UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday 9 June 2005 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***,...,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Let $\pi(x)$ be the number of primes $p \leqslant x$. State the Legendre formula, and prove that

$$\lim_{x \to \infty} \frac{\pi(x)}{x} = 0.$$

[*You may use the formula*

$$\prod_{p \leqslant x} (1 - 1/p)^{-1} \geqslant \log x$$

without proof.]

2F Topics in Analysis

Let $-1 \leqslant x_1 < x_2 < \ldots < x_n \leqslant 1$ and let $a_1, a_2, \ldots, a_n$ be real numbers such that

$$\int_{-1}^{1} p(t)\, dt = \sum_{i=1}^{n} a_i p(x_i)$$

for every polynomial p of degree less than $2n$. Prove the following three facts.

(i) $a_i > 0$ for every i.

(ii) $\sum_{i=1}^{n} a_i = 2$.

(iii) The numbers $x_1, x_2, \ldots, x_n$ are the roots of the Legendre polynomial of degree n.

[*You may assume standard orthogonality properties of the Legendre polynomials.*]

3G Geometry of Group Actions

By considering fixed points in $\mathbb{C} \cup \{\infty\}$, prove that any complex Möbius transformation is conjugate either to a map of the form $z \mapsto kz$ for some $k \in \mathbb{C}$ or to $z \mapsto z + 1$. Deduce that two Möbius transformations g,h (neither the identity) are conjugate if and only if $\mathrm{tr}^2(g) = \mathrm{tr}^2(h)$.

Does every Möbius transformation g also have a fixed point in $\mathbb{H}^3$? Briefly justify your answer.

4J Coding and Cryptography

Briefly explain how and why a signature scheme is used. Describe the el Gamal scheme.

5I Statistical Modelling

Consider the model $Y = X\beta + \epsilon$, where Y is an n-dimensional observation vector, X is an $n \times p$ matrix of rank p, ϵ is an n-dimensional vector with components $\epsilon_1, \ldots, \epsilon_n$, and $\epsilon_1, \ldots, \epsilon_n$ are independently and normally distributed, each with mean 0 and variance σ^2.

(a) Let $\hat{\beta}$ be the least-squares estimator of β. Show that

$$(X^T X)\hat{\beta} = X^T Y$$

and find the distribution of $\hat{\beta}$.

(b) Define $\hat{Y} = X\hat{\beta}$. Show that $\hat{Y}$ has distribution $N(X\beta, \sigma^2 H)$, where H is a matrix that you should define.

[*You may quote without proof any results you require about the multivariate normal distribution.*]

6E Mathematical Biology

Let x be the concentration of a binary master sequence of length L and let y be the total concentration of all mutant sequences. Master sequences try to self-replicate at a total rate ax, but each independent digit is only copied correctly with probability q. Mutant sequences self-replicate at a total rate by, where $a > b$, and the probability of mutation back to the master sequence is negligible.

(a) The evolution of x is given by

$$\frac{dx}{dt} = aq^L x \ .$$

Write down the corresponding equation for y and derive a differential equation for the master-to-mutant ratio $z = x/y$.

(b) What is the maximum length $L_{\max}$ for which there is a positive steady-state value of z? Is the positive steady state stable when it exists?

(c) Obtain a first-order approximation to $L_{\max}$ assuming that both $1 - q \ll 1$ and $s \ll 1$, where the selection coefficient s is defined by $b = a(1 - s)$.

7B Dynamical Systems

Define the stable and unstable invariant subspaces of the linearisation of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ at a saddle point located at the origin in $\mathbb{R}^n$. How, according to the Stable Manifold Theorem, are the stable and unstable manifolds related to the invariant subspaces?

Calculate the stable and unstable manifolds, correct to cubic order, for the system

$$\begin{aligned} \dot{x} &= x + x^2 + 2xy + 3y^2 \\ \dot{y} &= -y + 3x^2 \,. \end{aligned}$$

8A Further Complex Methods

The functions f and g have Laplace transforms $\widehat{f}$ and $\widehat{g}$, and satisfy $f(t) = 0 = g(t)$ for $t < 0$. The convolution h of f and g is defined by

$$h(u) = \int_0^u f(u-v)g(v)dv$$

and has Laplace transform $\widehat{h}$. Prove (the convolution theorem) that $\widehat{h}(p) = \widehat{f}(p)\widehat{g}(p)$.

Given that $\int_0^t (t-s)^{-1/2} s^{-1/2}\, ds = \pi \quad (t > 0)$, deduce the Laplace transform of the function $f(t)$, where

$$f(t) = \begin{cases} t^{-1/2}, & t > 0 \\ 0, & t \leqslant 0. \end{cases}$$

9C Classical Dynamics

Define the Poisson bracket $\{f, g\}$ between two functions $f(q_a, p_a)$ and $g(q_a, p_a)$ on phase space. If $f(q_a, p_a)$ has no explicit time dependence, and there is a Hamiltonian H, show that Hamilton's equations imply

$$\frac{df}{dt} = \{f, H\} \,.$$

A particle with position vector $\mathbf{x}$ and momentum $\mathbf{p}$ has angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. Compute $\{p_a, L_b\}$ and $\{L_a, L_b\}$.

10D Cosmology

(a) Define and discuss the concept of the *cosmological horizon* and the *Hubble radius* for a homogeneous isotropic universe. Illustrate your discussion with the specific examples of the Einstein–de Sitter universe ($a \propto t^{2/3}$ for $t > 0$) and a de Sitter universe ($a \propto e^{Ht}$ with H constant, $t > -\infty$).

(b) Explain the *horizon problem* for a decelerating universe in which $a(t) \propto t^{\alpha}$ with $\alpha < 1$. How can inflation cure the horizon problem?

(c) Consider a Tolman (radiation-filled) universe ($a(t) \propto t^{1/2}$) beginning at $t_{\rm r} \sim 10^{-35}$s and lasting until today at $t_0 \approx 10^{17}$s. Estimate the horizon size today $d_H(t_0)$ and project this lengthscale backwards in time to show that it had a physical size of about 1 metre at $t \approx t_{\rm r}$.

Prior to $t \approx t_{\rm r}$, assume an inflationary (de Sitter) epoch with constant Hubble parameter H given by its value at $t \approx t_{\rm r}$ for the Tolman universe. How much expansion during inflation is required for the observable universe today to have begun inside one Hubble radius?

SECTION II

11H Number Theory

Show that there are exactly two reduced positive definite integer binary quadratic forms with discriminant -20; write these forms down.

State a criterion for an odd integer n to be properly represented by a positive definite integer binary quadratic form of given discriminant d.

Describe, in terms of congruences modulo 20, which primes other than $2, 5$ are properly represented by the form $x^2 + 5y^2$, and justify your answer.

12J Coding and Cryptography

Define a *cyclic code*. Define the generator and check polynomials of a cyclic code and show that they exist.

Show that Hamming's original code is a cyclic code with check polynomial $X^4 + X^2 + X + 1$. What is its generator polynomial? Does Hamming's original code contain a subcode equivalent to its dual?

13E Mathematical Biology

Protein synthesis by RNA can be represented by the stochastic system

$$\begin{aligned} x_1 &\xrightarrow{\lambda_1} x_1 + 1 \quad \text{and} \quad x_1 \xrightarrow{\beta_1 x_1} x_1 - 1 \\ x_2 &\xrightarrow{\lambda_2 x_1} x_2 + 1 \quad \text{and} \quad x_2 \xrightarrow{\beta_2 x_2} x_2 - 1 \end{aligned} \tag{1}$$

in which x_1 is an environmental variable corresponding to the number of RNA molecules per cell and x_2 is a system variable, with birth rate proportional to x_1, corresponding to the number of protein molecules.

(a) Use the normalized stationary Fluctuation–Dissipation Theorem (FDT) to calculate the (exact) normalized stationary variances $\eta_{11} = \sigma_1^2/{<}x_1{>}^2$ and $\eta_{22} = \sigma_2^2/{<}x_2{>}^2$ in terms of the averages ${<}x_1{>}$ and ${<}x_2{>}$.

(b) Separate η_{22} into an intrinsic and an extrinsic term by considering the limits when x_1 does not fluctuate (intrinsic), and when x_2 responds deterministically to changes in x_1 (extrinsic). Explain how the extrinsic term represents the magnitude of environmental fluctuations and time-averaging.

(c) Assume now that the birth rate of x_2 is changed from the "constitutive" mechanism $\lambda_2 x_1$ in (1) to a "negative feedback" mechanism $\lambda_2 x_1 f(x_2)$, where f is a monotonically decreasing function of x_2. Use the stationary FDT to approximate η_{22} in terms of $h = |\partial \ln f / \partial \ln x_2|$. Apply your answer to the case $f(x_2) = k/x_2$.

[*Hint: To reduce the algebra introduce the elasticity* $H_{22} = \partial \ln(R_2^-/R_2^+)/\partial \ln x_2$*, where* R_2^- *and* R_2^+ *are the death and birth rates of* x_2 *respectively.*]

(d) Explain the extrinsic term for the negative feedback system in terms of environmental fluctuations, time-averaging, and static susceptibility.

(e) Explain why the FDT is exact for the constitutive system but approximate for the feedback system. When, generally speaking, does the FDT approximation work well?

(f) Consider the following three experimental observations: (i) Large changes in λ_2 have no effect on η_{22}; (ii) When x_2 is perturbed by 1% from its stationary average, perturbations are corrected more rapidly in the feedback system than in the constitutive system; (iii) The feedback system displays lower values η_{22} than the constitutive system.

What does (i) imply about the relative importance of the noise terms? Can (ii) be directly explained by (iii), i.e., does rapid adjustment reduce noise? Justify your answers.

14A Further Complex Methods

Show that the equation

$$zw'' + 2kw' + zw = 0\,,$$

where k is constant, has solutions of the form

$$w(z) = \int_\gamma (t^2+1)^{k-1} e^{zt} dt$$

provided that the path γ is chosen so that $\left[(t^2+1)^k e^{zt}\right]_\gamma = 0\,.$

(i) In the case $\operatorname{Re} k > 0$, show that there is a choice of γ for which $w(0) = iB(k, \frac{1}{2})$.

(ii) In the case $k = n/2$, where n is any integer, show that γ can be a finite contour and that the corresponding solution satisfies $w(0) = 0$ if $n \leqslant -1$.

15C Classical Dynamics

(i) A point mass m with position q and momentum p undergoes one-dimensional periodic motion. Define the action variable I in terms of q and p. Prove that an orbit of energy E has period

$$T = 2\pi \frac{dI}{dE}\,.$$

(ii) Such a system has Hamiltonian

$$H(q,p) = \frac{p^2+q^2}{\mu^2-q^2}\,,$$

where μ is a positive constant and $|q| < \mu$ during the motion. Sketch the orbits in phase space both for energies $E \gg 1$ and $E \ll 1$. Show that the action variable I is given in terms of the energy E by

$$I = \frac{\mu^2}{2} \frac{E}{\sqrt{E+1}}\,.$$

Hence show that for $E \gg 1$ the period of the orbit is $T \approx \frac{1}{2}\pi\mu^3/p_0$, where p_0 is the greatest value of the momentum during the orbit.

16F Logic and Set Theory

State the Axiom of Foundation and the Principle of $\in$-Induction, and show that they are equivalent (in the presence of the other axioms of ZF). [*You may assume the existence of transitive closures.*]

Explain briefly how the Principle of $\in$-Induction implies that every set is a member of some V_α.

For each natural number n, find the cardinality of V_n. For which ordinals α is the cardinality of V_α equal to that of the reals?

17F Graph Theory

Let X and Y be disjoint sets of $n \geqslant 6$ vertices each. Let G be a bipartite graph formed by adding edges between X and Y randomly and independently with probability $p = 1/100$. Let $e(U, V)$ be the number of edges of G between the subsets $U \subset X$ and $V \subset Y$. Let $k = \lceil n^{1/2} \rceil$. Consider three events $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, as follows.

$\mathcal{A}$: there exist $U \subset X$, $V \subset Y$ with $|U| = |V| = k$ and $e(U, V) = 0$

$\mathcal{B}$: there exist $x \in X$, $W \subset Y$ with $|W| = n - k$ and $e(\{x\}, W) = 0$

$\mathcal{C}$: there exist $Z \subset X$, $y \in Y$ with $|Z| = n - k$ and $e(Z, \{y\}) = 0$.

Show that $\Pr(\mathcal{A}) \leqslant n^{2k}(1-p)^{k^2}$ and $\Pr(\mathcal{B} \cup \mathcal{C}) \leqslant 2n^{k+1}(1-p)^{n-k}$. Hence show that $\Pr(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) < 3n^{2k}(1-p)^{n/2}$ and so show that, almost surely, none of $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ occur. Deduce that, almost surely, G has a matching from X to Y.

18G Galois Theory

Find the Galois group of the polynomial

$$x^4 + x + 1$$

over $\mathbb{F}_2$ and $\mathbb{F}_3$. Hence or otherwise determine the Galois group over $\mathbb{Q}$.

[*Standard general results from Galois theory may be assumed.*]

19G Representation Theory

Let G be the group with 21 elements generated by a and b, subject to the relations $a^7 = b^3 = 1$ and $ba = a^2b$.

(i) Find the conjugacy classes of G.

(ii) Find three non-isomorphic one-dimensional representations of G.

(iii) For a subgroup H of a finite group K, write down (without proof) the formula for the character of the K-representation induced from a representation of H.

(iv) By applying Part (iii) to the case when H is the subgroup $\langle a \rangle$ of $K = G$, find the remaining irreducible characters of G.

20H Algebraic Topology

Let X be a space that is triangulable as a simplicial complex with no n-simplices. Show that any continuous map from X to S^n is homotopic to a constant map.

[*General theorems from the course may be used without proof, provided they are clearly stated.*]

21F Linear Analysis

Let X be a normed vector space. Define the dual X^* of X. Define the normed vector spaces $l^s = l^s(\mathbb{C})$ for all $1 \leqslant s \leqslant \infty$. [*You are* **not** *required to prove that the norms you have given are indeed norms.*]

Now let $1 < p, q < \infty$ be such that $p^{-1} + q^{-1} = 1$. Show that $(l^q)^*$ is isometrically isomorphic to l^p as a normed vector space. [*You may assume any standard inequalities.*]

Show by a similar argument that $(l^1)^*$ is isomorphic to l^∞. Does your argument also show that $(l^\infty)^*$ is isomorphic to l^1? If not, where does it fail?

22H Riemann Surfaces

Explain what is meant by a *meromorphic differential* on a compact connected Riemann surface S. Show that if f is a meromorphic function on S then df defines a meromorphic differential on S. Show also that if η and ω are two meromorphic differentials on S which are not identically zero then $\eta = h\omega$ for some meromorphic function h. Show that zeros and poles of a meromorphic differential are well-defined and explain, without proof, how to obtain the genus of S by counting zeros and poles of ω.

Let $V_0 \subset \mathbb{C}^2$ be the affine curve with equation $u^2 = v^2 + 1$ and let $V \subset \mathbb{P}^2$ be the corresponding projective curve. Show that V is non-singular with two points at infinity, and that dv extends to a meromorphic differential on V.

[*You may assume without proof that that the map*

$$(u, v) = \left(\frac{t^2+1}{t^2-1}, \frac{2t}{t^2-1}\right), \qquad t \in \mathbb{C} \setminus \{-1, 1\},$$

is onto $V_0 \setminus \{(1,0)\}$ *and extends to a biholomorphic map from* $\mathbb{P}^1$ *onto* V.]

23H Differential Geometry

(i) Define geodesic curvature and state the Gauss–Bonnet theorem.

(ii) Let $\alpha : I \to \mathbb{R}^3$ be a closed regular curve parametrized by arc-length, and assume that α has non-zero curvature everywhere. Let $n : I \to S^2 \subset \mathbb{R}^3$ be the curve given by the normal vector $n(s)$ to $\alpha(s)$. Let $\bar{s}$ be the arc-length of the curve n on S^2. Show that the geodesic curvature k_g of n is given by

$$k_g = -\frac{d}{ds}\tan^{-1}(\tau/k)\,\frac{ds}{d\bar{s}}\,,$$

where k and τ are the curvature and torsion of α.

(iii) Suppose now that $n(s)$ is a simple curve (i.e. it has no self-intersections). Show that $n(I)$ divides S^2 into two regions of equal area.

24J Probability and Measure

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For a measurable function $f : \Omega \to \mathbb{R}$, and $p \in [1, \infty)$, let $||f||_p = [\mu(|f|^p)]^{1/p}$. Let L^p be the space of all such f with $||f||_p < \infty$. Explain what is meant by each of the following statements:

(a) A sequence of functions $(f_n : n \geqslant 1)$ is Cauchy in L^p.

(b) L^p is complete.

Show that L^p is complete for $p \in [1, \infty)$.

Take $\Omega = (1, \infty)$, $\mathcal{F}$ the Borel σ-field of Ω, and μ the Lebesgue measure on $(\Omega, \mathcal{F})$. For $p = 1, 2$, determine which if any of the following sequences of functions are Cauchy in L^p:

(i) $f_n(x) = x^{-1} 1_{(1,n)}(x)$,

(ii) $g_n(x) = x^{-2} 1_{(1,n)}(x)$,

where 1_A denotes the indicator function of the set A.

25I Applied Probability

Consider an M/G/r/0 loss system with arrival rate λ and service-time distribution F. Thus, arrivals form a Poisson process of rate λ, service times are independent with common distribution F, there are r servers and there is no space for waiting. Use Little's Lemma to obtain a relation between the long-run average occupancy L and the stationary probability π that the system is full.

Cafe–Bar Duo has 23 serving tables. Each table can be occupied either by one person or two. Customers arrive either singly or in a pair; if a table is empty they are seated and served immediately, otherwise, they leave. The times between arrivals are independent exponential random variables of mean 20/3. Each arrival is twice as likely to be a single person as a pair. A single customer stays for an exponential time of mean 20, whereas a pair stays for an exponential time of mean 30; all these times are independent of each other and of the process of arrivals. The value of orders taken at each table is a constant multiple 2/5 of the time that it is occupied.

Express the long-run rate of revenue of the cafe as a function of the probability π that an arriving customer or pair of customers finds the cafe full.

By imagining a cafe with infinitely many tables, show that $\pi \leqslant \mathbb{P}(N \geqslant 23)$ where N is a Poisson random variable of parameter 7/2. Deduce that π is very small. [*Credit will be given for any useful numerical estimate, an upper bound of* 10^{-3} *being sufficient for full credit.*]

26I Principles of Statistics

In the context of decision theory, explain the meaning of the following italicized terms: *loss function, decision rule*, the *risk* of a decision rule, a *Bayes rule* with respect to prior π, and an *admissible* rule. Explain how a Bayes rule with respect to a prior π can be constructed.

Suppose that $X_1, \ldots, X_n$ are independent with common $N(0, v)$ distribution, where $v > 0$ is supposed to have a prior density f_0. In a decision-theoretic approach to estimating v, we take a quadratic loss: $L(v, a) = (v - a)^2$. Write $X = (X_1, \ldots, X_n)$ and $|X| = (X_1^2 + \ldots + X_n^2)^{1/2}$.

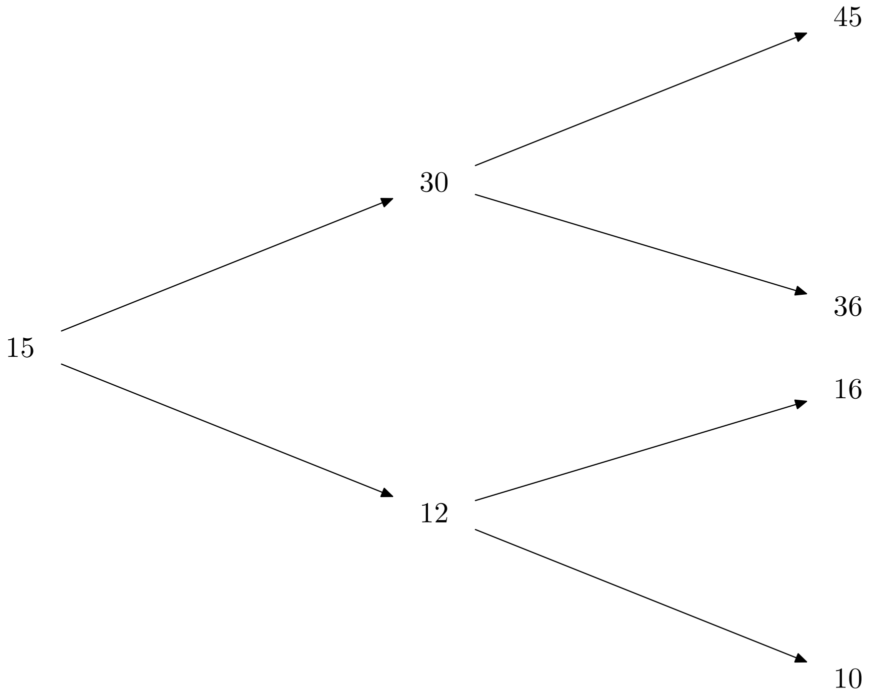
By considering decision rules (estimators) of the form $\hat{v}(X) = \alpha|X|^2$, prove that if $\alpha \neq 1/(n+2)$ then the estimator $\hat{v}(X) = \alpha|X|^2$ is not Bayes, for any choice of prior f_0.

By considering decision rules of the form $\hat{v}(X) = \alpha|X|^2 + \beta$, prove that if $\alpha \neq 1/n$ then the estimator $\hat{v}(X) = \alpha|X|^2$ is not Bayes, for any choice of prior f_0.

[*You may use without proof the fact that, if Z has a $N(0,1)$ distribution, then $EZ^4 = 3$.*]

27J Stochastic Financial Models

Suppose that over two periods a stock price moves on a binomial tree:



(a) Find an arbitrage opportunity when the riskless rate equals 1/10. Give precise details of when and how much you buy, borrow and sell.

(b) From here on, assume instead that the riskless rate equals 1/4. Determine the equivalent martingale measure. [*No proof is required.*]

(c) Determine the time-zero price of an American put with strike 15 and expiry 2. Assume you sell it at this price. Which hedge do you put on at time zero? Consider the scenario of two bad periods. How does your hedge work?

(d) The buyer of the American put turns out to be an unsophisticated investor who fails to use his early exercise right when he should. Assume the first period was bad. How much profit can you make out of this? You should detail your exact strategy.

28I Optimization and Control

Consider the problem

$$\text{minimize } E\left[x(T)^2 + \int_0^T u(t)^2\,dt\right]$$

where for $0 \leqslant t \leqslant T$,

$$\dot{x}(t) = y(t) \quad \text{and} \quad \dot{y}(t) = u(t) + \epsilon(t)\,,$$

$u(t)$ is the control variable, and $\epsilon(t)$ is Gaussian white noise. Show that the problem can be rewritten as one of controlling the scalar variable $z(t)$, where

$$z(t) = x(t) + (T-t)y(t)\,.$$

By guessing the form of the optimal value function and ensuring it satisfies an appropriate optimality equation, show that the optimal control is

$$u(t) = -\frac{(T-t)z(t)}{1+\frac{1}{3}(T-t)^3}\,.$$

Is this certainty equivalence control?

29C Partial Differential Equations

Write down a formula for the solution $u = u(t,x)$ of the n-dimensional heat equation

$$w_t(t,x) - \Delta w = 0, \qquad w(0,x) = g(x),$$

for $g : \mathbb{R}^n \to \mathbb{C}$ a given Schwartz function; here $w_t = \partial_t w$ and Δ is taken in the variables $x \in \mathbb{R}^n$. Show that

$$w(t,x) \leqslant \frac{\int |g(x)|\,dx}{(4\pi t)^{n/2}}\,.$$

Consider the equation

$$u_t - \Delta u = e^{it} f(x)\,, \tag{$*$}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a given Schwartz function. Show that $(*)$ has a solution of the form

$$u(t,x) = e^{it} v(x)\,,$$

where v is a Schwartz function.

Prove that the solution $u(t,x)$ of the initial value problem for $(*)$ with initial data $u(0,x) = g(x)$ satisfies

$$\lim_{t\to+\infty} \left|u(t,x) - e^{it}v(x)\right| = 0\,.$$

30A Asymptotic Methods

Explain, without proof, how to obtain an asymptotic expansion, as $x \to \infty$, of

$$I(x) = \int_0^\infty e^{-xt} f(t) dt \,,$$

if it is known that $f(t)$ possesses an asymptotic power series as $t \to 0$.

Indicate the modification required to obtain an asymptotic expansion, under suitable conditions, of

$$\int_{-\infty}^{\infty} e^{-xt^2} f(t) \; dt \,.$$

Find an asymptotic expansion as $z \to \infty$ of the function defined by

$$I(z) = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{(z-t)} dt \qquad (\mathrm{Im}(z) < 0)$$

and its analytic continuation to $\mathrm{Im}(z) \geqslant 0$. Where are the Stokes lines, that is, the critical lines separating the Stokes regions?

31A Integrable Systems

Let $Q(x,t)$ be an off-diagonal 2×2 matrix. The matrix NLS equation

$$iQ_t - Q_{xx}\sigma_3 + 2Q^3\sigma_3 = 0, \quad \sigma_3 = diag(1,-1),$$

admits the Lax pair

$$\begin{aligned}&\mu_x + ik[\sigma_3,\mu] = Q\mu,\\ &\mu_t + 2ik^2[\sigma_3,\mu] = (2kQ - iQ^2\sigma_3 - iQ_x\sigma_3)\mu,\end{aligned}$$

where $k \in \mathbb{C}$, $\mu(x,t,k)$ is a 2×2 matrix and $[\sigma_3,\mu]$ denotes the matrix commutator.

Let $S(k)$ be a 2×2 matrix-valued function decaying as $|k| \to \infty$. Let $\mu(x,t,k)$ satisfy the 2×2-matrix Riemann–Hilbert problem

$$\mu^+(x,t,k) = \mu^-(x,t,k)e^{-i(kx+2k^2t)\sigma_3}S(k)e^{i(kx+2k^2t)\sigma_3}, \quad k \in \mathbb{R},$$

$$\mu = diag(1,1) + \mathrm{O}\left(\frac{1}{k}\right), \quad k \to \infty.$$

(a) Find expressions for $Q(x,t)$, $A(x,t)$ and $B(x,t)$, in terms of the coefficients in the large k expansion of μ, so that μ solves

$$\mu_x + ik[\sigma_3,\mu] - Q\mu = 0,$$

and

$$\mu_t + 2ik^2[\sigma_3,\mu] - (kA + B)\mu = 0.$$

(b) Use the result of (a) to establish that

$$A = 2Q, \quad B = -i(Q^2 + Q_x)\sigma_3.$$

(c) Show that the above results provide a linearization of the matrix NLS equation. What is the disadvantage of this approach in comparison with the inverse scattering method?

32D Principles of Quantum Mechanics

The angular momentum operators $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ refer to independent systems, each with total angular momentum one. The combination of these systems has a basis of states which are of product form $|m_1; m_2\rangle = |1\, m_1\rangle|1\, m_2\rangle$ where m_1 and m_2 are the eigenvalues of $J_3^{(1)}$ and $J_3^{(2)}$ respectively. Let $|J\, M\rangle$ denote the alternative basis states which are simultaneous eigenstates of $\mathbf{J}^2$ and J_3, where $\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$ is the combined angular momentum. What are the possible values of J and M? Find expressions for all states with $J = 1$ in terms of product states. How do these states behave when the constituent systems are interchanged?

Two spin-one particles A and B have no mutual interaction but they each move in a potential $V(\mathbf{r})$ which is independent of spin. The single-particle energy levels E_i and the corresponding wavefunctions $\psi_i(\mathbf{r})$ $(i = 1, 2, \ldots)$ are the same for either A or B. Given that $E_1 < E_2 < \ldots$, explain how to construct the two-particle states of lowest energy and combined total spin $J = 1$ for the cases that (i) A and B are identical, and (ii) A and B are not identical.

[*You may assume* $\hbar = 1$ *and use the result* $J_\pm|j\, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, |j\;\, m{\pm}1\rangle$.]

33B Applications of Quantum Mechanics

Let $\{\mathbf{l}\}$ be the set of lattice vectors of some lattice. Define the reciprocal lattice. What is meant by a Bravais lattice?

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be mutually orthogonal unit vectors. A crystal has identical atoms at positions given by the vectors

$$\begin{array}{ll} a\big[n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}\big]\,, & a\big[(n_1 + \tfrac{1}{2})\mathbf{i} + (n_2 + \tfrac{1}{2})\mathbf{j} + n_3\mathbf{k}\big]\,, \\ a\big[(n_1 + \tfrac{1}{2})\mathbf{i} + \mathbf{j} + (n_3 + \tfrac{1}{2})\mathbf{k}\big]\,, & a\big[n_1\mathbf{i} + (n_2 + \tfrac{1}{2})\mathbf{j} + (n_3 + \tfrac{1}{2})\mathbf{k}\big]\,, \end{array}$$

where (n_1, n_2, n_3) are arbitrary integers and a is a constant. Show that these vectors define a Bravais lattice with basis vectors

$$\mathbf{a}_1 = a\tfrac{1}{2}(\mathbf{j} + \mathbf{k})\,, \qquad \mathbf{a}_2 = a\tfrac{1}{2}(\mathbf{i} + \mathbf{k})\,, \qquad \mathbf{a}_3 = a\tfrac{1}{2}(\mathbf{i} + \mathbf{j})\,.$$

Verify that a basis for the reciprocal lattice is

$$\mathbf{b}_1 = \frac{2\pi}{a}(\mathbf{j} + \mathbf{k} - \mathbf{i})\,, \qquad \mathbf{b}_2 = \frac{2\pi}{a}(\mathbf{i} + \mathbf{k} - \mathbf{j})\,, \qquad \mathbf{b}_3 = \frac{2\pi}{a}(\mathbf{i} + \mathbf{j} - \mathbf{k})\,.$$

In Bragg scattering, an incoming plane wave of wave-vector $\mathbf{k}$ is scattered to an outgoing wave of wave-vector $\mathbf{k}'$. Explain why $\mathbf{k}' = \mathbf{k} + \mathbf{g}$ for some reciprocal lattice vector $\mathbf{g}$. Given that θ is the scattering angle, show that

$$\sin\frac{1}{2}\theta = \frac{|\mathbf{g}|}{2\,|\mathbf{k}|}\,.$$

For the above lattice, explain why you would expect scattering through angles θ_1 and θ_2 such that

$$\frac{\sin\frac{1}{2}\theta_1}{\sin\frac{1}{2}\theta_2} = \frac{\sqrt{3}}{2}\,.$$

34D Statistical Physics

A free spinless particle moving in two dimensions is confined to a square box of side L. By imposing periodic boundary conditions show that the number of states in the energy range $\epsilon \to \epsilon + d\epsilon$ is $g(\epsilon)d\epsilon$, where

$$g(\epsilon) = \frac{mL^2}{2\pi\hbar^2} \, .$$

If, instead, the particle is an electron with magnetic moment μ moving in a constant external magnetic field H, show that

$$g(\epsilon) = \begin{cases} \dfrac{mL^2}{2\pi\hbar^2}, & -\mu H < \epsilon < \mu H \\ \dfrac{mL^2}{\pi\hbar^2}, & \mu H < \epsilon \, . \end{cases}$$

Let there be N electrons in the box. Explain briefly how to construct the ground state of the system. Let ϵ be the Fermi energy. Show that when $\epsilon > \mu H$

$$N \;=\; \frac{mL^2}{\pi\hbar^2}\epsilon \, .$$

Show also that the magnetic moment M of the system in its ground state is given by

$$M \;=\; \frac{\mu^2 mL^2}{\pi\hbar^2}H \, ,$$

and that the ground state energy is

$$\frac{1}{2}\frac{\pi\hbar^2}{mL^2}N^2 - \frac{1}{2}MH \, .$$

35B Electrodynamics

A non-relativistic particle of rest mass m and charge q is moving slowly with velocity $\mathbf{v}(t)$. The power $dP/d\Omega$ radiated per unit solid angle in the direction of a unit vector $\mathbf{n}$ is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2}\,|\mathbf{n} \times q\dot{\mathbf{v}}|^2 \,.$$

Obtain Larmor's formula

$$P = \frac{\mu_0\, q^2}{6\pi}\,|\dot{\mathbf{v}}|^2 \,.$$

The particle has energy $\mathcal{E}$ and, starting from afar, makes a head-on collision with a fixed central force described by a potential $V(r)$, where $V(r) > \mathcal{E}$ for $r < r_0$ and $V(r) < \mathcal{E}$ for $r > r_0$. Let W be the total energy radiated by the particle. Given that $W \ll \mathcal{E}$, show that

$$W \approx \frac{\mu_0\, q^2}{3\pi m^2}\sqrt{\frac{m}{2}}\int_{r_0}^{\infty}\left(\frac{dV}{dr}\right)^2 \frac{dr}{\sqrt{V(r_0) - V(r)}}\,.$$

36E Fluid Dynamics II

Write down the Navier–Stokes equations for an incompressible fluid.

Explain the concepts of the Euler and Prandtl limits applied to the Navier–Stokes equations near a rigid boundary.

A steady two-dimensional flow given by $(U, 0)$ far upstream flows past a semi-infinite flat plate, held at $y = 0, x > 0$. Derive the boundary layer equation

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu\frac{\partial^3\psi}{\partial y^3}$$

for the stream-function $\psi(x, y)$ near the plate, explaining any approximations made.

Show that the appropriate solution must be of the form

$$\psi(x, y) = (\nu U x)^{1/2} f(\eta),$$

and determine the dimensionless variable η.

Derive the equation and boundary conditions satisfied by $f(\eta)$. [*You need not solve them.*]

Suppose now that the plate has a *finite* length L in the direction of the flow. Show that the force F on the plate (per unit width perpendicular to the flow) is given by

$$F = \frac{4\rho U^2 L}{(UL/\nu)^{1/2}}\frac{f''(0)}{[f'(\infty)]^2}.$$

37E Waves

The real function $\phi(x,t)$ satisfies the equation

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} = \frac{\partial^3 \phi}{\partial x^3},$$

where $U > 0$ is a constant. Find the dispersion relation for waves of wavenumber k and deduce whether wave crests move faster or slower than a wave packet.

Suppose that $\phi(x,0)$ is given by a Fourier transform as

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}dk.$$

Use the method of stationary phase to find $\phi(Vt,t)$ as $t \to \infty$ for fixed $V > U$.

[*You may use the result that* $\int_{-\infty}^{\infty} e^{-a\xi^2}d\xi = (\pi/a)^{1/2}$ *if* $\mathrm{Re}(a) \geqslant 0$.]

What can be said if $V < U$? [*Detailed calculation is* **not** *required in this case.*]

38A Numerical Analysis

Consider the Runge–Kutta method

$$\begin{aligned} k_1 &= f(y_n), \\ k_2 &= f(y_n + (1-a)hk_1 + ahk_2), \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2) \end{aligned}$$

for the solution of the scalar ordinary differential equation $y' = f(y)$. Here a is a real parameter.

(a) Determine the order of the method.

(b) Find the range of values of a for which the method is A-stable.

END OF PAPER