(\mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}\right), \quad \mathbf{k}\cdot\mathbf{E}_0 = 0,$$

where $\mathbf{E}_0$ and $\mathbf{k}$ are constant vectors. Find the corresponding magnetic field $\mathbf{B}(\mathbf{x}, t)$ and the relationship between ω and $\mathbf{k}$.

Write down the relations giving the discontinuities (if any) in the normal and tangential components of $\mathbf{E}$ and $\mathbf{B}$ across a surface $z = 0$ which carries surface charge density σ and surface current density $\mathbf{j}$.

Suppose that a perfect conductor occupies the region $z < 0$, and that a plane wave with $\mathbf{k} = (0, 0, -k)$, $\mathbf{E}_0 = (E_0, 0, 0)$ is incident from the vacuum region $z > 0$. Show that the boundary conditions at $z = 0$ can be satisfied if a suitable reflected wave is present, and find the induced surface charge and surface current densities.

4/I/7G **Electromagnetism**

Starting from Maxwell's equations, deduce Faraday's law of induction

$$\frac{d\Phi}{dt} = -\varepsilon,$$

for a moving circuit C, where Φ is the flux of $\mathbf{B}$ through the circuit and where the EMF ε is defined to be

$$\varepsilon = \oint_C (\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot d\mathbf{r}$$

with $\mathbf{v}(\mathbf{r})$ denoting the velocity of a point $\mathbf{r}$ of C.

[*Hint: consider the closed surface consisting of the surface $S(t)$ bounded by $C(t)$, the surface $S(t+\delta t)$ bounded by $C(t+\delta t)$ and the surface S' stretching from $C(t)$ to $C(t+\delta t)$. Show that the flux of $\mathbf{B}$ through S' is $-\oint_C \mathbf{B}\cdot(\mathbf{v}\times d\mathbf{r})\delta t$.*]

1/I/4B **Special Relativity**

A ball of clay of mass m travels at speed v in the laboratory frame towards an identical ball at rest. After colliding head-on, the balls stick together, moving in the same direction as the first ball was moving before the collision. Calculate the mass m' and speed v' of the combined lump, justifying your answers carefully.

2/I/7B **Special Relativity**

A_1 moves at speed v_1 in the x-direction with respect to A_0. A_2 moves at speed v_2 in the x-direction with respect to A_1. By applying a Lorentz transformation between the rest frames of A_0, A_1, and A_2, calculate the speed at which A_0 observes A_2 to travel.

A_3 moves at speed v_3 in the x-direction with respect to A_2. Calculate the speed at which A_0 observes A_3 to travel.

4/II/17B **Special Relativity**

A javelin of length 4 metres is thrown at a speed of $\frac{12}{13}c$ horizontally and lengthwise through a barn of length 3 metres, which is open at both ends. (Here c denotes the speed of light.)

(a) What is the length of the javelin in the rest frame of the barn?

(b) What is the length of the barn in the rest frame of the javelin?

(c) Define the rest frame coordinates of the barn and of the javelin such that the point where the trailing end of the javelin enters the barn is the origin in both frames. Draw a space-time diagram in the rest frame coordinates (ct, x) of the barn, showing the world lines of both ends of the javelin and of the front and back of the barn. Draw a second space-time diagram in the rest frame coordinates (ct', x') of the javelin, again showing the world lines of both ends of the javelin and of the front and back of the barn.

(d) Clearly mark the space-time events corresponding to (A) the trailing end of the javelin entering the barn, and (B) the leading end of the javelin exiting the back of the barn. Give the corresponding (ct, x) and (ct', x') coordinates for (B).

Are the events (A) and (B) space-like, null, or time-like separated?

As the javelin is longer than the barn in one frame and shorter than the barn in another, it might be argued that the javelin is contained entirely within the barn for a period according to an observer in one frame, but not according to an observer in another. Explain how this apparent inconsistency is resolved.

1/I/5A **Fluid Dynamics**

Use the Euler equation for the motion of an inviscid fluid to derive the vorticity equation in the form

$$D\boldsymbol{\omega}/Dt = \boldsymbol{\omega} \cdot \nabla \mathbf{u}.$$

Give a physical interpretation of the terms in this equation and deduce that irrotational flows remain irrotational.

In a plane flow the vorticity at time $t = 0$ has the uniform value $\boldsymbol{\omega}_0 \neq \mathbf{0}$. Find the vorticity everywhere at times $t > 0$.

1/II/17A **Fluid Dynamics**

A point source of fluid of strength m is located at $\mathbf{x}_s = (0, 0, a)$ in inviscid fluid of density ρ. Gravity is negligible. The fluid is confined to the region $z \geqslant 0$ by the fixed boundary $z = 0$. Write down the equation and boundary conditions satisfied by the velocity potential ϕ. Find ϕ.

[*Hint: consider the flow generated in unbounded fluid by the source m together with an 'image source' of equal strength at* $\bar{\mathbf{x}}_s = (0, 0, -a)$.]

Use Bernoulli's theorem, which may be stated without proof, to find the fluid pressure everywhere on $z = 0$. Deduce the magnitude of the hydrodynamic force on the boundary $z = 0$. Determine whether the boundary is attracted toward the source or repelled from it.

2/I/8A **Fluid Dynamics**

Explain what is meant by a *material time derivative*, D/Dt. Show that if the material velocity is $\mathbf{u}(\mathbf{x}, t)$ then

$$D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla.$$

When glass is processed in its liquid state, its temperature, $\theta(\mathbf{x}, t)$, satisfies the equation

$$D\theta/Dt = -\theta.$$

The glass flows in a two-dimensional channel $-1 < y < 1, \quad x > 0$ with steady velocity $\mathbf{u} = (1 - y^2, 0)$. At $x = 0$ the glass temperature is maintained at the constant value θ_0. Find the steady temperature distribution throughout the channel.

3/II/18A Fluid Dynamics

State and prove Bernoulli's theorem for a time-dependent irrotational flow of an inviscid fluid.

A large vessel is part-filled with inviscid liquid of density ρ. The pressure in the air above the liquid is maintained at the constant value $P + p_a$, where p_a is atmospheric pressure and $P > 0$. Liquid can flow out of the vessel along a cylindrical tube of length L. The radius a of the tube is much smaller than both L and the linear dimensions of the vessel. Initially the tube is sealed and is full of liquid. At time $t = 0$ the tube is opened and the liquid starts to flow. Assuming that the tube remains full of liquid, that the pressure at the open end of the tube is atmospheric and that P is so large that gravity is negligible, determine the flux of liquid along the tube at time t.

4/II/18A Fluid Dynamics

A rectangular tank has a horizontal base and vertical sides. Viewed from above, the cross-section of the tank is a square of side a. At rest, the depth of water in the tank is h. Suppose that the free-surface is disturbed in such a way that the flow in the water is irrotational. Take the pressure at the free surface as atmospheric. Starting from the appropriate non-linear expressions, obtain free-surface boundary conditions for the velocity potential appropriate for small-amplitude disturbances of the surface.

Show that the governing equations and boundary conditions admit small-amplitude normal mode solutions for which the free-surface elevation above its equilibrium level is everywhere proportional to $e^{i\omega t}$, and find the frequencies, ω, of such modes.

1/I/6D **Numerical Analysis**

(a) Perform the LU-factorization with column pivoting of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

(b) Explain briefly why every nonsingular matrix A admits an LU-factorization with column pivoting.

2/II/18D **Numerical Analysis**

(a) For a positive weight function w, let

$$\int_{-1}^{1} f(x)w(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

be the corresponding Gaussian quadrature with $n+1$ nodes. Prove that all the coefficients a_i are positive.

(b) The integral

$$I(f) = \int_{-1}^{1} f(x)w(x)\,dx$$

is approximated by a quadrature

$$I_n(f) = \sum_{i=0}^{n} a_i^{(n)} f(x_i^{(n)})$$

which is exact on polynomials of degree $\leqslant n$ and has positive coefficients $a_i^{(n)}$. Prove that, for any f continuous on $[-1,1]$, the quadrature converges to the integral, i.e.,

$$|I(f) - I_n(f)| \to 0 \quad \text{as} \quad n \to \infty.$$

[*You may use the Weierstrass theorem: for any f continuous on $[-1,1]$, and for any $\epsilon > 0$, there exists a polynomial Q of degree $n = n(\epsilon, f)$ such that* $\max_{x\in[-1,1]} |f(x) - Q(x)| < \epsilon$.]

3/II/19D **Numerical Analysis**

(a) Define the QR factorization of a rectangular matrix and explain how it can be used to solve the least squares problem of finding an $x^* \in \mathbb{R}^n$ such that

$$\|Ax^* - b\| = \min_{x \in \mathbb{R}^n} \|Ax - b\|, \quad \text{where} \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad m \geqslant n,$$

and the norm is the Euclidean distance $\|y\| = \sqrt{\sum_{i=1}^m |y_i|^2}$.

(b) Define a Householder transformation (reflection) H and prove that H is an orthogonal matrix.

(c) Using Householder reflection, solve the least squares problem for the case

$$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix},$$

and find the value of $\|Ax^* - b\| = \min_{x \in \mathbb{R}^2} \|Ax - b\|$.

4/I/8D **Numerical Analysis**

(a) Given the data

x_i	-1	0	1	3
$f(x_i)$	-7	-3	-3	9

find the interpolating cubic polynomial $p \in \mathcal{P}_3$ in the Newton form, and transform it to the power form.

(b) We add to the data one more value $f(x_i)$ at $x_i = 2$. Find the power form of the interpolating quartic polynomial $q \in \mathcal{P}_4$ to the extended data

x_i	-1	0	1	2	3
$f(x_i)$	-7	-3	-3	-7	9

1/I/7C **Statistics**

A random sample $X_1, \ldots, X_n$ is taken from a normal distribution having unknown mean θ and variance 1. Find the maximum likelihood estimate $\hat{\theta}_M$ for θ based on $X_1, \ldots, X_n$.

Suppose that we now take a Bayesian point of view and regard θ itself as a normal random variable of known mean μ and variance τ^{-1}. Find the Bayes' estimate $\hat{\theta}_B$ for θ based on $X_1, \ldots, X_n$, corresponding to the quadratic loss function $(\theta - a)^2$.

1/II/18C **Statistics**

Let X be a random variable whose distribution depends on an unknown parameter θ. Explain what is meant by a sufficient statistic $T(X)$ for θ.

In the case where X is discrete, with probability mass function $f(x|\theta)$, explain, with justification, how a sufficient statistic may be found.

Assume now that $X = (X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are independent non-negative random variables with common density function

$$f(x|\theta) = \begin{cases} \lambda e^{-\lambda(x-\theta)} & \text{if } x \geqslant \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\theta \geq 0$ is unknown and λ is a known positive parameter. Find a sufficient statistic for θ and hence obtain an unbiased estimator $\hat{\theta}$ for θ of variance $(n\lambda)^{-2}$.

[*You may use without proof the following facts: for independent exponential random variables X and Y, having parameters λ and μ respectively, X has mean λ^{-1} and variance λ^{-2} and min$\{X, Y\}$ has exponential distribution of parameter $\lambda + \mu$.*]

2/II/19C **Statistics**

Suppose that $X_1, \ldots, X_n$ are independent normal random variables of unknown mean θ and variance 1. It is desired to test the hypothesis $H_0 : \theta \leq 0$ against the alternative $H_1 : \theta > 0$. Show that there is a uniformly most powerful test of size $\alpha = 1/20$ and identify a critical region for such a test in the case $n = 9$. If you appeal to any theoretical result from the course you should also prove it.

[*The 95th percentile of the standard normal distribution is 1.65.*]

3/I/8C Statistics

One hundred children were asked whether they preferred crisps, fruit or chocolate. Of the boys, 12 stated a preference for crisps, 11 for fruit, and 17 for chocolate. Of the girls, 13 stated a preference for crisps, 14 for fruit, and 33 for chocolate. Answer each of the following questions by carrying out an appropriate statistical test.

(a) Are the data consistent with the hypothesis that girls find all three types of snack equally attractive?

(b) Are the data consistent with the hypothesis that boys and girls show the same distribution of preferences?

4/II/19C Statistics

Two series of experiments are performed, the first resulting in observations $X_1, \ldots, X_m$, the second resulting in observations $Y_1, \ldots, Y_n$. We assume that all observations are independent and normally distributed, with unknown means μ_X in the first series and μ_Y in the second series. We assume further that the variances of the observations are unknown but are all equal.

Write down the distributions of the sample mean $\bar{X} = m^{-1} \sum_{i=1}^{m} X_i$ and sum of squares $S_{XX} = \sum_{i=1}^{m} (X_i - \bar{X})^2$.

Hence obtain a statistic $T(X, Y)$ to test the hypothesis $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X > \mu_Y$ and derive its distribution under H_0. Explain how you would carry out a test of size $\alpha = 1/100$.

1/I/8C **Optimization**

State the Lagrangian sufficiency theorem.

Let $p \in (1, \infty)$ and let $a_1, \ldots, a_n \in \mathbb{R}$. Maximize

$$\sum_{i=1}^{n} a_i x_i$$

subject to

$$\sum_{i=1}^{n} |x_i|^p \leqslant 1, \quad x_1, \ldots, x_n \in \mathbb{R}.$$

2/I/9C **Optimization**

Consider the maximal flow problem on a finite set N, with source A, sink B and capacity constraints c_{ij} for $i, j \in N$. Explain what is meant by a cut and by the capacity of a cut.

Show that the maximal flow value cannot exceed the minimal cut capacity.

Take $N = \{0, 1, 2, 3, 4\}^2$ and suppose that, for $i = (i_1, i_2)$ and $j = (j_1, j_2)$,

$$c_{ij} = \max\{|i_1 - i_2|, |j_1 - j_2|\} \quad \text{if} \quad |i_1 - j_1| + |i_2 - j_2| = 1,$$

and $c_{ij} = 0$ otherwise. Thus the node set is a square grid of 25 points, with positive flow capacity only between nearest neighbours, and where the capacity of an edge in the grid equals the larger of the distances of its two endpoints from the diagonal. Find a maximal flow from $(0, 3)$ to $(3, 0)$. Justify your answer.

3/II/20C **Optimization**

Explain what is meant by a two-person zero-sum game with payoff matrix $A = (a_{ij} : 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n)$ and what is meant by an optimal strategy $p = (p_i : 1 \leqslant i \leqslant m)$.

Consider the following betting game between two players: each player bets an amount $1, 2, 3$ or 4; if both bets are the same, then the game is void; a bet of 1 beats a bet of 4 but otherwise the larger bet wins; the winning player collects both bets. Write down the payoff matrix A and explain why the optimal strategy $p = (p_1, p_2, p_3, p_4)^T$ must satisfy $(Ap)_i \leqslant 0$ for all i. Hence find p.

4/II/20C **Optimization**

Use a suitable version of the simplex algorithm to solve the following linear programming problem:

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 50x_1 & - & 30x_2 & + & x_3 & & \\
\text{subject to} & x_1 & + & x_2 & + & x_3 & \leq & 30 \\
 & 2x_1 & - & x_2 & & & \leq & 35 \\
 & x_1 & + & 2x_2 & - & x_3 & \geq & 40 \\
\text{and} & & & x_1, x_2, x_3 & & & \geq & 0.
\end{array}
$$

1/II/19C Markov Chains

Explain what is meant by a stopping time of a Markov chain $(X_n)_{n\geq 0}$. State the strong Markov property.

Show that, for any state i, the probability, starting from i, that $(X_n)_{n\geq 0}$ makes infinitely many visits to i can take only the values 0 or 1.

Show moreover that, if

$$\sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) = \infty,$$

then $(X_n)_{n\geq 0}$ makes infinitely many visits to i with probability 1.

2/II/20C Markov Chains

Consider the Markov chain $(X_n)_{n\geq 0}$ on the integers $\mathbb{Z}$ whose non-zero transition probabilities are given by $p_{0,1} = p_{0,-1} = 1/2$ and

$$p_{n,n-1} = 1/3, \quad p_{n,n+1} = 2/3, \qquad \text{for } n \geq 1,$$

$$p_{n,n-1} = 3/4, \quad p_{n,n+1} = 1/4, \qquad \text{for } n \leqslant -1.$$

(a) Show that, if $X_0 = 1$, then $(X_n)_{n\geq 0}$ hits 0 with probability 1/2.

(b) Suppose now that $X_0 = 0$. Show that, with probability 1, as $n \to \infty$ either $X_n \to \infty$ or $X_n \to -\infty$.

(c) In the case $X_0 = 0$ compute $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty)$.

3/I/9C Markov Chains

A hungry student always chooses one of three places to get his lunch, basing his choice for one day on his gastronomic experience the day before. He sometimes tries a sandwich from Natasha's Patisserie: with probability 1/2 this is delicious so he returns the next day; if the sandwich is less than delicious, he chooses with equal probability 1/4 either to eat in Hall or to cook for himself. Food in Hall leaves no strong impression, so he chooses the next day each of the options with equal probability 1/3. However, since he is a hopeless cook, he never tries his own cooking two days running, always preferring to buy a sandwich the next day. On the first day of term the student has lunch in Hall. What is the probability that 60 days later he is again having lunch in Hall?

[*Note* $0^0 = 1$.]

4/I/9C **Markov Chains**

A game of chance is played as follows. At each turn the player tosses a coin, which lands heads or tails with equal probability 1/2. The outcome determines a score for that turn, which depends also on the cumulative score so far. Write S_n for the cumulative score after n turns. In particular $S_0 = 0$. When S_n is odd, a head scores 1 but a tail scores 0. When S_n is a multiple of 4, a head scores 4 and a tail scores 1. When S_n is even but is not a multiple of 4, a head scores 2 and a tail scores 1. By considering a suitable four-state Markov chain, determine the long run proportion of turns for which S_n is a multiple of 4. State clearly any general theorems to which you appeal.