UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 1st June, 2006 9 to 12

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt **all four** *questions from Section I and* **at most five** *questions from Section II. In Section II, no more than* **three** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **A**, **B**, **C**, **D**, **E** *and* **F** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Algebra and Geometry

Consider the cone K in $\mathbb{R}^3$ defined by

$$x_3^2 = x_1^2 + x_2^2, \quad x_3 > 0.$$

Find a unit normal $\mathbf{n} = (n_1, n_2, n_3)$ to K at the point $\mathbf{x} = (x_1, x_2, x_3)$ such that $n_3 \geqslant 0$. Show that if $\mathbf{p} = (p_1, p_2, p_3)$ satisfies

$$p_3^2 \geqslant p_1^2 + p_2^2$$

and $p_3 \geqslant 0$ then

$$\mathbf{p} \cdot \mathbf{n} \geqslant 0.$$

2A Algebra and Geometry

Express the unit vector $\mathbf{e}_r$ of spherical polar coordinates in terms of the orthonormal Cartesian basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Express the equation for the paraboloid $z = x^2 + y^2$ in (i) cylindrical polar coordinates (ρ, ϕ, z) and (ii) spherical polar coordinates (r, θ, ϕ).

In spherical polar coordinates, a surface is defined by $r^2 \cos 2\theta = a$, where a is a real non-zero constant. Find the corresponding equation for this surface in Cartesian coordinates and sketch the surfaces in the two cases $a > 0$ and $a < 0$.

3F Analysis

Let $a_n \in \mathbb{R}$ for $n \geqslant 1$. What does it mean to say that the infinite series $\sum_n a_n$ converges to some value A? Let $s_n = a_1 + \cdots + a_n$ for all $n \geqslant 1$. Show that if $\sum_n a_n$ converges to some value A, then the sequence whose n-th term is

$$(s_1 + \cdots + s_n)/n$$

converges to some value $\tilde{A}$ as $n \to \infty$. Is it always true that $A = \tilde{A}$? Give an example where $(s_1 + \cdots + s_n)/n$ converges but $\sum_n a_n$ does not.

4D Analysis

Let $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ be power series in the complex plane with radii of convergence R and S respectively. Show that if $R \neq S$ then $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ has radius of convergence $\min(R, S)$. [*Any results on absolute convergence that you use should be clearly stated.*]

SECTION II

5C Algebra and Geometry

Prove the Cauchy–Schwarz inequality,

$$|\mathbf{x} \cdot \mathbf{y}| \leqslant |\mathbf{x}||\mathbf{y}|,$$

for two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$. Under what condition does equality hold?

Consider a pyramid in $\mathbb{R}^n$ with vertices at the origin O and at $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, where $\mathbf{e}_1 = (1, 0, 0, \ldots)$, $\mathbf{e}_2 = (0, 1, 0, \ldots)$, and so on. The "base" of the pyramid is the $(n-1)$-dimensional object B specified by $(\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n) \cdot \mathbf{x} = 1$, $\mathbf{e}_i \cdot \mathbf{x} \geqslant 0$ for $i = 1, \ldots, n$.

Find the point C in B equidistant from each vertex of B and find the length of OC. (C is the centroid of B.)

Show, using the Cauchy–Schwarz inequality, that this is the closest point in B to the origin O.

Calculate the angle between OC and any edge of the pyramid connected to O. What happens to this angle and to the length of OC as n tends to infinity?

6C Algebra and Geometry

Given a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, write down the vector $\mathbf{x}'$ obtained by rotating $\mathbf{x}$ through an angle θ.

Given a unit vector $\mathbf{n} \in \mathbb{R}^3$, any vector $\mathbf{x} \in \mathbb{R}^3$ may be written as $\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}$ where $\mathbf{x}_{\parallel}$ is parallel to $\mathbf{n}$ and $\mathbf{x}_{\perp}$ is perpendicular to $\mathbf{n}$. Write down explicit formulae for $\mathbf{x}_{\parallel}$ and $\mathbf{x}_{\perp}$, in terms of $\mathbf{n}$ and $\mathbf{x}$. Hence, or otherwise, show that the linear map

$$\mathbf{x} \mapsto \mathbf{x}' = (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos\theta(\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}) + \sin\theta(\mathbf{n} \times \mathbf{x}) \qquad (*)$$

describes a rotation about $\mathbf{n}$ through an angle θ, in the positive sense defined by the right hand rule.

Write equation $(*)$ in matrix form, $x'_i = R_{ij}x_j$. Show that the trace $R_{ii} = 1 + 2\cos\theta$.

Given the rotation matrix

$$R = \frac{1}{2}\begin{pmatrix} 1+r & 1-r & 1 \\ 1-r & 1+r & -1 \\ -1 & 1 & 2r \end{pmatrix},$$

where $r = 1/\sqrt{2}$, find the two pairs $(\theta, \mathbf{n})$, with $-\pi \leqslant \theta < \pi$, giving rise to R. Explain why both represent the same rotation.

7B Algebra and Geometry

(i) Let $\mathbf{u}, \mathbf{v}$ be unit vectors in $\mathbb{R}^3$. Write the transformation on vectors $\mathbf{x} \in \mathbb{R}^3$

$$\mathbf{x} \mapsto (\mathbf{u} \cdot \mathbf{x})\mathbf{u} + \mathbf{v} \times \mathbf{x}$$

in matrix form as $\mathbf{x} \mapsto A\mathbf{x}$ for a matrix A. Find the eigenvalues in the two cases (a) when $\mathbf{u} \cdot \mathbf{v} = 0$, and (b) when $\mathbf{u}, \mathbf{v}$ are parallel.

(ii) Let $\mathcal{M}$ be the set of 2×2 complex hermitian matrices with trace zero. Show that if $A \in \mathcal{M}$ there is a unique vector $\mathbf{x} \in \mathbb{R}^3$ such that

$$A = \mathcal{R}(\mathbf{x}) = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}.$$

Show that if U is a 2×2 unitary matrix, the transformation

$$A \mapsto U^{-1}AU$$

maps $\mathcal{M}$ to $\mathcal{M}$, and that if $U^{-1}\mathcal{R}(\mathbf{x})U = \mathcal{R}(\mathbf{y})$, then $\|\mathbf{x}\| = \|\mathbf{y}\|$ where $\| \cdot \|$ means ordinary Euclidean length. [*Hint: Consider determinants.*]

8A Algebra and Geometry

(i) State de Moivre's theorem. Use it to express $\cos 5\theta$ as a polynomial in $\cos\theta$.

(ii) Find the two fixed points of the Möbius transformation

$$z \longmapsto \omega = \frac{3z+1}{z+3}\,,$$

that is, find the two values of z for which $\omega = z$.

Given that $c \neq 0$ and $(a-d)^2+4bc \neq 0$, show that a general Möbius transformation

$$z \longmapsto \omega = \frac{az+b}{cz+d}\,, \qquad ad-bc \neq 0\,,$$

has two fixed points α, β given by

$$\alpha = \frac{a-d+m}{2c}\,, \qquad \beta = \frac{a-d-m}{2c}\,,$$

where $\pm m$ are the square roots of $(a-d)^2+4bc$.

Show that such a transformation can be expressed in the form

$$\frac{\omega-\alpha}{\omega-\beta} = k\frac{z-\alpha}{z-\beta}\,,$$

where k is a constant that you should determine.

9E Analysis

State and prove the Intermediate Value Theorem.

Suppose that the function f is differentiable everywhere in some open interval containing $[a,b]$, and that $f'(a) < k < f'(b)$. By considering the functions g and h defined by

$$g(x) = \frac{f(x)-f(a)}{x-a} \quad (a < x \leqslant b)\,, \quad g(a) = f'(a)$$

and

$$h(x) = \frac{f(b)-f(x)}{b-x} \quad (a \leqslant x < b)\,, \quad h(b) = f'(b),$$

or otherwise, show that there is a subinterval $[a',b'] \subseteq [a,b]$ such that

$$\frac{f(b')-f(a')}{b'-a'} = k.$$

Deduce that there exists $c \in (a,b)$ with $f'(c) = k$. [*You may assume the Mean Value Theorem.*]

10E Analysis

Prove that if the function f is infinitely differentiable on an interval (r, s) containing a, then for any $x \in (r, s)$ and any positive integer n we may expand $f(x)$ in the form

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(f, a, x),$$

where the remainder term $R_n(f, a, x)$ should be specified explicitly in terms of $f^{(n+1)}$.

Let $p(t)$ be a nonzero polynomial in t, and let f be the real function defined by

$$f(x) = p\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right) \quad (x \neq 0)\ , \quad f(0) = 0.$$

Show that f is differentiable everywhere and that

$$f'(x) = q\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right) \quad (x \neq 0)\ , \quad f'(0) = 0,$$

where $q(t) = 2t^3 p(t) - t^2 p'(t)$. Deduce that f is infinitely differentiable, but that there exist arbitrarily small values of x for which the remainder term $R_n(f, 0, x)$ in the Taylor expansion of f about 0 does not tend to 0 as $n \to \infty$.

11F Analysis

Consider a sequence $(a_n)_{n \geqslant 1}$ of real numbers. What does it mean to say that $a_n \to a \in \mathbb{R}$ as $n \to \infty$? What does it mean to say that $a_n \to \infty$ as $n \to \infty$? What does it mean to say that $a_n \to -\infty$ as $n \to \infty$? Show that for every sequence of real numbers there exists a subsequence which converges to a value in $\mathbb{R} \cup \{\infty, -\infty\}$. [*You may use the Bolzano–Weierstrass theorem provided it is clearly stated.*]

Give an example of a bounded sequence $(a_n)_{n \geqslant 1}$ which is not convergent, but for which

$$a_{n+1} - a_n \to 0 \quad \text{as} \quad n \to \infty.$$

12D Analysis

Let f_1 and f_2 be Riemann integrable functions on $[a, b]$. Show that $f_1 + f_2$ is Riemann integrable.

Let f be a Riemann integrable function on $[a, b]$ and set $f^+(x) = \max(f(x), 0)$. Show that f^+ and $|f|$ are Riemann integrable.

Let f be a function on $[a, b]$ such that $|f|$ is Riemann integrable. Is it true that f is Riemann integrable? Justify your answer.

Show that if f_1 and f_2 are Riemann integrable on $[a, b]$, then so is $\max(f_1, f_2)$. Suppose now $f_1, f_2, \ldots$ is a sequence of Riemann integrable functions on $[a, b]$ and $f(x) = \sup_n f_n(x)$; is it true that f is Riemann integrable? Justify your answer.

END OF PAPER