UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday 6th June, 2006 9 to 12

PAPER 1

Before you begin read these instructions carefully.

*Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most **four** questions from Section I and at most **six** questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in separate bundles labelled **A, B, . . . , H** according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Linear Algebra

Define what is meant by the *minimal polynomial* of a complex $n \times n$ matrix, and show that it is unique. Deduce that the minimal polynomial of a real $n \times n$ matrix has real coefficients.

For $n > 2$, find an $n \times n$ matrix with minimal polynomial $(t-1)^2(t+1)$.

2H Geometry

Define the hyperbolic metric in the upper half-plane model H of the hyperbolic plane. How does one define the hyperbolic area of a region in H? State the Gauss–Bonnet theorem for hyperbolic triangles.

Let R be the region in H defined by

$$0 < x < \frac{1}{2}, \quad \sqrt{1-x^2} < y < 1.$$

Calculate the hyperbolic area of R.

3D Complex Analysis or Complex Methods

Let L be the Laplace operator, i.e., $L(g) = g_{xx} + g_{yy}$. Prove that if $f : \Omega \to \mathbf{C}$ is analytic in a domain Ω, then

$$L\big(|f(z)|^2\big) = 4|f'(z)|^2 , \quad z \in \Omega.$$

4B Special Relativity

A ball of clay of mass m travels at speed v in the laboratory frame towards an identical ball at rest. After colliding head-on, the balls stick together, moving in the same direction as the first ball was moving before the collision. Calculate the mass m' and speed v' of the combined lump, justifying your answers carefully.

5A Fluid Dynamics

Use the Euler equation for the motion of an inviscid fluid to derive the vorticity equation in the form

$$D\boldsymbol{\omega}/Dt = \boldsymbol{\omega} \cdot \nabla \mathbf{u}.$$

Give a physical interpretation of the terms in this equation and deduce that irrotational flows remain irrotational.

In a plane flow the vorticity at time $t = 0$ has the uniform value $\boldsymbol{\omega}_0 \neq \mathbf{0}$. Find the vorticity everywhere at times $t > 0$.

6D Numerical Analysis

(a) Perform the LU-factorization with column pivoting of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

(b) Explain briefly why every nonsingular matrix A admits an LU-factorization with column pivoting.

7C Statistics

A random sample $X_1, \ldots, X_n$ is taken from a normal distribution having unknown mean θ and variance 1. Find the maximum likelihood estimate $\hat{\theta}_M$ for θ based on $X_1, \ldots, X_n$.

Suppose that we now take a Bayesian point of view and regard θ itself as a normal random variable of known mean μ and variance τ^{-1}. Find the Bayes' estimate $\hat{\theta}_B$ for θ based on $X_1, \ldots, X_n$, corresponding to the quadratic loss function $(\theta - a)^2$.

8C Optimization

State the Lagrangian sufficiency theorem.

Let $p \in (1, \infty)$ and let $a_1, \ldots, a_n \in \mathbb{R}$. Maximize

$$\sum_{i=1}^{n} a_i x_i$$

subject to

$$\sum_{i=1}^{n} |x_i|^p \leqslant 1, \quad x_1, \ldots, x_n \in \mathbb{R}.$$

SECTION II

9H Linear Algebra

Let U, V be finite-dimensional vector spaces, and let θ be a linear map of U into V. Define the *rank* $r(\theta)$ and the *nullity* $n(\theta)$ of θ, and prove that

$$r(\theta) + n(\theta) = \dim U.$$

Now let θ, ϕ be endomorphisms of a vector space U. Define the endomorphisms $\theta + \phi$ and $\theta\phi$, and prove that

$$\begin{aligned} r(\theta + \phi) &\leqslant r(\theta) + r(\phi) \\ n(\theta\phi) &\leqslant n(\theta) + n(\phi). \end{aligned}$$

Prove that equality holds in **both** inequalities if and only if $\theta + \phi$ is an isomorphism and $\theta\phi$ is zero.

10E Groups, Rings and Modules

Find all subgroups of indices 2, 3, 4 and 5 in the alternating group A_5 on 5 letters. You may use any general result that you choose, provided that you state it clearly, but you must justify your answers.

[*You may take for granted the fact that A_4 has no subgroup of index 2.*]

11F Analysis II

Let a_n and b_n be sequences of real numbers for $n \geqslant 1$ such that $|a_n| \leqslant c/n^{1+\epsilon}$ and $|b_n| \leqslant c/n^{1+\epsilon}$ for all $n \geqslant 1$, for some constants $c > 0$ and $\epsilon > 0$. Show that the series

$$f(x) = \sum_{n \geqslant 1} a_n \cos nx + \sum_{n \geqslant 1} b_n \sin nx$$

converges uniformly to a continuous function on the real line. Show that f is periodic in the sense that $f(x + 2\pi) = f(x)$.

Now suppose that $|a_n| \leqslant c/n^{2+\epsilon}$ and $|b_n| \leqslant c/n^{2+\epsilon}$ for all $n \geqslant 1$, for some constants $c > 0$ and $\epsilon > 0$. Show that f is differentiable on the real line, with derivative

$$f'(x) = -\sum_{n \geqslant 1} n a_n \sin nx + \sum_{n \geqslant 1} n b_n \cos nx.$$

[*You may assume the convergence of standard series.*]

12F Metric and Topological Spaces

(i) Define the product topology on $X \times Y$ for topological spaces X and Y, proving that your definition does define a topology.

(ii) Let X be the logarithmic spiral defined in polar coordinates by $r = e^{\theta}$, where $-\infty < \theta < \infty$. Show that X (with the subspace topology from $\mathbf{R}^2$) is homeomorphic to the real line.

13D Complex Analysis or Complex Methods

By integrating round the contour involving the real axis and the line $\mathrm{Im}(z) = 2\pi$, or otherwise, evaluate

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\, dx\,, \quad 0 < a < 1.$$

Explain why the given restriction on the value a is necessary.

14A Methods

Define a *second rank tensor*. Show from your definition that if M_{ij} is a second rank tensor then M_{ii} is a scalar.

A rigid body consists of a thin flat plate of material having density $\rho(\mathbf{x})$ per unit area, where $\mathbf{x}$ is the position vector. The body occupies a region D of the (x, y)-plane; its thickness in the z-direction is negligible. The moment of inertia tensor of the body is given as

$$M_{ij} = \int_D (x_k x_k \delta_{ij} - x_i x_j)\rho\, dS.$$

Show that the z-direction is an eigenvector of M_{ij} and write down an integral expression for the corresponding eigenvalue $M_{\perp}$.

Hence or otherwise show that if the remaining eigenvalues of M_{ij} are M_1 and M_2 then

$$M_{\perp} = M_1 + M_2.$$

Find M_{ij} for a circular disc of radius a and uniform density having its centre at the origin.

15B Quantum Mechanics

Let $V_1(x)$ and $V_2(x)$ be two real potential functions of one space dimension, and let a be a positive constant. Suppose also that $V_1(x) \leqslant V_2(x) \leqslant 0$ for all x and that $V_1(x) = V_2(x) = 0$ for all x such that $|x| \geqslant a$. Consider an incoming beam of particles described by the plane wave $\exp(ikx)$, for some $k > 0$, scattering off one of the potentials $V_1(x)$ or $V_2(x)$. Let p_i be the probability that a particle in the beam is reflected by the potential $V_i(x)$. Is it necessarily the case that $p_1 \leqslant p_2$? Justify your answer carefully, either by giving a rigorous proof or by presenting a counterexample with explicit calculations of p_1 and p_2.

16G Electromagnetism

Three concentric conducting spherical shells of radii a, b and c $(a < b < c)$ carry charges $q, -2q$ and $3q$ respectively. Find the electric field and electric potential at all points of space.

Calculate the total energy of the electric field.

17A Fluid Dynamics

A point source of fluid of strength m is located at $\mathbf{x}_s = (0, 0, a)$ in inviscid fluid of density ρ. Gravity is negligible. The fluid is confined to the region $z \geqslant 0$ by the fixed boundary $z = 0$. Write down the equation and boundary conditions satisfied by the velocity potential ϕ. Find ϕ.

[*Hint: consider the flow generated in unbounded fluid by the source m together with an 'image source' of equal strength at* $\bar{\mathbf{x}}_s = (0, 0, -a)$.]

Use Bernoulli's theorem, which may be stated without proof, to find the fluid pressure everywhere on $z = 0$. Deduce the magnitude of the hydrodynamic force on the boundary $z = 0$. Determine whether the boundary is attracted toward the source or repelled from it.

18C Statistics

Let X be a random variable whose distribution depends on an unknown parameter θ. Explain what is meant by a sufficient statistic $T(X)$ for θ.

In the case where X is discrete, with probability mass function $f(x|\theta)$, explain, with justification, how a sufficient statistic may be found.

Assume now that $X = (X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are independent non-negative random variables with common density function

$$f(x|\theta) = \begin{cases} \lambda e^{-\lambda(x-\theta)} & \text{if } x \geqslant \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\theta \geq 0$ is unknown and λ is a known positive parameter. Find a sufficient statistic for θ and hence obtain an unbiased estimator $\hat{\theta}$ for θ of variance $(n\lambda)^{-2}$.

[*You may use without proof the following facts: for independent exponential random variables X and Y, having parameters λ and μ respectively, X has mean λ^{-1} and variance λ^{-2} and min$\{X, Y\}$ has exponential distribution of parameter $\lambda + \mu$.*]

19C Markov Chains

Explain what is meant by a stopping time of a Markov chain $(X_n)_{n \geq 0}$. State the strong Markov property.

Show that, for any state i, the probability, starting from i, that $(X_n)_{n \geq 0}$ makes infinitely many visits to i can take only the values 0 or 1.

Show moreover that, if

$$\sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) = \infty,$$

then $(X_n)_{n \geq 0}$ makes infinitely many visits to i with probability 1.

END OF PAPER