UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 5th June, 2006 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt **all four** *questions from Section I and* **at most five** *questions from Section II. In Section II, no more than* **three** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **C** *and* **E** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

Explain what is meant by a prime number.

By considering numbers of the form $6p_1p_2\cdots p_n - 1$, show that there are infinitely many prime numbers of the form $6k-1$.

By considering numbers of the form $(2p_1p_2\cdots p_n)^2+3$, show that there are infinitely many prime numbers of the form $6k+1$. [*You may assume the result that, for a prime* $p>3$, *the congruence* $x^2 \equiv -3 \pmod{p}$ *is soluble only if* $p \equiv 1 \pmod{6}$.]

2E Numbers and Sets

Define the binomial coefficient $\binom{n}{r}$ and prove that

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \qquad \text{for } 0 < r \leqslant n.$$

Show also that if p is prime then $\binom{p}{r}$ is divisible by p for $0<r<p$.

Deduce that if $0 \leqslant k < p$ and $0 \leqslant r \leqslant k$ then

$$\binom{p+k}{r} \equiv \binom{k}{r} \qquad \pmod{p}.$$

3C Dynamics

A car is at rest on a horizontal surface. The engine is switched on and suddenly sets the wheels spinning at a constant angular velocity Ω. The wheels have radius r and the coefficient of friction between the ground and the surface of the wheels is μ. Calculate the time T when the wheels start rolling without slipping. If the car is started on an upward slope in a similar manner, explain whether T is increased or decreased relative to the case where the car starts on a horizontal surface.

4C Dynamics

For the dynamical system

$$\ddot{x} = -\sin x,$$

find the stable and unstable fixed points and the equation determining the separatrix. Sketch the phase diagram. If the system starts on the separatrix at $x=0$, write down an integral determining the time taken for the velocity $\dot{x}$ to reach zero. Show that the integral is infinite.

SECTION II

5E Numbers and Sets

Explain what is meant by an *equivalence relation* on a set A.

If R and S are two equivalence relations on the same set A, we define

$$R \circ S = \{(x, z) \in A \times A : \text{there exists } y \in A \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}.$$

Show that the following conditions are equivalent:

(i) $R \circ S$ is a symmetric relation on A;

(ii) $R \circ S$ is a transitive relation on A;

(iii) $S \circ R \subseteq R \circ S$;

(iv) $R \circ S$ is the unique smallest equivalence relation on A containing both R and S.

Show also that these conditions hold if $A = \mathbb{Z}$ and R and S are the relations of congruence modulo m and modulo n, for some positive integers m and n.

6E Numbers and Sets

State and prove the Inclusion–Exclusion Principle.

A permutation σ of $\{1, 2, \ldots, n\}$ is called a *derangement* if $\sigma(j) \neq j$ for every $j \leqslant n$. Use the Inclusion–Exclusion Principle to find a formula for the number $f(n)$ of derangements of $\{1, 2, \ldots, n\}$. Show also that $f(n)/n!$ converges to $1/e$ as $n \to \infty$.

7E Numbers and Sets

State and prove Fermat's Little Theorem.

An odd number n is called a *Carmichael number* if it is not prime, but every positive integer a satisfies $a^n \equiv a \pmod{n}$. Show that a Carmichael number cannot be divisible by the square of a prime. Show also that a product of two distinct odd primes cannot be a Carmichael number, and that a product of three distinct odd primes p, q, r is a Carmichael number if and only if $p - 1$ divides $qr - 1$, $q - 1$ divides $pr - 1$ and $r - 1$ divides $pq - 1$. Deduce that 1729 is a Carmichael number.

[*You may assume the result that, for any prime p, there exists a number g prime to p such that the congruence $g^d \equiv 1 \pmod{p}$ holds only when d is a multiple of $p - 1$. The prime factors of* 1729 *are* $7, 13$ *and* 19.]

8E Numbers and Sets

Explain what it means for a set to be countable. Prove that a countable union of countable sets is countable, and that the set of all subsets of $\mathbb{N}$ is uncountable.

A function $f\colon \mathbb{N} \to \mathbb{N}$ is said to be increasing if $f(m) \leqslant f(n)$ whenever $m \leqslant n$, and decreasing if $f(m) \geqslant f(n)$ whenever $m \leqslant n$. Show that the set of all increasing functions $\mathbb{N} \to \mathbb{N}$ is uncountable, but that the set of decreasing functions is countable.

[*Standard results on countability, other than those you are asked to prove, may be assumed.*]

9C Dynamics

A motorcycle of mass M moves on a bowl-shaped surface specified by its height $h(r)$ where $r = \sqrt{x^2+y^2}$ is the radius in cylindrical polar coordinates (r,ϕ,z). The torque exerted by the motorcycle engine on the rear wheel results in a force $\mathbf{F}(t)$ pushing the motorcycle forward. Assuming $\mathbf{F}(t)$ is directed along the motorcycle's velocity and that the motorcycle's vertical velocity and acceleration are small, show that the motion is described by

$$\ddot{r} - r\dot{\phi}^2 = -g\frac{dh}{dr} + \frac{F(t)}{M}\frac{\dot{r}}{\sqrt{\dot{r}^2 + r^2\dot{\phi}^2}}\,,$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = \frac{F(t)}{M}\frac{r\dot{\phi}}{\sqrt{\dot{r}^2 + r^2\dot{\phi}^2}}\,,$$

where dots denote time derivatives, $F(t) = |\mathbf{F}(t)|$ and g is the acceleration due to gravity.

The motorcycle rider can adjust $F(t)$ to produce the desired trajectory. If the rider wants to move on a curve $r(\phi)$, show that $\phi(t)$ must obey

$$\dot{\phi}^2 = g\,\frac{dh}{dr}\bigg/\left(r + \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - \frac{d^2r}{d\phi^2}\right).$$

Now assume that $h(r) = r^2/\ell$, with ℓ a constant, and $r(\phi) = \epsilon\phi$ with ϵ a positive constant, and $0 \leqslant \phi < \infty$ so that the desired trajectory is a spiral curve. Assuming that $\phi(t)$ tends to infinity as t tends to infinity, show that $\dot{\phi}(t)$ tends to $\sqrt{2g/\ell}$ and $F(t)$ tends to $4\epsilon Mg/\ell$ as t tends to infinity.

10C Dynamics

A particle of mass m bounces back and forth between two walls of mass M moving towards each other in one dimension. The walls are separated by a distance $\ell(t)$. The wall on the left has velocity $+V(t)$ and the wall on the right has velocity $-V(t)$. The particle has speed $v(t)$. Friction is negligible and the particle–wall collisions are elastic.

Consider a collision between the particle and the wall on the right. Show that the centre–of–mass velocity of the particle–wall system is $v_{\rm cm} = (mv - MV)/(m + M)$. Calculate the particle's speed following the collision.

Assume that the particle is much lighter than the walls, i.e., $m \ll M$. Show that the particle's speed increases by approximately $2V$ every time it collides with a wall.

Assume also that $v \gg V$ (so that particle–wall collisions are frequent) and that the velocities of the two walls remain nearly equal and opposite. Show that in a time interval Δt, over which the change in V is negligible, the wall separation changes by $\Delta\ell \approx -2V\Delta t$. Show that the number of particle–wall collisions during Δt is approximately $v\Delta t/\ell$ and that the particle's speed increases by $\Delta v \approx -(\Delta\ell/\ell)v$ during this time interval.

Hence show that under the given conditions the particle speed v is approximately proportional to ℓ^{-1}.

11C Dynamics

Two light, rigid rods of length 2ℓ have a mass m attached to each end. Both are free to move in two dimensions. The two rods are placed so that their two ends are located at $(-d, +2\ell)$, $(-d, 0)$, and $(+d, 0)$, $(+d, -2\ell)$ respectively, where d is positive. They are set in motion with no rotation, with centre–of–mass velocities $(+V, 0)$ and $(-V, 0)$, so that the lower mass on the first rod collides head on with the upper mass on the second rod at the origin $(0, 0)$. [*You may assume that the impulse is directed along the x-axis.*]

Assuming the collision is elastic, calculate the centre–of–mass velocity $\boldsymbol{v}$ and the angular velocity $\boldsymbol{\omega}$ of each rod immediately after the collision.

Assuming a coefficient of restitution e, compute $\boldsymbol{v}$ and $\boldsymbol{\omega}$ for each rod after the collision.

12C Dynamics

A particle of mass m and charge $q > 0$ moves in a time-dependent magnetic field $\mathbf{B} = (0, 0, B_z(t))$.

Write down the equations of motion governing the particle's x, y and z coordinates.

Show that the speed of the particle in the (x, y) plane, $V = \sqrt{\dot{x}^2 + \dot{y}^2}$, is a constant.

Show that the general solution of the equations of motion is

$$\begin{aligned} x(t) &= x_0 + V \int_0^t dt' \cos\left(-\int_0^{t'} dt'' q \frac{B_z(t'')}{m} + \phi \right), \\ y(t) &= y_0 + V \int_0^t dt' \sin\left(-\int_0^{t'} dt'' q \frac{B_z(t'')}{m} + \phi \right), \\ z(t) &= z_0 + v_z t, \end{aligned}$$

and interpret each of the six constants of integration, x_0, y_0, z_0, v_z, V and ϕ. [*Hint: Solve the equations for the particle's* **velocity** *in cylindrical polars.*]

Let $B_z(t) = \beta t$, where β is a positive constant. Assuming that $x_0 = y_0 = z_0 = v_z = \phi = 0$ and $V = 1$, calculate the position of the particle in the limit $t \to \infty$ (you may assume this limit exists). [*Hint: You may use the results* $\int_0^\infty dx \, \cos(x^2) = \int_0^\infty dx \, \sin(x^2) = \sqrt{\pi/8}$.]

END OF PAPER