UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday 9 June 2006 9 to 12

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A****,* ***B****,* ***C****,...,****J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Let x be a real number greater than or equal to 2, and define

$$P(x) = \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right),$$

where the product is taken over all primes p which are less than or equal to x. Prove that $P(x) \to 0$ as $x \to \infty$, and deduce that $\displaystyle\sum_p \frac{1}{p}$ diverges when the summation is taken over all primes p.

2G Topics in Analysis

(a) State the Baire category theorem, in its closed-sets version.

(b) For every $n \in \mathbb{N}$ let f_n be a continuous function from $\mathbb{R}$ to $\mathbb{R}$, and let $g(x) = 1$ when x is rational and 0 otherwise. For each $N \in \mathbb{N}$, let

$$F_N = \left\{x \in \mathbb{R} : \forall n \geqslant N \quad f_n(x) \leqslant \tfrac{1}{3} \quad \text{or} \quad f_n(x) \geqslant \tfrac{2}{3}\right\}.$$

By applying the Baire category theorem, prove that the functions f_n cannot converge pointwise to g. (That is, it is not the case that $f_n(x) \to g(x)$ for every $x \in \mathbb{R}$.)

3F Geometry and Groups

What is a crystallographic group in the Euclidean plane? Prove that, if G is crystallographic and g is a nontrivial rotation in G, then g has order 2, 3, 4, or 6.

4G Coding and Cryptography

A binary erasure channel with erasure probability p is a discrete memoryless channel with channel matrix

$$\begin{pmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{pmatrix}.$$

State Shannon's second coding theorem, and use it to compute the capacity of this channel.

5I Statistical Modelling

The table below summarises the yearly numbers of named storms in the Atlantic basin over the period 1944–2004, and also gives an index of average July ocean temperature in the northern hemisphere over the same period. To save space, only the data for the first four and last four years are shown.

Year	Storms	Temp
1944	11	0.165
1945	11	0.080
1946	6	0.000
1947	9	-0.024
⋮	⋮	⋮
2001	15	0.592
2002	12	0.627
2003	16	0.608
2004	15	0.546

Explain and interpret the `R` commands and (slightly abbreviated) output below.

```
> Mod <- glm(Storms~Temp,family=poisson)

> summary(Mod)

Coefficients:

             Estimate Std. Error z value Pr(>|z|)

(Intercept)  2.26061    0.04841  46.697  < 2e-16 ***

Temp         0.48870    0.16973   2.879  0.00399 **


Residual deviance: 51.499  on 59  degrees of freedom
```

In 2005, the ocean temperature index was 0.743. Explain how you would predict the number of named storms for that year.

6B Mathematical Biology

A nonlinear model of insect dispersal with exponential death rate takes the form (for insect population $n(x,t)$)

$$\frac{\partial n}{\partial t} = -\mu n + \frac{\partial}{\partial x}\left(n\frac{\partial n}{\partial x}\right) . \qquad (*)$$

At time $t = 0$ the total insect population is Q, and all the insects are at the origin. A solution is sought in the form

$$n = \frac{e^{-\mu t}}{\lambda(t)} f(\eta); \quad \eta = \frac{x}{\lambda(t)}, \quad \lambda(0) = 0 . \qquad (\dagger)$$

(a) Verify that $\int_{-\infty}^{\infty} f \, d\eta = Q$, provided f decays sufficiently rapidly as $|x| \to \infty$.

(b) Show, by substituting the form of n given in equation ($\dagger$) into equation ($*$), that ($*$) is satisfied, for nonzero f, when

$$\frac{d\lambda}{dt} = \lambda^{-2} e^{-\mu t} \quad \text{and} \quad \frac{df}{d\eta} = -\eta .$$

Hence find the complete solution and show that the insect population is always confined to a finite region that never exceeds the range

$$|x| \leqslant \left(\frac{9Q}{2\mu}\right)^{1/3} .$$

7E Dynamical Systems

Consider the logistic map $F(x) = \mu x(1-x)$ for $0 \leqslant x \leqslant 1$, $0 \leqslant \mu \leqslant 4$. Show that there is a period-doubling bifurcation of the nontrivial fixed point at $\mu = 3$. Show further that the bifurcating 2-cycle (x_1, x_2) is given by the roots of

$$\mu^2 x^2 - \mu(\mu+1)x + \mu + 1 = 0 .$$

Show that there is a second period-doubling bifurcation at $\mu = 1 + \sqrt{6}$.

8E Further Complex Methods

By means of the change of variable $u = rs$, $v = r(1-s)$ in a suitable double integral, or otherwise, show that for Re $z > 0$

$$\left[\Gamma\left(\tfrac{1}{2}z\right)\right]^2 = B\left(\tfrac{1}{2}z, \tfrac{1}{2}z\right)\Gamma(z) .$$

Deduce that, if $\Gamma(z) = 0$ for some z with Re $z > 0$, then $\Gamma\left(z/2^m\right) = 0$ for any positive integer m.

Prove that $\Gamma(z) \neq 0$ for any z.

9C Classical Dynamics

Calculate the principal moments of inertia for a uniform cylinder, of mass M, radius R and height $2h$, about its centre of mass. For what height-to-radius ratio does the cylinder spin like a sphere?

10D Cosmology

The number density of fermions of mass m at equilibrium in the early universe with temperature T, is given by the integral

$$n \;=\; \frac{4\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{\exp[(\mathcal{E}(p)-\mu)/kT]+1}$$

where $\mathcal{E}(p) = c\sqrt{p^2 + m^2c^2}$, and μ is the chemical potential. Assuming that the fermions remain in equilibrium when they become non-relativistic (kT, $\mu \ll mc^2$), show that the number density can be expressed as

$$n \;=\; \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \exp\left[(\mu - mc^2)/kT\right] \;.$$

[*Hint: You may assume* $\int_0^\infty dx\, e^{-\sigma^2 x^2} = \sqrt{\pi}/(2\sigma)\,, \quad (\sigma > 0)$.]

Suppose that the fermions decouple at a temperature given by $kT = mc^2/\alpha$ where $\alpha \gg 1$. Assume also that $\mu = 0$. By comparing with the photon number density at $n_\gamma = 16\pi\zeta(3)(kT/hc)^3$, where $\zeta(3) = \sum_{n=1}^\infty n^{-3} = 1.202\ldots$, show that the ratio of number densities at decoupling is given by

$$\frac{n}{n_\gamma} \;=\; \frac{\sqrt{2\pi}}{8\zeta(3)}\,\alpha^{3/2}\,e^{-\alpha} \;.$$

Now assume that $\alpha \approx 20$, (which implies $n/n_\gamma \approx 5\times 10^{-8}$), and that the fermion mass $m = m_p/20$, where m_p is the proton mass. Explain clearly why this new fermion would be a good candidate for solving the dark matter problem of the standard cosmology.

SECTION II

11H Number Theory

Define the notion of a Fermat, Euler, and strong pseudo-prime to the base b, where b is an integer greater than 1.

Let N be an odd integer greater than 1. Prove that:

(a) If N is a prime number, then N is a strong pseudo-prime for every base b with $(b, N) = 1$.

(b) If there exists a base b_1 with $1 < b_1 < N$ and $(b_1, N) = 1$ for which N is not a pseudo-prime, then in fact N is not a pseudo-prime for at least half of all bases b with $1 < b < N$ and $(b, N) = 1$.

Prove that 341 is a Fermat pseudo-prime, but not an Euler pseudo-prime, to the base 2.

12F Geometry and Groups

Let G be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Show that G is countable. Let $G = \{g_1, g_2, \ldots\}$ be some enumeration of the elements of G. Show that for any point p in hyperbolic 3-space $\mathbb{H}^3$, the distance $d_{hyp}(p, g_n(p))$ tends to infinity. Deduce that a subgroup G of $\mathrm{PSL}_2(\mathbb{C})$ is discrete if and only if it acts properly discontinuously on $\mathbb{H}^3$.

13I Statistical Modelling

Consider a linear model for $Y = (Y_1, \ldots, Y_n)^T$ given by

$$Y = X\beta + \epsilon,$$

where X is a known $n \times p$ matrix of full rank $p < n$, where β is an unknown vector and $\epsilon \sim N_n(0, \sigma^2 I)$. Derive an expression for the maximum likelihood estimator $\hat{\beta}$ of β, and write down its distribution.

Find also the maximum likelihood estimator $\hat{\sigma}^2$ of σ^2, and derive its distribution.

[*You may use Cochran's theorem, provided that it is stated carefully. You may also assume that the matrix* $P = X(X^TX)^{-1}X^T$ *has rank* p, *and that* $I - P$ *has rank* $n - p$.]

14E Further Complex Methods

Let $I = \displaystyle\int_0^1 \left[x(1-x^2)\right]^{1/3}\ dx$.

(a) Express I in terms of an integral of the form $\displaystyle\oint (z^3 - z)^{1/3}\, dz$, where the path of integration is a large circle. You should explain carefully which branch of $(z^3 - z)^{1/3}$ you choose, by using polar co-ordinates with respect to the branch points. Hence show that $I = \frac{1}{6}\pi \operatorname{cosec} \frac{1}{3}\pi$.

(b) Give an alternative method of evaluating I, by making a suitable change of variable and expressing I in terms of a beta function.

15D Cosmology

The perturbed motion of cold dark matter particles (pressure-free, $P = 0$) in an expanding universe can be parametrized by the trajectories

$$\mathbf{r}(\mathbf{q}, t) = a(t)\left[\mathbf{q} + \boldsymbol{\psi}(\mathbf{q}, t)\right] \ ,$$

where $a(t)$ is the scale factor of the universe, $\mathbf{q}$ is the unperturbed comoving trajectory and $\boldsymbol{\psi}$ is the comoving displacement. The particle equation of motion is $\ddot{\mathbf{r}} = -\nabla\Phi$, where the Newtonian potential satisfies the Poisson equation $\nabla^2\Phi = 4\pi G\rho$ with mass density $\rho(\mathbf{r}, t)$.

(a) Discuss how matter conservation in a small volume $d^3\mathbf{r}$ ensures that the perturbed density $\rho(\mathbf{r}, t)$ and the unperturbed background density $\bar{\rho}(t)$ are related by

$$\rho(\mathbf{r}, t) d^3\mathbf{r} = \bar{\rho}(t) a^3(t) d^3\mathbf{q} \ .$$

By changing co-ordinates with the Jacobian

$$|\partial r_i/\partial q_j|^{-1} = |a\delta_{ij} + a\,\partial\psi_i/\partial q_j|^{-1} \approx a^{-3}(1 - \nabla_q \cdot \boldsymbol{\psi}) \ ,$$

show that the fractional density perturbation $\delta(\mathbf{q}, t)$ can be written to leading order as

$$\delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} = -\nabla_q \cdot \boldsymbol{\psi} \ ,$$

where $\nabla_q \cdot \boldsymbol{\psi} = \sum_i \partial\psi_i/\partial q_i$.

Use this result to integrate the Poisson equation once. Hence, express the particle equation of motion in terms of the comoving displacement as

$$\ddot{\boldsymbol{\psi}} + 2\frac{\dot{a}}{a}\dot{\boldsymbol{\psi}} - 4\pi G\bar{\rho}\boldsymbol{\psi} = 0 \ .$$

Infer that the density perturbation evolution equation is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\bar{\rho}\delta = 0 \ . \qquad (*)$$

[*Hint: You may assume that the integral of* $\nabla^2\Phi = 4\pi G\bar{\rho}$ *is* $\nabla\Phi = -4\pi G\bar{\rho}\mathbf{r}/3$. *Note also that the Raychaudhuri equation (for* $P = 0$*) is* $\ddot{a}/a = -4\pi G\bar{\rho}/3$.]

(b) Find the general solution of equation $(*)$ in a flat ($k = 0$) universe dominated by cold dark matter ($P = 0$). Discuss the effect of late-time Λ or dark energy domination on the growth of density perturbations.

16H Logic and Set Theory

Explain carefully what is meant by a *well-founded* relation on a set. State the recursion theorem, and use it to prove that a binary relation r on a set a is well-founded if and only if there exists a function f from a to some ordinal α such that $(x, y) \in r$ implies $f(x) < f(y)$.

Deduce, using the axiom of choice, that any well-founded relation on a set may be extended to a well-ordering.

17F Graph Theory

What is meant by a graph G of order n being *strongly regular* with parameters (d, a, b)? Show that, if such a graph G exists and $b > 0$, then

$$\frac{1}{2}\left\{ n - 1 + \frac{(n-1)(b-a) - 2d}{\sqrt{(a-b)^2 + 4(d-b)}} \right\}$$

is an integer.

Let G be a graph containing no triangles, in which every pair of non-adjacent vertices has exactly three common neighbours. Show that G must be d-regular and $|G| = 1 + d(d+2)/3$ for some $d \in \{1, 3, 21\}$. Show that such a graph exists for $d = 3$.

18H Galois Theory

Let K be a field of characteristic different from 2.

Show that if L/K is an extension of degree 2, then $L = K(x)$ for some $x \in L$ such that $x^2 = a \in K$. Show also that if $L' = K(y)$ with $0 \neq y^2 = b \in K$ then L and L' are isomorphic (as extensions of K) if and only b/a is a square in K.

Now suppose that $F = K(x_1, \ldots, x_n)$ where $0 \neq x_i^2 = a_i \in K$. Show that F/K is a Galois extension, with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$ for some $m \leqslant n$. By considering the subgroups of $\mathrm{Gal}(F/K)$, show that if $K \subset L \subset F$ and $[L : K] = 2$, then $L = K(y)$ where $y = \prod_{i \in I} x_i$ for some subset $I \subset \{1, \ldots, n\}$.

19F Representation Theory

In this question, all vector spaces will be complex.

(a) Let A be a finite abelian group.

(i) Show directly from the definitions that any irreducible representation must be one-dimensional.

(ii) Show that A has a faithful one-dimensional representation if and only if A is cyclic.

(b) Now let G be an arbitrary finite group and suppose that the centre of G is non-trivial. Write $Z = \{z \in G \mid zg = gz \quad \forall g \in G\}$ for this centre.

(i) Let W be an irreducible representation of G. Show that $\mathrm{Res}_Z^G W = \dim W.\chi$, where χ is an irreducible representation of Z.

(ii) Show that every irreducible representation of Z occurs in this way.

(iii) Suppose that Z is not a cyclic group. Show that there does not exist an irreducible representation W of G such that every irreducible representation V occurs as a summand of $W^{\otimes n}$ for some n.

20G Number Fields

Let $\zeta = e^{2\pi i/5}$ and let $K = \mathbb{Q}(\zeta)$. Show that the discriminant of K is 125. Hence prove that the ideals in K are all principal.

Verify that $(1-\zeta^n)/(1-\zeta)$ is a unit in K for each integer n with $1 \leqslant n \leqslant 4$. Deduce that $5/(1-\zeta)^4$ is a unit in K. Hence show that the ideal $[1-\zeta]$ is prime and totally ramified in K. Indicate briefly why there are no other ramified prime ideals in K.

[*It can be assumed that* ζ, ζ^2, ζ^3, ζ^4 *is an integral basis for* K *and that the Minkowski constant for* K *is* $3/(2\pi^2)$.]

21H Algebraic Topology

Fix a point p in the torus $S^1 \times S^1$. Let G be the group of homeomorphisms f from the torus $S^1 \times S^1$ to itself such that $f(p) = p$. Determine a non-trivial homomorphism ϕ from G to the group $\mathrm{GL}(2, \mathbb{Z})$.
[*The group* $\mathrm{GL}(2,\mathbb{Z})$ *consists of* 2×2 *matrices with integer coefficients that have an inverse which also has integer coefficients.*]

Establish whether each f in the kernel of ϕ is homotopic to the identity map.

22G Linear Analysis

Let H be a complex Hilbert space. Define what it means for a linear operator $T : H \to H$ to be *self-adjoint*. State a version of the spectral theorem for compact self-adjoint operators on a Hilbert space. Give an example of a Hilbert space H and a compact self-adjoint operator on H with infinite dimensional range. Define the notions *spectrum, point spectrum*, and *resolvent set*, and describe these in the case of the operator you wrote down. Justify your answers.

23F Riemann Surfaces

Define what is meant by a *divisor* on a compact Riemann surface, the *degree* of a divisor, and a *linear equivalence* between divisors. For a divisor D, define $\ell(D)$ and show that if a divisor D' is linearly equivalent to D then $\ell(D) = \ell(D')$. Determine, without using the Riemann–Roch theorem, the value $\ell(P)$ in the case when P is a point on the Riemann sphere S^2.

[*You may use without proof any results about holomorphic maps on S^2 provided that these are accurately stated.*]

State the Riemann–Roch theorem for a compact connected Riemann surface C. (You are *not* required to give a definition of a canonical divisor.) Show, by considering an appropriate divisor, that if C has genus g then C admits a non-constant meromorphic function (that is a holomorphic map $C \to S^2$) of degree at most $g + 1$.

24H Differential Geometry

(a) Let $S \subset \mathbb{R}^3$ be an oriented surface and let λ be a real number. Given a point $p \in S$ and a vector $v \in T_pS$ with unit norm, show that there exist $\varepsilon > 0$ and a unique curve $\gamma : (-\varepsilon, \varepsilon) \to S$ parametrized by arc-length and with constant geodesic curvature λ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

[*You may use the theorem on existence and uniqueness of solutions of ordinary differential equations.*]

(b) Let S be an oriented surface with positive Gaussian curvature and diffeomorphic to S^2. Show that two simple closed geodesics in S must intersect. Is it true that two smooth simple closed curves in S with constant geodesic curvature $\lambda \neq 0$ must intersect?

25J Probability and Measure

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ a measurable function.

(a) Explain what is meant by saying that f is *integrable*, and how the integral $\int_\Omega f \, d\mu$ is defined, starting with integrals of $\mathcal{A}$-simple functions.

[*Your answer should consist of clear definitions, including the ones for $\mathcal{A}$-simple functions and their integrals.*]

(b) For $f : \Omega \to [0, \infty)$ give a specific sequence $(g_n)_{n \in \mathbb{N}}$ of $\mathcal{A}$-simple functions such that $0 \leqslant g_n \leqslant f$ and $g_n(x) \to f(x)$ for all $x \in \Omega$. Justify your answer.

(c) Suppose that that $\mu(\Omega) < \infty$ and let $f_1, f_2, \ldots : \Omega \to \mathbb{R}$ be measurable functions such that $f_n(x) \to 0$ for all $x \in \Omega$. Prove that, if

$$\lim_{c \to \infty} \sup_{n \in \mathbb{N}} \int_{|f_n| > c} |f_n| \, d\mu = 0,$$

then $\int_\Omega f_n \, d\mu \to 0$.

Give an example with $\mu(\Omega) < \infty$ such that $f_n(x) \to 0$ for all $x \in \Omega$, but $\int_\Omega f_n \, d\mu \not\to 0$, and justify your answer.

(d) State and prove Fatou's Lemma for a sequence of non-negative measurable functions.

[*Standard results on measurability and integration may be used without proof.*]

26J Applied Probability

(a) Let $(N_t)_{t\geqslant 0}$ be a Poisson process of rate $\lambda > 0$. Let p be a number between 0 and 1 and suppose that each jump in (N_t) is counted as type one with probability p and type two with probability $1-p$, independently for different jumps and independently of the Poisson process. Let $M_t^{(1)}$ be the number of type-one jumps and $M_t^{(2)} = N_t - M_t^{(1)}$ the number of type-two jumps by time t. What can you say about the pair of processes $(M_t^{(1)})_{t\geqslant 0}$ and $(M_t^{(2)})_{t\geqslant 0}$? What if we fix probabilities p_1, ..., p_m with $p_1 + ... + p_m = 1$ and consider m types instead of two?

(b) A person collects coupons one at a time, at jump times of a Poisson process $(N_t)_{t\geqslant 0}$ of rate λ. There are m types of coupons, and each time a coupon of type j is obtained with probability p_j, independently of the previously collected coupons and independently of the Poisson process. Let T be the first time when a complete set of coupon types is collected. Show that

$$\mathbb{P}(T < t) = \prod_{j=1}^{m} (1 - e^{-p_j \lambda t}) \ .$$

Let $L = N_T$ be the total number of coupons collected by the time the complete set of coupon types is obtained. Show that $\lambda \mathbb{E} T = \mathbb{E} L$. Hence, or otherwise, deduce that $\mathbb{E} L$ does not depend on λ.

27J Principles of Statistics

(a) State the strong law of large numbers. State the central limit theorem.

(b) Assuming whatever regularity conditions you require, show that if $\hat{\theta}_n \equiv \hat{\theta}_n(X_1, \ldots, X_n)$ is the maximum-likelihood estimator of the unknown parameter θ based on an independent identically distributed sample of size n, then under P_θ

$$\sqrt{n}(\hat{\theta}_n - \theta) \to N(0, J(\theta)^{-1}) \qquad \text{in distribution}$$

as $n \to \infty$, where $J(\theta)$ is a matrix which you should identify. A rigorous derivation is not required.

(c) Suppose that $X_1, X_2, \ldots$ are independent binomial $\mathrm{Bin}(1, \theta)$ random variables. It is required to test $H_0 : \theta = \frac{1}{2}$ against the alternative $H_1 : \theta \in (0, 1)$. Show that the construction of a likelihood-ratio test leads us to the statistic

$$T_n = 2n\{\hat{\theta}_n \log \hat{\theta}_n + (1 - \hat{\theta}_n)\log(1 - \hat{\theta}_n) + \log 2\},$$

where $\hat{\theta}_n \equiv n^{-1}\sum_{k=1}^{n} X_k$. Stating clearly any result to which you appeal, for large n, what approximately is the distribution of T_n under H_0? Writing $\hat{\theta}_n = \frac{1}{2} + Z_n$, and assuming that Z_n is small, show that

$$T_n \simeq 4nZ_n^2.$$

Using this and the central limit theorem, briefly justify the approximate distribution of T_n given by asymptotic maximum-likelihood theory. What could you say if the assumption that Z_n is small failed?

28I Stochastic Financial Models

State the definitions of a *martingale* and a *stopping time*.

State and prove the optional sampling theorem.

If $(M_n, \mathcal{F}_n)_{n \geqslant 0}$ is a martingale, under what conditions is it true that M_n converges with probability 1 as $n \to \infty$? Show by an example that some condition is necessary.

A market consists of $K > 1$ agents, each of whom is either optimistic or pessimistic. At each time $n = 0, 1, \ldots$, one of the agents is selected at random, and chooses to talk to one of the other agents (again selected at random), whose type he then adopts. If choices in different periods are independent, show that the proportion of pessimists is a martingale. What can you say about the limiting behaviour of the proportion of pessimists as time n tends to infinity?

29I Optimization and Control

An investor has a (possibly negative) bank balance $x(t)$ at time t. For given positive $x(0), T, \mu, A$ and r, he wishes to choose his spending rate $u(t) \geqslant 0$ so as to maximize

$$\Phi(u;\mu) \equiv \int_0^T e^{-\beta t} \log u(t)\, dt + \mu e^{-\beta T} x(T),$$

where $dx(t)/dt = A + rx(t) - u(t)$. Find the investor's optimal choice of control $u(t) = u_*(t;\mu)$.

Let $x_*(t;\mu)$ denote the optimally-controlled bank balance. By considering next how $x_*(T;\mu)$ depends on μ, show that there is a unique positive μ_* such that $x_*(T;\mu_*) = 0$. If the original problem is modified by setting $\mu = 0$, but requiring that $x(T) \geqslant 0$, show that the optimal control for this modified problem is $u(t) = u_*(t;\mu_*)$.

30A Partial Differential Equations

(a) State the Fourier inversion theorem for Schwartz functions $\mathcal{S}(\mathbb{R})$ on the real line. Define the Fourier transform of a tempered distribution and compute the Fourier transform of the distribution defined by the function $F(x) = \frac{1}{2}$ for $-t \leqslant x \leqslant +t$ and $F(x) = 0$ otherwise. (Here t is any positive number.)

Use the Fourier transform in the x variable to deduce a formula for the solution to the one dimensional wave equation

$$u_{tt} - u_{xx} = 0\ , \qquad \text{with initial data} \qquad u(0,x) = 0\ , \quad u_t(0,x) = g(x)\ , \qquad (*)$$

for g a Schwartz function. Explain what is meant by "finite propagation speed" and briefly explain why the formula you have derived is in fact valid for arbitrary smooth $g \in C^\infty(\mathbb{R})$.

(b) State a theorem on the representation of a smooth 2π-periodic function g as a Fourier series

$$g(x) = \sum_{\alpha \in \mathbb{Z}} \hat{g}(\alpha) e^{i\alpha x}$$

and derive a representation for solutions to $(*)$ as Fourier series in x.

(c) Verify that the formulae obtained in (a) and (b) agree for the case of smooth 2π-periodic g.

31B Asymptotic Methods

(a) Outline the Liouville–Green approximation to solutions $w(z)$ of the ordinary differential equation

$$\frac{d^2w}{dz^2} = f(z)w$$

in a neighbourhood of infinity, in the case that, near infinity, $f(z)$ has the convergent series expansion

$$f(z) = \sum_{s=0}^{\infty} \frac{f_s}{z^s} \; ,$$

with $f_0 \neq 0$.

In the case

$$f(z) = 1 + \frac{1}{z} + \frac{2}{z^2} \; ,$$

explain why you expect a basis of two asymptotic solutions $w_1(z)$, $w_2(z)$, with

$$\begin{aligned} w_1(z) &\sim z^{\frac{1}{2}} e^{z} \left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\right), \\ w_2(z) &\sim z^{-\frac{1}{2}} e^{-z} \left(1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots\right), \end{aligned}$$

as $z \to +\infty$, and show that $a_1 = -\frac{9}{8}$.

(b) Determine, at leading order in the large positive real parameter λ, an approximation to the solution $u(x)$ of the eigenvalue problem:

$$u''(x) + \lambda^2 g(x)u(x) = 0; \quad u(0) = u(1) = 0;$$

where $g(x)$ is greater than a positive constant for $x \in [0, 1]$.

32A Principles of Quantum Mechanics

Define the Heisenberg picture of quantum mechanics in relation to the Schrödinger picture and explain how these formulations give rise to identical physical predictions. Derive an equation of motion for an operator in the Heisenberg picture, assuming the operator is independent of time in the Schrödinger picture.

State clearly the form of the unitary operator corresponding to a rotation through an angle θ about an axis $\mathbf{n}$ (a unit vector) for a general quantum system. Verify your statement for the case in which the system is a single particle by considering the effect of an infinitesimal rotation on the particle's position $\hat{\mathbf{x}}$ and on its spin $\mathbf{S}$.

Show that if the Hamiltonian for a particle is of the form

$$H = \frac{1}{2m}\hat{\mathbf{p}}^2 + U(\hat{\mathbf{x}}^2)\hat{\mathbf{x}}\cdot\mathbf{S}$$

then all components of the total angular momentum are independent of time in the Heisenberg picture. Is the same true for either orbital or spin angular momentum?

[*You may quote commutation relations involving components of* $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$, $\mathbf{L}$ *and* $\mathbf{S}$.]

33D Applications of Quantum Mechanics

For the one-dimensional potential

$$V(x) = -\frac{\hbar^2\lambda}{m}\sum_n \delta(x-na)\,,$$

solve the Schrödinger equation for negative energy and obtain an equation that determines possible energy bands. Show that the results agree with the tight-binding model in appropriate limits.

$$\left[\textit{It may be useful to note that } V(x) = -\frac{\hbar^2\lambda}{ma}\sum_n e^{2\pi i n x/a}.\right]$$

34D Statistical Physics

Two examples of phenomenological temperature measurements are (i) the mark reached along the length of a liquid-in-glass thermometer; and (ii) the wavelength of the brightest colour of electromagnetic radiation emitted by a hot body (used, for example, to measure the surface temperature of a star).

Give the definition of temperature in statistical physics, and explain how the analysis of ideal gases and black body radiation is used to calibrate and improve phenomenological temperature measurements like those mentioned above. You should give brief derivations of any key results that you use.

35E Electrodynamics

The retarded scalar potential produced by a charge distribution $\rho(t', \mathbf{x}')$ is

$$\varphi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\rho(t - R, \mathbf{x}')}{R},$$

where $R = |\mathbf{R}|$ and $\mathbf{R} = \mathbf{x} - \mathbf{x}'$. By use of an appropriate delta function rewrite the integral as an integral over both d^3x' and dt' involving $\rho(t', \mathbf{x}')$.

Now specialize to a point charge q moving on a path $\mathbf{x}' = \mathbf{x}_0(t')$ so that we may set

$$\rho(t', \mathbf{x}') = q \, \delta^{(3)}(\mathbf{x}' - \mathbf{x}_0(t')).$$

By performing the volume integral first obtain the Liénard–Wiechert potential

$$\varphi(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0} \frac{1}{(R^* - \mathbf{v} \cdot \mathbf{R}^*)} \ ,$$

where $\mathbf{R}^*$ and $\mathbf{v}$ should be specified.

Obtain the corresponding result for the magnetic potential.

36A General Relativity

What are local inertial co-ordinates? What is their physical significance and how are they related to the equivalence principle?

If V_a are the components of a covariant vector field, show that

$$\partial_a V_b - \partial_b V_a$$

are the components of an anti-symmetric second rank covariant tensor field.

If K^a are the components of a contravariant vector field and g_{ab} the components of a metric tensor, let

$$Q_{ab} = K^c \partial_c g_{ab} + g_{ac} \partial_b K^c + g_{cb} \partial_a K^c \ .$$

Show that

$$Q_{ab} = 2\nabla_{(a} K_{b)} \ ,$$

where $K_a = g_{ab} K^b$, and ∇_a is the Levi–Civita covariant derivative operator of the metric g_{ab}.

In a particular co-ordinate system (x^1, x^2, x^3, x^4), it is given that $K^a = (0,0,0,1)$, $Q_{ab} = 0$. Deduce that, in this co-ordinate system, the metric tensor g_{ab} is independent of the co-ordinate x^4. Hence show that

$$\nabla_a K_b = \tfrac{1}{2}\big(\partial_a K_b - \partial_b K_a\big) \ ,$$

and that

$$E = -K_a \frac{dx^a}{d\lambda} \ ,$$

is constant along every geodesic $x^a(\lambda)$ in every co-ordinate system.

What further conditions must one impose on K^a and $dx^a/d\lambda$ to ensure that the metric is stationary and that E is proportional to the energy of a particle moving along the geodesic?

37B Fluid Dynamics II

A line force of magnitude F is applied in the positive x-direction to an unbounded fluid, generating a thin two-dimensional jet along the positive x-axis. The fluid is at rest at $y = \pm\infty$ and there is negligible motion in $x < 0$. Write down the pressure gradient within the boundary layer. Deduce that the function $M(x)$ defined by

$$M(x) = \int_{-\infty}^{\infty} \rho u^2(x,y)\, dy$$

is independent of x for $x > 0$. Interpret this result, and explain why $M = F$. Use scaling arguments to deduce that there is a similarity solution having stream function

$$\psi = (F\nu x/\rho)^{1/3} f(\eta) \quad \text{where} \quad \eta = y(F/\rho\nu^2 x^2)^{1/3} \ .$$

Hence show that f satisfies

$$3f''' + f'^2 + ff'' = 0 \ . \qquad (*)$$

Show that a solution of $(*)$ is

$$f(\eta) = A\tanh(A\eta/6) \ ,$$

where A is a constant to be determined by requiring that M is independent of x. Find the volume flux, $Q(x)$, in the jet. Briefly indicate why $Q(x)$ increases as x increases.

[*Hint: You may use* $\int_{-\infty}^{\infty} \operatorname{sech}^4(x)dx = 4/3$.]

38C Waves

Obtain an expression for the compressive energy $W(\rho)$ per unit volume for adiabatic motion of a perfect gas, for which the pressure p is given in terms of the density ρ by a relation of the form

$$p = p_0(\rho/\rho_0)^{\gamma} \ , \qquad (*)$$

where p_0, ρ_0 and γ are positive constants.

For one-dimensional motion with speed u write down expressions for the mass flux and the momentum flux. Deduce from the energy flux $u\big(p + W + \frac{1}{2}\rho u^2\big)$ together with the mass flux that if the motion is steady then

$$\frac{\gamma}{\gamma - 1}\,\frac{p}{\rho} \ + \ \tfrac{1}{2}u^2 \ = \ \text{constant.} \qquad (\dagger)$$

A one-dimensional shock wave propagates at constant speed along a tube containing the gas. Ahead of the shock the gas is at rest with pressure p_0 and density ρ_0. Behind the shock the pressure is maintained at the constant value $(1+\beta)p_0$ with $\beta > 0$. Determine the density ρ_1 behind the shock, assuming that $(\dagger)$ holds throughout the flow.

For small β show that the changes in pressure and density across the shock satisfy the adiabatic relation $(*)$ approximately, correeect to order β^2.

39C Numerical Analysis

The difference equation

$$u_m^{n+1} = u_m^n + \tfrac{3}{2}\mu\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right) - \tfrac{1}{2}\mu\left(u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1}\right) ,$$

where $\mu = \Delta t/(\Delta x)^2$, is used to approximate a solution of the diffusion equation $u_t = u_{xx}$.

(a) Prove that, as $\Delta t \to 0$ with constant μ, the local error of the method is $\mathcal{O}(\Delta t)^2$.

(b) Applying the Fourier stability test, show that the method is stable if and only if $\mu \leqslant \frac{1}{4}$.

END OF PAPER