UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Wednesday 8 June 2005 9 to 12

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***,...,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Recall that, if p is an odd prime, a *primitive root* modulo p is a generator of the cyclic (multiplicative) group $(\mathbb{Z}/p\mathbb{Z})^{\times}$. Let p be an odd prime of the form $2^{2^n}+1$; show that a is a primitive root mod p if and only if a is not a quadratic residue mod p. Use this result to prove that 7 is a primitive root modulo every such prime.

2F Topics in Analysis

(i) Let α be an algebraic number and let p and q be integers with $q \neq 0$. What does Liouville's theorem say about α and p/q?

(ii) Let p and q be integers with $q \neq 0$. Prove that

$$\left|\sqrt{2}-\frac{p}{q}\right| \geqslant \frac{1}{4q^2}\ .$$

[*In (ii), you may not use Liouville's theorem unless you prove it.*]

3G Geometry of Group Actions

Describe the *geodesics* in the disc model of the hyperbolic plane $\mathbb{H}^2$.

Define the *area* of a region in $\mathbb{H}^2$. Compute the area $A(r)$ of a hyperbolic circle of radius r from the definition just given. Compute the circumference $C(r)$ of a hyperbolic circle of radius r, and check explicitly that $dA(r)/dr = C(r)$.

How could you define π geometrically if you lived in $\mathbb{H}^2$? Briefly justify your answer.

4J Coding and Cryptography

What is a *linear binary code*? What is the *weight* $w(C)$ of a linear binary code C? Define the bar product $C_1|C_2$ of two binary linear codes C_1 and C_2, stating the conditions that C_1 and C_2 must satisfy. Under these conditions show that

$$w(C_1|C_2) \geqslant \min(2w(C_1), w(C_2)).$$

5I Statistical Modelling

You see below three R commands, and the corresponding output (which is slightly abbreviated). Explain the effects of the commands. How is the deviance defined, and why do we have d.f.=7 in this case? Interpret the numerical values found in the output.

```
> n = scan()
  3 5 16 12 11 34 37 51 56

> i = scan ()
  1 2 3 4 5 6 7 8 9

> summary(glm(n~i,poisson))
  deviance = 13.218
      d.f. = 7
  Coefficients:
                 Value     Std.Error
  (intercept)    1.363     0.2210
  i              0.3106    0.0382
```

6E Mathematical Biology

Consider a system with stochastic reaction events

$$x \xrightarrow{\lambda} x+1 \quad \text{and} \quad x \xrightarrow{\beta x^2} x-2\ ,$$

where λ and β are rate constants.

(a) State or derive the exact differential equation satisfied by the average number of molecules $<x>$. Assuming that fluctuations are negligible, approximate the differential equation to obtain the steady-state value of $<x>$.

(b) Using this approximation, calculate the elasticity H, the average lifetime τ, and the average chemical event size $<r>$ (averaged over fluxes).

(c) State the stationary Fluctuation Dissipation Theorem for the normalised variance η. Hence show that

$$\eta = \frac{3}{4<x>}\ .$$

7B Dynamical Systems

Define Lyapunov stability and quasi-asymptotic stability of a fixed point $\mathbf{x}_0$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

By considering a Lyapunov function of the form $V = g(x)+y^2$, show that the origin is an asymptotically stable fixed point of

$$\begin{aligned} \dot{x} &= -y - x^3 \\ \dot{y} &= x^5\ . \end{aligned}$$

[*Lyapunov's Second Theorem may be used without proof, provided you show that its conditions apply.*]

8A Further Complex Methods

The Hankel representation of the gamma function is

$$\Gamma(z) = \frac{1}{2i\sin(\pi z)} \int_{-\infty}^{(0^+)} t^{z-1} e^t dt \, ,$$

where the path of integration is the Hankel contour.

Use this representation to find the residue of $\Gamma(z)$ at $z = -n$, where n is a non-negative integer.

Is there a pole at $z = n$, where n is a positive integer? Justify your answer carefully, working only from the above representation of $\Gamma(z)$.

9C Classical Dynamics

A rigid body has principal moments of inertia I_1, I_2 and I_3 and is moving under the action of no forces with angular velocity components $(\omega_1, \omega_2, \omega_3)$. Its motion is described by Euler's equations

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= 0 \, . \end{aligned}$$

Are the components of the angular momentum to be evaluated in the body frame or the space frame?

Now suppose that an asymmetric body is moving with constant angular velocity $(\Omega, 0, 0)$. Show that this motion is stable if and only if I_1 is the largest or smallest principal moment.

10D Cosmology

(a) A spherically symmetric star obeys the pressure-support equation

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,, \qquad (*)$$

where $P(r)$ is the pressure at a distance r from the centre, $\rho(r)$ is the density, and the mass $m(r)$ is defined through the relation $dm/dr = 4\pi r^2 \rho(r)$. Multiply $(*)$ by $4\pi r^3$ and integrate over the total volume V of the star to derive the virial theorem

$$\langle P \rangle V = -\tfrac{1}{3} E_{\rm grav}\,,$$

where $\langle P \rangle$ is the average pressure and $E_{\rm grav}$ is the total gravitational potential energy.

(b) Consider a white dwarf supported by electron Fermi degeneracy pressure $P \approx h^2 n^{5/3}/m_{\rm e}$, where $m_{\rm e}$ is the electron mass and n is the number density. Assume a uniform density $\rho(r) = m_{\rm p} n(r) \approx m_{\rm p}\langle n \rangle$, so the total mass of the star is given by $M = (4\pi/3)\langle n \rangle m_{\rm p} R^3$ where R is the star radius and $m_{\rm p}$ is the proton mass. Show that the total energy of the white dwarf can be written in the form

$$E_{\rm total} \;=\; E_{\rm kin} + E_{\rm grav} \;=\; \frac{\alpha}{R^2} \;-\; \frac{\beta}{R}\,,$$

where α, β are positive constants which you should determine. [*You may assume that for an ideal gas* $E_{\rm kin} = \frac{3}{2}\langle P \rangle V$.] Use this expression to explain briefly why a white dwarf is stable.

SECTION II

11F Topics in Analysis

(i) State the Baire category theorem. Deduce from it a statement about nowhere dense sets.

(ii) Let X be the set of all real numbers with decimal expansions consisting of the digits 4 and 5 only. Prove that there is a real number t that cannot be written in the form $x+y$ with $x \in X$ and y rational.

12J Coding and Cryptography

What does it means to say that $f : \mathbb{F}_2^d \to \mathbb{F}_2^d$ is a linear feedback shift register? Let $(x_n)_{n \geqslant 0}$ be a stream produced by such a register. Show that there exist N, M with $N + M \leqslant 2^d - 1$ such that $x_{r+N} = x_r$ for all $r \geqslant M$.

Explain and justify the Berlekamp–Massey method for 'breaking' a cipher stream arising from a linear feedback register of unknown length.

Let x_n, y_n, z_n be three streams produced by linear feedback registers. Set

$$k_n = x_n \text{ if } y_n = z_n$$

$$k_n = y_n \text{ if } y_n \neq z_n \, .$$

Show that k_n is also a stream produced by a linear feedback register. Sketch proofs of any theorems that you use.

13E Mathematical Biology

Consider the reaction-diffusion system

$$\begin{aligned}\frac{\partial u}{\partial \tau} &= \beta_u \left(\frac{u^2}{v} - u\right) + D_u \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial v}{\partial \tau} &= \beta_v \left(u^2 - v\right) + D_v \frac{\partial^2 v}{\partial x^2}\end{aligned}$$

for an activator u and inhibitor v, where β_u and β_v are degradation rate constants and D_u and D_v are diffusion rate constants.

(a) Find a suitably scaled time t and length s such that

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{u^2}{v} - u + \frac{\partial^2 u}{\partial s^2} \\ \frac{1}{Q}\frac{\partial v}{\partial t} &= u^2 - v + P\,\frac{\partial^2 v}{\partial s^2}\ ,\end{aligned} \qquad (*)$$

and find expressions for P and Q.

(b) Show that the Jacobian matrix for small spatially homogenous deviations from a nonzero steady state of $(*)$ is

$$J = \begin{pmatrix} 1 & -1 \\ 2Q & -Q \end{pmatrix}$$

and find the values of Q for which the steady state is stable.
[*Hint: The eigenvalues of a* 2×2 *real matrix both have positive real parts iff the matrix has a positive trace and determinant.*]

(c) Derive linearised ordinary differential equations for the amplitudes $\hat{u}(t)$ and $\hat{v}(t)$ of small spatially inhomogeneous deviations from a steady state of $(*)$ that are proportional to $\cos(s/L)$, where L is a constant.

(d) Assuming that the system is stable to homogeneous perturbations, derive the condition for inhomogeneous *instability*. Interpret this condition in terms of how far activator and inhibitor molecules diffuse on average before they are degraded.

(e) Calculate the lengthscale L_{crit} of disturbances that are expected to be observed when the condition for inhomogeneous instability is just satisfied. What are the dominant mechanisms for stabilising disturbances on lengthscales (i) much less than and (ii) much greater than L_{crit}?

14B Dynamical Systems

Prove that if a continuous map F of an interval into itself has a periodic orbit of period three then it also has periodic orbits of least period n for all positive integers n.

Explain briefly why there must be at least two periodic orbits of least period 5.

[*You may assume without proof:*

(i) If U and V are non-empty closed bounded intervals such that $V \subseteq F(U)$ then there is a closed bounded interval $K \subseteq U$ such that $F(K) = V$.

(ii) The Intermediate Value Theorem.]

15D Cosmology

(a) Consider a homogeneous and isotropic universe with scale factor $a(t)$ and filled with mass density $\rho(t)$. Show how the conservation of kinetic energy plus gravitational potential energy for a test particle on the edge of a spherical region in this universe can be used to derive the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho\,, \tag{$*$}$$

where k is a constant. State clearly any assumptions you have made.

(b) Now suppose that the universe was filled throughout its history with radiation with equation of state $P = \rho c^2/3$. Using the fluid conservation equation and the definition of the relative density Ω, show that the density of this radiation can be expressed as

$$\rho \;=\; \frac{3H_0^2}{8\pi G}\,\frac{\Omega_0}{a^4}\,,$$

where H_0 is the Hubble parameter today and Ω_0 is the relative density today ($t = t_0$) and $a_0 \equiv a(t_0) = 1$ is assumed. Show also that $kc^2 = H_0^2(\Omega_0 - 1)$ and hence rewrite the Friedmann equation ($*$) as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\Omega_0\left(\frac{1}{a^4} - \frac{\beta}{a^2}\right)\,, \tag{$\dagger$}$$

where $\beta \equiv (\Omega_0 - 1)/\Omega_0$.

(c) Now consider a closed model with $k > 0$ (or $\Omega > 1$). Rewrite ($\dagger$) using the new time variable τ defined by

$$\frac{dt}{d\tau} = a\,.$$

Hence, or otherwise, solve ($\dagger$) to find the parametric solution

$$a(\tau) = \frac{1}{\sqrt{\beta}}(\sin\alpha\tau)\,, \qquad t(\tau) = \frac{1}{\alpha\sqrt{\beta}}(1 - \cos\alpha\tau)\,,$$

where $\alpha \equiv H_0\sqrt{(\Omega_0 - 1)}$. [*Recall that* $\int dx/\sqrt{1-x^2} = \sin^{-1} x$.]

Using the solution for $a(\tau)$, find the value of the new time variable $\tau = \tau_0$ today and hence deduce that the age of the universe in this model is

$$t_0 = H_0^{-1}\frac{\sqrt{\Omega_0} - 1}{\Omega_0 - 1}\,.$$

16F Logic and Set Theory

Give the inductive and the synthetic definitions of ordinal addition, and prove that they are equivalent. Give an example to show that ordinal addition is not commutative.

Which of the following assertions about ordinals α, β and γ are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(ii) If α and β are limit ordinals then $\alpha + \beta = \beta + \alpha$.

(iii) If $\alpha + \beta = \omega_1$ then $\alpha = 0$ or $\alpha = \omega_1$.

(iv) If $\alpha + \beta = \omega_1$ then $\beta = 0$ or $\beta = \omega_1$.

17F Graph Theory

Brooks' Theorem states that if G is a connected graph then $\chi(G) \leqslant \Delta(G)$ unless G is complete or is an odd cycle. Prove the theorem for 3-connected graphs G.

Let G be a graph, and let $d_1 + d_2 = \Delta(G) - 1$. By considering a partition V_1, V_2 of $V(G)$ that minimizes the quantity $d_2 e(G[V_1]) + d_1 e(G[V_2])$, show that there is a partition with $\Delta(G[V_i]) \leqslant d_i$, $i = 1, 2$.

By taking $d_1 = 3$, show that if a graph G contains no K_4 then $\chi(G) \leqslant \frac{3}{4}\Delta(G) + \frac{3}{2}$.

18G Galois Theory

Let K be a field of characteristic 0 containing all roots of unity.

(i) Let L be the splitting field of the polynomial $X^n - a$ where $a \in K$. Show that the Galois group of L/K is cyclic.

(ii) Suppose that M/K is a cyclic extension of degree m over K. Let g be a generator of the Galois group and $\zeta \in K$ a primitive m-th root of 1. By considering the resolvent

$$R(w) = \sum_{i=0}^{m-1} \frac{g^i(w)}{\zeta^i}$$

of elements $w \in M$, show that M is the splitting field of a polynomial $X^m - a$ for some $a \in K$.

19G Representation Theory

Let G be a finite group and $\{\chi_i\}$ the set of its irreducible characters. Also choose representatives g_j for the conjugacy classes, and denote by $Z(g_j)$ their centralisers.

(i) State the orthogonality and completeness relations for the χ_k.

(ii) Using Part (i), or otherwise, show that

$$\sum_i \overline{\chi_i(g_j)} \cdot \chi_i(g_k) = \delta_{jk} \cdot |Z(g_j)|\,.$$

(iii) Let A be the matrix with $A_{ij} = \chi_i(g_j)$. Prove that

$$|\det A|^2 = \prod_j |Z(g_j)|.$$

(iv) Show that $\det A$ is either real or purely imaginary, explaining when each situation occurs.

[*Hint for (iv): Consider the effect of complex conjugation on the rows of the matrix A.*]

20G Number Fields

Show that $\varepsilon = (3+\sqrt{7})/(3-\sqrt{7})$ is a unit in $k = \mathbb{Q}(\sqrt{7})$. Show further that 2 is the square of the principal ideal in k generated by $3+\sqrt{7}$.

Assuming that the Minkowski constant for k is $\frac{1}{2}$, deduce that k has class number 1.

Assuming further that ε is the fundamental unit in k, show that the complete solution in integers x, y of the equation $x^2 - 7y^2 = 2$ is given by

$$x + \sqrt{7}y = \pm\varepsilon^n(3+\sqrt{7}) \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Calculate the particular solution in positive integers x, y when $n = 1$.

21H Algebraic Topology

State the Van Kampen Theorem. Use this theorem and the fact that $\pi_1 S^1 = \mathbb{Z}$ to compute the fundamental groups of the torus $T = S^1 \times S^1$, the punctured torus $T \setminus \{p\}$, for some point $p \in T$, and the connected sum $T \# T$ of two copies of T.

22F Linear Analysis

Let X and Y be Banach spaces. Define what it means for a linear operator $T : X \to Y$ to be *compact*. For a linear operator $T : X \to X$, define the *spectrum*, *point spectrum*, and *resolvent set* of T.

Now let H be a complex Hilbert space. Define what it means for a linear operator $T : H \to H$ to be *self-adjoint*. Suppose $e_1, e_2, \ldots$ is an orthonormal basis for H. Define a linear operator $T : H \to H$ by setting $Te_i = \frac{1}{i} e_i$. Is T compact? Is T self-adjoint? Justify your answers. Describe, with proof, the spectrum, point spectrum, and resolvent set of T.

23H Riemann Surfaces

Define the terms *function element* and *complete analytic function*.

Let (f, D) be a function element such that $f(z)^n = p(z)$, for some integer $n \geqslant 2$, where $p(z)$ is a complex polynomial with no multiple roots. Let F be the complete analytic function containing (f, D). Show that every function element $(\tilde{f}, \tilde{D})$ in F satisfies $\tilde{f}(z)^n = p(z)$.

Describe how the non-singular complex algebraic curve

$$C = \{(z, w) \in \mathbb{C}^2 \mid w^n - p(z) = 0\}$$

can be made into a Riemann surface such that the first and second projections $\mathbb{C}^2 \to \mathbb{C}$ define, by restriction, holomorphic maps $f_1, f_2 : C \to \mathbb{C}$.

Explain briefly the relation between C and the Riemann surface $S(F)$ for the complete analytic function F given earlier.

[*You do not need to prove the Inverse Function Theorem, provided that you state it accurately.*]

24H Differential Geometry

State the isoperimetric inequality in the plane.

Let $S \subset \mathbb{R}^3$ be a surface. Let $p \in S$ and let $S_r(p)$ be a geodesic circle of centre p and radius r (r small). Let L be the length of $S_r(p)$ and A be the area of the region bounded by $S_r(p)$. Prove that

$$4\pi A - L^2 = \pi^2 r^4 K(p) + \varepsilon(r),$$

where $K(p)$ is the Gaussian curvature of S at p and

$$\lim_{r \to 0} \frac{\varepsilon(r)}{r^4} = 0.$$

When $K(p) > 0$ and r is small, compare this briefly with the isoperimetric inequality in the plane.

25J Probability and Measure

Let $\mathcal{R}$ be a family of random variables on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. What is meant by saying that $\mathcal{R}$ is uniformly integrable? Explain the use of uniform integrability in the study of convergence in probability and in L^1. [*Clear definitions should be given of any terms used, but proofs may be omitted.*]

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be uniformly integrable families of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the family $\mathcal{R}$ given by

$$\mathcal{R} = \{X + Y : X \in \mathcal{R}_1,\ Y \in \mathcal{R}_2\}$$

is uniformly integrable.

26I Applied Probability

What does it mean to say that (X_t) is a renewal process?

Let (X_t) be a renewal process with holding times $S_1, S_2, \ldots$ and let $s > 0$. For $n \geqslant 1$, set $T_n = S_{X_s+n}$. Show that

$$\mathbb{P}(T_n > t) \geqslant \mathbb{P}(S_n > t), \quad t \geqslant 0,$$

for all n, with equality if $n \geqslant 2$.

Consider now the case where $S_1, S_2, \ldots$ are exponential random variables. Show that

$$\mathbb{P}(T_1 > t) > \mathbb{P}(S_1 > t), \quad t > 0,$$

and that, as $s \to \infty$,

$$\mathbb{P}(T_1 > t) \to \mathbb{P}(S_1 + S_2 > t), \quad t \geqslant 0\,.$$

27I Principles of Statistics

(i) Suppose that X is a multivariate normal vector with mean $\mu \in \mathbb{R}^d$ and covariance matrix $\sigma^2 I$, where μ and σ^2 are both unknown, and I denotes the $d \times d$ identity matrix. Suppose that $\Theta_0 \subset \Theta_1$ are linear subspaces of $\mathbb{R}^d$ of dimensions d_0 and d_1, where $d_0 < d_1 < d$. Let P_i denote orthogonal projection onto Θ_i $(i = 0, 1)$. Carefully derive the joint distribution of $(|X - P_1 X|^2, |P_1 X - P_0 X|^2)$ under the hypothesis $H_0 : \mu \in \Theta_0$. How could you use this to make a test of H_0 against $H_1 : \mu \in \Theta_1$?

(ii) Suppose that I students take J exams, and that the mark X_{ij} of student i in exam j is modelled as

$$X_{ij} = m + \alpha_i + \beta_j + \varepsilon_{ij}$$

where $\sum_i \alpha_i = 0 = \sum_j \beta_j$, the ε_{ij} are independent $N(0, \sigma^2)$, and the parameters m, α, β and σ are unknown. Construct a test of $H_0 : \beta_j = 0$ for all j against $H_1 : \sum_j \beta_j = 0$.

28J Stochastic Financial Models

(i) At the beginning of year n, an investor makes decisions about his investment and consumption for the coming year. He first takes out an amount c_n from his current wealth w_n, and sets this aside for consumption. He splits his remaining wealth between a bank account (unit wealth invested at the start of the year will have grown to a sure amount $r > 1$ by the end of the year), and the stock market. Unit wealth invested in the stock market will have become the random amount $X_{n+1} > 0$ by the end of the year.

The investor's objective is to invest and consume so as to maximise the expected value of $\sum_{n=1}^{N} U(c_n)$, where U is strictly increasing and strictly convex. Consider the dynamic programming equation (Bellman equation) for his problem,

$$\begin{aligned} V_n(w) &= \sup_{c,\theta} \Big\{ U(c) + E_n\big[V_{n+1}(\theta(w-c)X_{n+1} + (1-\theta)(w-c)r) \big] \Big\} \quad (0 \leqslant n < N), \\ V_N(w) &= U(w). \end{aligned}$$

Explain all undefined notation, and explain briefly why the equation holds.

(ii) Supposing that the X_i are independent and identically distributed, and that $U(x) = x^{1-R}/(1-R)$, where $R > 0$ is different from 1, find as explicitly as you can the form of the agent's optimal policy.

(iii) Return to the general problem of (i). Assuming that the sample space Ω is finite, and that all suprema are attained, show that

$$\begin{aligned} E_n[\, V'_{n+1}(w^*_{n+1})(X_{n+1} - r) \,] &= 0, \\ r E_n[\, V'_{n+1}(w^*_{n+1}) \,] &= U'(c^*_n), \\ r E_n[\, V'_{n+1}(w^*_{n+1}) \,] &= V'_n(w^*_n), \end{aligned}$$

where $(c^*_n, w^*_n)_{0 \leqslant n \leqslant N}$ denotes the optimal consumption and wealth process for the problem. Explain the significance of each of these equalities.

29I Optimization and Control

Explain what is meant by a time-homogeneous discrete time Markov decision problem.

What is the positive programming case?

A discrete time Markov decision problem has state space $\{0, 1, \ldots, N\}$. In state i, $i \neq 0, N$, two actions are possible. We may either stop and obtain a terminal reward $r(i) \geqslant 0$, or may continue, in which case the subsequent state is equally likely to be $i-1$ or $i+1$. In states 0 and N stopping is automatic (with terminal rewards $r(0)$ and $r(N)$ respectively). Starting in state i, denote by $V_n(i)$ and $V(i)$ the maximal expected terminal reward that can be obtained over the first n steps and over the infinite horizon, respectively. Prove that $\lim_{n\to\infty} V_n = V$.

Prove that V is the smallest concave function such that $V(i) \geqslant r(i)$ for all i.

Describe an optimal policy.

Suppose $r(0), \ldots, r(N)$ are distinct numbers. Show that the optimal policy is unique, or give a counter-example.

30C Partial Differential Equations

Define a *fundamental solution* of a linear partial differential operator P. Prove that the function

$$G(x) = \tfrac{1}{2} e^{-|x|}$$

defines a distribution which is a fundamental solution of the operator P given by

$$P\,u = -\frac{d^2 u}{dx^2} + u\,.$$

Hence find a solution u_0 to the equation

$$-\frac{d^2 u_0}{dx^2} + u_0 = V(x)\,,$$

where $V(x) = 0$ for $|x| > 1$ and $V(x) = 1$ for $|x| \leqslant 1$.

Consider the functional

$$I[u] = \int_{\mathbb{R}} \left\{ \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + u^2 \right] - V u \right\} dx\,.$$

Show that $I[u_0 + \phi] > I[u_0]$ for all Schwartz functions ϕ that are not identically zero.

31C Integrable Systems

Suppose $q(x,t)$ satisfies the mKdV equation

$$q_t + q_{xxx} + 6q^2 q_x = 0\,,$$

where $q_t = \partial q/\partial t$ etc.

(a) Find the 1-soliton solution.

[*You may use, without proof, the indefinite integral* $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}} = -\mathrm{arcsech}\, x\,.$]

(b) Express the self-similar solution of the mKdV equation in terms of a solution, denoted by $v(z)$, of the Painlevé II equation.

(c) Using the Ansatz

$$\frac{dv}{dz} + iv^2 - \frac{i}{6}z = 0\,,$$

find a particular solution of the mKdV equation in terms of a solution of the Airy equation

$$\frac{d^2\Psi}{dz^2} + \frac{z}{6}\Psi = 0\,.$$

32D Principles of Quantum Mechanics

The components of $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are 2×2 hermitian matrices obeying

$$[\,\sigma_i,\, \sigma_j\,] \;=\; 2i\varepsilon_{ijk}\sigma_k \qquad \text{and} \qquad (\mathbf{n}\cdot\boldsymbol{\sigma})^2 = 1 \tag{$*$}$$

for any unit vector $\mathbf{n}$. Show that these properties imply

$$(\mathbf{a}\cdot\boldsymbol{\sigma})\,(\mathbf{b}\cdot\boldsymbol{\sigma}) \;=\; \mathbf{a}\cdot\mathbf{b} \,+\, i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}$$

for any constant vectors $\mathbf{a}$ and $\mathbf{b}$. Assuming that θ is real, explain why the matrix $U = \exp(-i\mathbf{n}\cdot\boldsymbol{\sigma}\,\theta/2)$ is unitary, and show that

$$U \;=\; \cos(\theta/2) \,-\, i\mathbf{n}\cdot\boldsymbol{\sigma}\sin(\theta/2)\ .$$

Hence deduce that

$$U\mathbf{m}\cdot\boldsymbol{\sigma}U^{-1} \;=\; \mathbf{m}\cdot\boldsymbol{\sigma}\cos\theta \,+\, (\mathbf{n}\times\mathbf{m})\cdot\boldsymbol{\sigma}\sin\theta$$

where $\mathbf{m}$ is any unit vector orthogonal to $\mathbf{n}$.

Write down an equation relating the matrices $\boldsymbol{\sigma}$ and the angular momentum operator $\mathbf{S}$ for a particle of spin one half, and explain *briefly* the significance of the conditions $(*)$. Show that if $|\chi\rangle$ is a state with spin 'up' measured along the direction $(0,0,1)$ then, for a certain choice of $\mathbf{n}$, $U|\chi\rangle$ is a state with spin 'up' measured along the direction $(\sin\theta, 0, \cos\theta)$.

33B Applications of Quantum Mechanics

Describe briefly the variational approach to the determination of an approximate ground state energy E_0 of a Hamiltonian H.

Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two states, and consider the trial state

$$|\psi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle$$

for real constants a_1 and a_2. Given that

$$\begin{aligned} \langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1\,, \qquad \langle\psi_2|\psi_1\rangle = \langle\psi_1|\psi_2\rangle = s\,, \\ \langle\psi_1|H|\psi_1\rangle = \langle\psi_2|H|\psi_2\rangle = \mathcal{E}\,, \quad \langle\psi_2|H|\psi_1\rangle = \langle\psi_1|H|\psi_2\rangle = \epsilon\,, \end{aligned} \qquad (*)$$

and that $\epsilon < s\mathcal{E}$, obtain an upper bound on E_0 in terms of $\mathcal{E}$, ϵ and s.

The normalized ground-state wavefunction of the Hamiltonian

$$H_1 = \frac{p^2}{2m} - K\delta(x)\,, \qquad K > 0,$$

is

$$\psi_1(x) = \sqrt{\lambda}\,e^{-\lambda|x|}\,, \qquad \lambda = \frac{mK}{\hbar^2}\,.$$

Verify that the ground state energy of H_1 is

$$E_B \equiv \langle\psi_1|H|\psi_1\rangle = -\frac{1}{2}K\lambda\,.$$

Now consider the Hamiltonian

$$H = \frac{p^2}{2m} - K\delta(x) - K\delta(x-R)\,,$$

and let $E_0(R)$ be its ground-state energy as a function of R. Assuming that

$$\psi_2(x) = \sqrt{\lambda}\,e^{-\lambda|x-R|}\,,$$

use $(*)$ to compute s, $\mathcal{E}$ and ϵ for ψ_1 and ψ_2 as given. Hence show that

$$E_0(R) \leqslant E_B\left[1 + 2\,\frac{e^{-\lambda R}\left(1+e^{-\lambda R}\right)}{1+(1+\lambda R)\,e^{-\lambda R}}\right].$$

Why should you expect this inequality to become an approximate equality for sufficiently large R? Describe briefly how this is relevant to molecular binding.

34D Statistical Physics

Write down the first law of thermodynamics in differential form applied to an infinitesimal reversible change.

Explain what is meant by an adiabatic change.

Starting with the first law in differential form, derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V .$$

Hence show that

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P .$$

For radiation in thermal equilibrium at temperature T in volume V, it is given that $E = Ve(T)$ and $P = e(T)/3$. Hence deduce Stefan's Law,

$$E = aVT^4 ,$$

where a is a constant.

The radiation is allowed to expand adiabatically. Show that VT^3 is constant during the expansion.

35C General Relativity

State without proof the properties of local inertial coordinates x^a centred on an arbitrary spacetime event p. Explain their physical significance.

Obtain an expression for $\partial_a \Gamma_b{}^c{}_d$ at p in inertial coordinates. Use it to derive the formula

$$R_{abcd} = \tfrac{1}{2}\big(\partial_b\partial_c g_{ad} + \partial_a\partial_d g_{bc} - \partial_b\partial_d g_{ac} - \partial_a\partial_c g_{bd}\big)$$

for the components of the Riemann tensor at p in local inertial coordinates. Hence deduce that at any point in any chart $R_{abcd} = R_{cdab}$.

Consider the metric

$$ds^2 = \frac{\eta_{ab}\, dx^a\, dx^b}{(1 + L^{-2}\eta_{ab}x^a x^b)^2} ,$$

where $\eta_{ab} = \mathrm{diag}(1,1,1,-1)$ is the Minkowski metric tensor and L is a constant. Compute the Ricci scalar $R = R^{ab}{}_{ab}$ at the origin $x^a = 0$.

36E Fluid Dynamics II

A volume V of very viscous fluid of density ρ and dynamic viscosity μ is released at the origin on a rigid horizontal boundary at time $t = 0$. Using lubrication theory, determine the velocity profile in the gravity current once it has spread sufficiently that the axisymmetric thickness $h(r,t)$ of the current is much less than the radius $R(t)$ of the front.

Derive the differential equation

$$\frac{\partial h}{\partial t} = \frac{\beta}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right),$$

where β is to be determined.

Write down the other equations that are needed to determine the appropriate similarity solution for this problem.

Determine the similarity solution and calculate $R(t)$.

37E Waves

Show that, in the standard notation for a one-dimensional flow of a perfect gas at constant entropy, the quantity $u + 2(c - c_0)/(\gamma - 1)$ remains constant along characteristics $dx/dt = u + c$.

A perfect gas is initially at rest and occupies a tube in $x > 0$. A piston is pushed into the gas so that its position at time t is $x(t) = \frac{1}{2}ft^2$, where $f > 0$ is a constant. Find the time and position at which a shock first forms in the gas.

38A Numerical Analysis

Define a *Krylov subspace* $\mathcal{K}_n(A, v)$.

Let d_n be the dimension of $\mathcal{K}_n(A, v)$. Prove that the sequence $\{d_m\}_{m=1,2,\ldots}$ increases monotonically. Show that, moreover, there exists an integer k with the following property: $d_m = m$ for $m = 1, 2, \ldots, k$, while $d_m = k$ for $m \geqslant k$. Assuming that A has a full set of eigenvectors, show that k is equal to the number of eigenvectors of A required to represent the vector v.

END OF PAPER