UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Monday 6 June 2005 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A****,* ***B****,* ***C****,...,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Define the Legendre symbol $\left(\frac{a}{p}\right)$. Prove that, if p is an odd prime, then

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

Use the law of quadratic reciprocity to calculate $\left(\frac{91}{167}\right)$.

[*You may use the Gauss Lemma without proof.*]

2F Topics in Analysis

Prove that $\cosh(1/2)$ is irrational.

3G Geometry of Group Actions

Let G be a subgroup of the group of isometries $\mathrm{Isom}(\mathbb{R}^2)$ of the Euclidean plane. What does it mean to say that G is *discrete*?

Supposing that G is discrete, show that the subgroup G_T of G consisting of all translations in G is generated by translations in at most two linearly independent vectors in $\mathbb{R}^2$. Show that there is a homomorphism $G \to O(2)$ with kernel G_T.

Draw, and briefly explain, pictures which illustrate two different possibilities for G when G_T is isomorphic to the additive group $\mathbb{Z}$.

4J Coding and Cryptography

Briefly describe the methods of Shannon-Fano and Huffman for economical coding. Illustrate both methods by finding decipherable binary codings in the case where messages $\mu_1, \ldots, \mu_5$ are emitted with probabilities $0.45, 0.25, 0.2, 0.05, 0.05$. Compute the expected word length in each case.

5I Statistical Modelling

Suppose that $Y_1, \ldots, Y_n$ are independent random variables, and that Y_i has probability density function

$$f(y_i|\theta_i, \phi) = \exp\left[\frac{(y_i\theta_i - b(\theta_i))}{\phi} + c(y_i, \phi)\right] .$$

Assume that $\mathbb{E}(Y_i) = \mu_i$ and that there is a known link function $g(.)$ such that

$$g(\mu_i) = \beta^T x_i \ ,$$

where $x_1, \ldots, x_n$ are known p-dimensional vectors and β is an unknown p-dimensional parameter. Show that $\mathbb{E}(Y_i) = b'(\theta_i)$ and that, if $\ell(\beta, \phi)$ is the log-likelihood function from the observations $(y_1, \ldots, y_n)$, then

$$\frac{\partial \ell(\beta, \phi)}{\partial \beta} = \sum_1^n \frac{(y_i - \mu_i)x_i}{g'(\mu_i)V_i} \ ,$$

where V_i is to be defined.

6E Mathematical Biology

Consider a biological system in which concentrations $x(t)$ and $y(t)$ satisfy

$$\frac{dx}{dt} = f(y) - x \quad \text{and} \quad \frac{dy}{dt} = g(x) - y \ ,$$

where f and g are positive and monotonically decreasing functions of their arguments, so that x represses the synthesis of y and vice versa.

(a) Suppose the functions f and g are bounded. Sketch the phase plane and explain why there is always at least one steady state. Show that if there is a steady state with

$$\frac{\partial \ln f}{\partial \ln y} \frac{\partial \ln g}{\partial \ln x} > 1$$

then the system is multistable.

(b) If $f = \lambda/(1+y^m)$ and $g = \lambda/(1+x^n)$, where λ, m and n are positive constants, what values of m and n allow the system to display multistability for some value of λ?

Can $f = \lambda/y^m$ and $g = \lambda/x^n$ generate multistability? Explain your answer carefully.

7B Dynamical Systems

State Dulac's Criterion and the Poincaré–Bendixson Theorem regarding the existence of periodic solutions to the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. Hence show that

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y(\mu - 2x^2 - y^2) \end{aligned}$$

has no periodic solutions if $\mu < 0$ and at least one periodic solution if $\mu > 0$.

8A Further Complex Methods

Explain what is meant by the Papperitz symbol

$$P \left\{ \begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\}.$$

The hypergeometric function $F(a, b; c; z)$ is defined as the solution of the equation determined by the Papperitz symbol

$$P \left\{ \begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix} \right\}$$

that is analytic at $z = 0$ and satisfies $F(a, b; c; 0) = 1$.

Show, explaining each step, that

$$F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z).$$

9C Classical Dynamics

A particle of mass m_1 is constrained to move on a circle of radius r_1, centre $x = y = 0$ in a horizontal plane $z = 0$. A second particle of mass m_2 moves on a circle of radius r_2, centre $x = y = 0$ in a horizontal plane $z = c$. The two particles are connected by a spring whose potential energy is

$$V = \tfrac{1}{2}\omega^2 d^2,$$

where d is the distance between the particles. How many degrees of freedom are there? Identify suitable generalized coordinates and write down the Lagrangian of the system in terms of them.

10D Cosmology

(a) Around $t \approx 1\,\mathrm{s}$ after the big bang ($kT \approx 1\,\mathrm{MeV}$), neutrons and protons are kept in equilibrium by weak interactions such as

$$n + \nu_e \leftrightarrow p + e^- \,. \qquad (*)$$

Show that, in equilibrium, the neutron-to-proton ratio is given by

$$\frac{n_n}{n_p} \approx e^{-Q/kT} \,,$$

where $Q = (m_n - m_p)c^2 = 1.29\,\mathrm{MeV}$ corresponds to the mass difference between the neutron and the proton. Explain briefly why we can neglect the difference $\mu_n - \mu_p$ in the chemical potentials.

(b) The ratio of the weak interaction rate $\Gamma_W \propto T^5$ which maintains $(*)$ to the Hubble expansion rate $H \propto T^2$ is given by

$$\frac{\Gamma_W}{H} \approx \left(\frac{kT}{0.8\,\mathrm{MeV}}\right)^3 \,. \qquad (\dagger)$$

Explain why the neutron-to-proton ratio effectively "freezes out" once $kT < 0.8\,\mathrm{MeV}$, except for some slow neutron decay. Also explain why almost all neutrons are subsequently captured in ${}^4He$; estimate the value of the relative mass density $Y_{{}^4He} = \rho_{{}^4He}/\rho_{\mathrm{B}}$ (with $\rho_{\mathrm{B}} = \rho_n + \rho_p$) given a final ratio $n_n/n_p \approx 1/8$.

(c) Suppose instead that the weak interaction rate were very much weaker than that described by equation $(\dagger)$. Describe the effect on the relative helium density $Y_{{}^4He}$. Briefly discuss the wider implications of this primordial helium-to-hydrogen ratio on stellar lifetimes and life on earth.

SECTION II

11F Topics in Analysis

State and prove a discrete form of Brouwer's theorem, concerning colourings of points in triangular grids. Use it to deduce that there is no continuous retraction from a disc to its boundary.

12G Geometry of Group Actions

What is the *limit set* of a subgroup G of Möbius transformations?

Suppose that G is complicated and has no finite orbit in $\mathbb{C} \cup \{\infty\}$. Prove that the limit set of G is infinite. Can the limit set be countable?

State Jørgensen's inequality, and deduce that not every two-generator subgroup $G = \langle A, B\rangle$ of Möbius transformations is discrete. Briefly describe two examples of discrete two-generator subgroups, one for which the limit set is connected and one for which it is disconnected.

13I Statistical Modelling

The Independent, June 1999, under the headline 'Tourists get hidden costs warnings' gave the following table of prices in pounds, called 'How the resorts compared'.

Algarve	8.00	0.50	3.50	3.00	4.00	100.00
CostaDelSol	6.95	1.30	4.10	12.30	4.10	130.85
Majorca	10.25	1.45	5.35	6.15	3.30	122.20
Tenerife	12.30	1.25	4.90	3.70	2.90	130.85
Florida	15.60	1.90	5.05	5.00	2.50	114.00
Tunisia	10.90	1.40	5.45	1.90	2.75	218.10
Cyprus	11.60	1.20	5.95	3.00	3.60	149.45
Turkey	6.50	1.05	6.50	4.90	2.85	263.00
Corfu	5.20	1.05	3.75	4.20	2.50	137.60
Sorrento	7.70	1.40	6.30	8.75	4.75	215.40
Malta	11.20	0.70	4.55	8.00	4.80	87.85
Rhodes	6.30	1.05	5.20	3.15	2.70	261.30
Sicily	13.25	1.75	4.20	7.00	3.85	174.40
Madeira	10.25	0.70	5.10	6.85	6.85	153.70

Here the column headings are, respectively: Three-course meal, Bottle of Beer, Suntan Lotion, Taxi (5km), Film (24 exp), Car Hire (per week). Interpret the R commands, and explain how to interpret the corresponding (slightly abbreviated) R output given below. Your solution should include a careful statement of the underlying statistical model, but you may quote without proof any distributional results required.

```
> price = scan("dresorts") ; price

> Goods = gl(6,1,length=84); Resort=gl(14,6,length=84)

> first.lm = lm(log(price) ~ Goods + Resort)

> summary(first.lm)

 Coefficients:

              Estimate Std. Error t value Pr(>|t|)

(Intercept)   1.8778     0.1629  11.527  < 2e-16

Goods2       -2.1084     0.1295 -16.286  < 2e-16

Goods3       -0.6343     0.1295  -4.900 6.69e-06

Goods4       -0.6284     0.1295  -4.854 7.92e-06

Goods5       -0.9679     0.1295  -7.476 2.49e-10

Goods6        2.8016     0.1295  21.640  < 2e-16

Resort2       0.4463     0.1978   2.257  0.02740

Resort3       0.4105     0.1978   2.076  0.04189

Resort4       0.3067     0.1978   1.551  0.12584

Resort5       0.4235     0.1978   2.142  0.03597

Resort6       0.2883     0.1978   1.458  0.14963

Resort7       0.3457     0.1978   1.748  0.08519

Resort8       0.3787     0.1978   1.915  0.05993

Resort9       0.0943     0.1978   0.477  0.63508

Resort10      0.5981     0.1978   3.025  0.00356

Resort11      0.3281     0.1978   1.659  0.10187

Resort12      0.2525     0.1978   1.277  0.20616

Resort13      0.5508     0.1978   2.785  0.00700

Resort14      0.4590     0.1978   2.321  0.02343


 Residual standard error: 0.3425 on 65 degrees of freedom

 Multiple R-Squared: 0.962
```

14B Dynamical Systems

Consider the equations

$$\begin{aligned}\dot{x} &= (a - x^2)(a^2 - y)\\ \dot{y} &= x - y\end{aligned}$$

as a function of the parameter a. Find the fixed points and plot their location in the (a, x) plane. Hence, or otherwise, deduce that there are bifurcations at $a = 0$ and $a = 1$.

Investigate the bifurcation at $a = 1$ by making the substitutions $u = x - 1$, $v = y - x$ and $\mu = a - 1$. Find the equation of the extended centre manifold to second order. Find the evolution equation on the centre manifold to second order, and determine the stability of its fixed points.

Show which branches of fixed points in the (a, x) plane are stable and which are unstable, and state, without calculation, the type of bifurcation at $a = 0$. Hence sketch the structure of the (x, y) phase plane very near the origin for $|a| \ll 1$ in the cases (i) $a < 0$ and (ii) $a > 0$.

The system is perturbed to $\dot{x} = (a - x^2)(a^2 - y) + \epsilon$, where $0 < \epsilon \ll 1$, with $\dot{y} = x - y$ still. Sketch the possible changes to the bifurcation diagram near $a = 0$ and $a = 1$. [*Calculation is not required.*]

15C Classical Dynamics

(i) The action for a system with generalized coordinates (q_a) is given by

$$S = \int_{t_1}^{t_2} L(q_a, \dot{q}_b)\, dt.$$

Derive Lagrange's equations from the principle of least action by considering all paths with fixed endpoints, $\delta q_a(t_1) = \delta q_a(t_2) = 0$.

(ii) A pendulum consists of a point mass m at the end of a light rod of length l. The pivot of the pendulum is attached to a mass M which is free to slide without friction along a horizontal rail. Choose as generalized coordinates the position x of the pivot and the angle θ that the pendulum makes with the vertical.

Write down the Lagrangian and derive the equations of motion.

Find the frequency of small oscillations around the stable equilibrium.

Now suppose that a force acts on the pivot causing it to travel with constant acceleration in the x-direction. Find the equilibrium angle θ of the pendulum.

16F Logic and Set Theory

State and prove Zorn's Lemma. [*You may assume Hartogs' Lemma.*] Where in your argument have you made use of the Axiom of Choice?

Show that $\mathbb{R}$, considered as a rational vector space, has a basis.

Prove that $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic as rational vector spaces.

17F Graph Theory

Show that an acyclic graph has a vertex of degree at most one. Prove that a tree (that is, a connected acyclic graph) of order n has size $n-1$, and deduce that every connected graph of order n and size $n-1$ is a tree.

Let T be a tree of order t. Show that if G is a graph with $\delta(G) \geqslant t-1$ then T is a subgraph of G, but that this need not happen if $\delta(G) \geqslant t-2$.

18G Galois Theory

Let L/K be a field extension. State what it means for an element $x \in L$ to be *algebraic* over K. Show that x is algebraic over K if and only if the field $K(x)$ is finite dimensional as a vector space over K.

State what it means for a field extension L/K to be *algebraic*. Show that, if M/L is algebraic and L/K is algebraic, then M/K is algebraic.

19G Representation Theory

Let the finite group G act on finite sets X and Y, and denote by $\mathbb{C}[X]$, $\mathbb{C}[Y]$ the associated permutation representations on the spaces of complex functions on X and Y. Call their characters χ_X and χ_Y.

(i) Show that the inner product $\langle \chi_X | \chi_Y \rangle$ is the number of orbits for the diagonal action of G on $X \times Y$.

(ii) Assume that $|X| > 1$, and let $S \subset \mathbb{C}[X]$ be the subspace of those functions whose values sum to zero. By considering $\|\chi_X\|^2$, show that S is irreducible if and only if the G-action on X is *doubly transitive*: this means that for any two pairs (x_1, x_2) and (x_1', x_2') of points in X with $x_1 \neq x_2$ and $x_1' \neq x_2'$, there exists some $g \in G$ with $gx_1 = x_1'$ and $gx_2 = x_2'$.

(iii) Let now $G = S_n$ acting on the set $X = \{1, 2, \ldots, n\}$. Call Y the set of 2-element subsets of X, with the natural action of S_n. If $n \geqslant 4$, show that $\mathbb{C}[Y]$ decomposes under S_n into three irreducible representations, one of which is the trivial representation and another of which is S. What happens when $n = 3$?

[*Hint: Consider* $\langle 1 | \chi_Y \rangle$, $\langle \chi_X | \chi_Y \rangle$ *and* $\|\chi_Y\|^2$.]

20G Number Fields

Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{p})$ where p is a prime with $p \equiv 3 \pmod 4$. By computing the relative traces $\mathrm{Tr}_{K/k}(\theta)$ where k runs through the three quadratic subfields of K, show that the algebraic integers θ in K have the form

$$\theta = \tfrac{1}{2}(a + b\sqrt{p}) + \tfrac{1}{2}(c + d\sqrt{p})\sqrt{2}\ ,$$

where a, b, c, d are rational integers. By further computing the relative norm $\mathrm{N}_{K/k}(\theta)$ where $k = \mathbb{Q}(\sqrt{2})$, show that 4 divides

$$a^2 + pb^2 - 2\big(c^2 + pd^2\big) \quad \text{and} \quad 2\big(ab - 2cd\big).$$

Deduce that a and b are even and $c \equiv d \pmod 2$. Hence verify that an integral basis for K is

$$1, \quad \sqrt{2}, \quad \sqrt{p}, \quad \tfrac{1}{2}\big(1 + \sqrt{p}\big)\sqrt{2}.$$

21H Algebraic Topology

(i) Show that if $E \to T$ is a covering map for the torus $T = S^1 \times S^1$, then E is homeomorphic to one of the following: the plane $\mathbb{R}^2$, the cylinder $\mathbb{R} \times S^1$, or the torus T.

(ii) Show that any continuous map from a sphere S^n $(n \geqslant 2)$ to the torus T is homotopic to a constant map.

[*General theorems from the course may be used without proof, provided that they are clearly stated.*]

22F Linear Analysis

Let K be a compact Hausdorff space, and let $C(K)$ denote the Banach space of continuous, complex-valued functions on K, with the supremum norm. Define what it means for a set $S \subset C(K)$ to be *totally bounded*, *uniformly bounded*, and *equicontinuous*.

Show that S is totally bounded if and only if it is both uniformly bounded and equicontinuous.

Give, with justification, an example of a Banach space X and a subset $S \subset X$ such that S is bounded but not totally bounded.

23H Riemann Surfaces

Let Λ be a lattice in $\mathbb{C}$ generated by 1 and τ, where τ is a fixed complex number with $\mathrm{Im}\tau > 0$. The Weierstrass $\wp$-function is defined as a Λ-periodic meromorphic function such that

(1) the only poles of $\wp$ are at points of Λ, and

(2) there exist positive constants ε and M such that for all $|z| < \varepsilon$, we have

$$|\wp(z) - 1/z^2| < M|z|.$$

Show that $\wp$ is uniquely determined by the above properties and that $\wp(-z) = \wp(z)$. By considering the valency of $\wp$ at $z = 1/2$, show that $\wp''(1/2) \neq 0$.

Show that $\wp$ satisfies the differential equation

$$\wp''(z) = 6\wp^2(z) + A,$$

for some complex constant A.

[*Standard theorems about doubly-periodic meromorphic functions may be used without proof provided they are accurately stated, but any properties of the $\wp$-function that you use must be deduced from first principles.*]

24H Differential Geometry

Let $f : X \to Y$ be a smooth map between manifolds without boundary.

(i) Define what is meant by a *critical point*, *critical value* and *regular value* of f.

(ii) Show that if y is a regular value of f and $\dim X \geqslant \dim Y$, then the set $f^{-1}(y)$ is a submanifold of X with $\dim f^{-1}(y) = \dim X - \dim Y$.

[*You may assume the inverse function theorem.*]

(iii) Let $SL(n, \mathbb{R})$ be the group of all $n \times n$ real matrices with determinant 1. Prove that $SL(n, \mathbb{R})$ is a submanifold of the set of all $n \times n$ real matrices. Find the tangent space to $SL(n, \mathbb{R})$ at the identity matrix.

25J Probability and Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $\mathcal{G} \subseteq \mathcal{F}$, what is meant by saying that $\mathcal{G}$ is a π-system? State the 'uniqueness of extension' theorem for measures on $\sigma(\mathcal{G})$ having given values on $\mathcal{G}$.

For $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$, we call $\mathcal{G}$, $\mathcal{H}$ independent if

$$\mathbb{P}(G \cap H) = \mathbb{P}(G)\mathbb{P}(H) \quad \text{for all} \quad G \in \mathcal{G},\ H \in \mathcal{H}.$$

If $\mathcal{G}$ and $\mathcal{H}$ are independent π-systems, show that $\sigma(\mathcal{G})$ and $\sigma(\mathcal{H})$ are independent.

Let $Y_1, Y_2, \ldots, Y_m,\ Z_1, Z_2, \ldots, Z_n$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the σ-fields $\sigma(Y) = \sigma(Y_1, Y_2, \ldots, Y_m)$ and $\sigma(Z) = \sigma(Z_1, Z_2, \ldots, Z_n)$ are independent.

26I Applied Probability

A cell has been placed in a biological solution at time $t = 0$. After an exponential time of rate μ, it is divided, producing k cells with probability p_k, $k = 0, 1, \ldots$, with the mean value $\rho = \sum_{k=1}^{\infty} kp_k$ ($k = 0$ means that the cell dies). The same mechanism is applied to each of the living cells, independently.

(a) Let M_t be the number of living cells in the solution by time $t > 0$. Prove that $\mathbb{E}M_t = \exp\left[t\mu(\rho - 1)\right]$. [*You may use without proof, if you wish, the fact that, if a positive function $a(t)$ satisfies $a(t+s) = a(t)a(s)$ for $t, s \geqslant 0$ and is differentiable at zero, then $a(t) = e^{\alpha t}$, $t \geqslant 0$, for some α.*]

Let $\phi_t(s) = \mathbb{E}\ s^{M_t}$ be the probability generating function of M_t. Prove that it satisfies the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(s) = \mu\left(-\phi_t(s) + \sum_{k=0}^{\infty} p_k\left[\phi_t(s)\right]^k\right), \quad \text{with} \quad \phi_0(s) = s.$$

(b) Now consider the case where each cell is divided in two cells ($p_2 = 1$). Let $N_t = M_t - 1$ be the number of cells produced in the solution by time t.

Calculate the distribution of N_t. Is (N_t) an inhomogeneous Poisson process? If so, what is its rate $\lambda(t)$? Justify your answer.

27I Principles of Statistics

State *Wilks' Theorem* on the asymptotic distribution of likelihood-ratio test statistics.

Suppose that $X_1, \ldots, X_n$ are independent with common $N(\mu, \sigma^2)$ distribution, where the parameters μ and σ are both unknown. Find the likelihood-ratio test statistic for testing $H_0 : \mu = 0$ against $H_1 : \mu$ unrestricted, and state its (approximate) distribution.

What is the form of the t-test of H_0 against H_1? Explain why for large n the likelihood-ratio test and the t-test are nearly the same.

28J Stochastic Financial Models

Let $X \equiv (X_0, X_1, \ldots, X_J)^T$ be a zero-mean Gaussian vector, with covariance matrix $V = (v_{jk})$. In a simple single-period economy with J agents, agent i will receive X_i at time 1 ($i = 1, \ldots, J$). If Y is a contingent claim to be paid at time 1, define agent i's *reservation bid price* for Y, assuming his preferences are given by $E[U_i(X_i + Z)]$ for any contingent claim Z.

Assuming that $U_i(x) \equiv -\exp(-\gamma_i x)$ for each i, where $\gamma_i > 0$, show that agent i's reservation bid price for λ units of X_0 is

$$p_i(\lambda) = -\frac{1}{2}\gamma_i(\lambda^2 v_{00} + 2\lambda v_{0i}).$$

As $\lambda \to 0$, find the limit of agent i's per-unit reservation bid price for X_0, and comment on the expression you obtain.

The agents bargain, and reach an equilibrium. Assuming that the contingent claim X_0 is in zero net supply, show that the equilibrium price of X_0 will be

$$p = -\Gamma v_{0\bullet},$$

where $\Gamma^{-1} = \sum_{i=1}^J \gamma_i^{-1}$ and $v_{0\bullet} = \sum_{i=1}^J v_{0i}$. Show that at that price agent i will choose to buy

$$\theta_i = (\Gamma v_{0\bullet} - \gamma_i v_{0i})/(\gamma_i v_{00})$$

units of X_0.

By computing the improvement in agent i's expected utility, show that the value to agent i of access to this market is equal to a fixed payment of

$$\frac{(\gamma_i v_{0i} - \Gamma v_{0\bullet})^2}{2\gamma_i v_{00}}.$$

29C Partial Differential Equations

Consider the equation

$$x_2 \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_2} + a \frac{\partial u}{\partial x_3} = u, \qquad (*)$$

where $a \in \mathbb{R}$, to be solved for $u = u(x_1, x_2, x_3)$. State clearly what it means for a hypersurface

$$S_\phi = \big\{(x_1, x_2, x_3) : \ \phi(x_1, x_2, x_3) = 0\big\},$$

defined by a C^1 function ϕ, to be *non-characteristic for* $(*)$. Does the non-characteristic condition hold when $\phi(x_1, x_2, x_3) = x_3$?

Solve $(*)$ for $a > 0$ with initial condition $u(x_1, x_2, 0) = f(x_1, x_2)$ where $f \in C^1(\mathbb{R}^2)$. For the case $f(x_1, x_2) = x_1^2 + x_2^2$ discuss the limiting behaviour as $a \to 0_+$.

30A Asymptotic Methods

Explain what is meant by an asymptotic power series about $x = a$ for a real function $f(x)$ of a real variable. Show that a convergent power series is also asymptotic.

Show further that an asymptotic power series is unique (assuming that it exists).

Let the function $f(t)$ be defined for $t \geqslant 0$ by

$$f(t) = \frac{1}{\pi^{1/2}} \int_0^\infty \frac{e^{-x}}{x^{1/2}(1 + 2xt)} dx \,.$$

By suitably expanding the denominator of the integrand, or otherwise, show that, as $t \to 0_+$,

$$f(t) \sim \sum_{k=0}^{\infty} (-1)^k 1.3 \ldots (2k-1) t^k$$

and that the error, when the series is stopped after n terms, does not exceed the absolute value of the $(n+1)$th term of the series.

31D Integrable Systems

Let $\phi(t)$ satisfy the linear singular integral equation

$$(t^2+t-1)\phi(t) - \frac{t^2-t-1}{\pi i}\oint_L \frac{\phi(\tau)d\tau}{\tau - t} - \frac{1}{\pi i}\int_L \left(\tau + \frac{1}{\tau}\right)\phi(\tau)d\tau = t-1, \quad t \in L,$$

where $\oint$ denotes the principal value integral and L denotes a counterclockwise smooth closed contour, enclosing the origin but not the points ± 1.

(a) Formulate the associated Riemann–Hilbert problem.

(b) For this Riemann–Hilbert problem, find the index, the homogeneous canonical solution and the solvability condition.

(c) Find $\phi(t)$.

32D Principles of Quantum Mechanics

A one-dimensional harmonic oscillator has Hamiltonian

$$H \;=\; \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 \;=\; \hbar\omega\left(a^\dagger a + \frac{1}{2}\right),$$

where

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \quad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \text{ obey } [a, a^\dagger] = 1.$$

Assuming the existence of a normalised state $|0\rangle$ with $a|0\rangle = 0$, verify that

$$|n\rangle = \frac{1}{\sqrt{n!}}a^{\dagger\, n}|0\rangle, \qquad n = 0, 1, 2, \ldots$$

are eigenstates of H with energies E_n, to be determined, and that these states all have unit norm.

The Hamiltonian is now modified by a term

$$\lambda V = \lambda\hbar\omega(\, a^r + a^{\dagger\, r}\,)$$

where r is a positive integer. Use perturbation theory to find the change in the lowest energy level to order λ^2 for any r. [*You may quote any standard formula you need.*]

Compute by perturbation theory, again to order λ^2, the change in the first excited energy level when $r = 1$. Show that in this special case, $r = 1$, the *exact* change in *all* energy levels as a result of the perturbation is $-\lambda^2\hbar\omega$.

33B Applications of Quantum Mechanics

A beam of particles is incident on a central potential $V(r)$ $(r = |\mathbf{x}|)$ that vanishes for $r > R$. Define the differential cross-section $d\sigma/d\Omega$.

Given that each incoming particle has momentum $\hbar\mathbf{k}$, explain the relevance of solutions to the time-independent Schrödinger equation with the asymptotic form

$$\psi(\mathbf{x}) \sim e^{i\mathbf{k}\cdot\mathbf{x}} + f(\hat{\mathbf{x}})\,\frac{e^{ikr}}{r} \qquad (*)$$

as $r \to \infty$, where $k = |\mathbf{k}|$ and $\hat{\mathbf{x}} = \mathbf{x}/r$. Write down a formula that determines $d\sigma/d\Omega$ in this case.

Write down the time-independent Schrödinger equation for a particle of mass m and energy $E = \dfrac{\hbar^2 k^2}{2m}$ in a central potential $V(r)$, and show that it allows a solution of the form

$$\psi(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{m}{2\pi\hbar^2}\int d^3x'\,\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,V(r')\psi(\mathbf{x}')\ .$$

Show that this is consistent with $(*)$ and deduce an expression for $f(\hat{\mathbf{x}})$. Obtain the Born approximation for $f(\hat{\mathbf{x}})$, and show that $f(\hat{\mathbf{x}}) = F(k\hat{\mathbf{x}} - \mathbf{k})$, where

$$F(\mathbf{q}) = -\frac{m}{2\pi\hbar^2}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\,V(r)\,.$$

Under what conditions is the Born approximation valid?

Obtain a formula for $f(\hat{\mathbf{x}})$ in terms of the scattering angle θ in the case that

$$V(r) = K\,\frac{e^{-\mu r}}{r}\,,$$

for constants K and μ. Hence show that $f(\hat{\mathbf{x}})$ is independent of $\hbar$ in the limit $\mu \to 0$, when expressed in terms of θ and the energy E.

[*You may assume that* $(\nabla^2 + k^2)\left(\dfrac{e^{ikr}}{r}\right) = -4\pi\delta^3(\mathbf{x})$.]

34B Electrodynamics

In a frame $\mathcal{F}$ the electromagnetic fields $(\mathbf{E}, \mathbf{B})$ are encoded into the Maxwell field 4-tensor F^{ab} and its dual ${}^*F^{ab}$, where

$$F^{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and

$${}^*F^{ab} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} .$$

[*Here the signature is* $(+ \; - \; - \; -)$ *and units are chosen so that* $c = 1$.] Obtain two independent Lorentz scalars of the electromagnetic field in terms of $\mathbf{E}$ and $\mathbf{B}$.

Suppose that $\mathbf{E}\cdot\mathbf{B} > 0$ in the frame $\mathcal{F}$. Given that there exists a frame $\mathcal{F}'$ in which $\mathbf{E}' \times \mathbf{B}' = \mathbf{0}$, show that

$$E' = \left[(\mathbf{E}\cdot\mathbf{B})\big(K + \sqrt{1+K^2}\big)\right]^{1/2}, \qquad B' = \left[\frac{\mathbf{E}\cdot\mathbf{B}}{K + \sqrt{1+K^2}}\right]^{1/2},$$

where (E', B') are the magnitudes of $(\mathbf{E}', \mathbf{B}')$, and

$$K = \frac{1}{2}\big(|\mathbf{E}|^2 - |\mathbf{B}|^2\big)/\big(\mathbf{E}\cdot\mathbf{B}\big) .$$

[*Hint: there is no need to consider the Lorentz transformations for* $\mathbf{E}'$ *and* $\mathbf{B}'$.]

35C General Relativity

Suppose $(x(\tau), t(\tau))$ is a timelike geodesic of the metric

$$ds^2 = \frac{dx^2}{1+x^2} - (1+x^2)\, dt^2\,,$$

where τ is proper time along the world line. Show that $dt/d\tau = E/(1+x^2)$, where $E > 1$ is a constant whose physical significance should be stated. Setting $a^2 = E^2 - 1$, show that

$$d\tau = \frac{dx}{\sqrt{a^2 - x^2}}\,, \qquad dt = \frac{E\ dx}{(1+x^2)\sqrt{a^2-x^2}}\,. \tag{$*$}$$

Deduce that x is a periodic function of proper time τ with period 2π. Sketch $dx/d\tau$ as a function of x and superpose on this a sketch of dx/dt as a function of x. Given the identity

$$\int_{-a}^{a} \frac{E\, dx}{(1+x^2)\sqrt{a^2-x^2}} = \pi\,,$$

deduce that x is also a periodic function of t with period 2π.

Next consider the family of metrics

$$ds^2 = \frac{[1+f(x)]^2\, dx^2}{1+x^2} - (1+x^2)\, dt^2\,,$$

where f is an odd function of x, $f(-x) = -f(x)$, and $|f(x)| < 1$ for all x. Derive expressions analogous to $(*)$ above. Deduce that x is a periodic function of τ and also that x is a periodic function of t. What are the periods?

36E Fluid Dynamics II

Consider a unidirectional flow with dynamic viscosity μ along a straight rigid-walled channel of uniform cross-sectional shape $\mathcal{D}$ driven by a uniform applied pressure gradient G. Write down the differential equation and boundary conditions governing the velocity w along the channel.

Consider the situation when the boundary includes a sharp corner of angle 2α. Explain why one might expect that, sufficiently close to the corner, the solution should be of the form

$$w = (G/\mu)r^2 f(\theta)\,,$$

where r and θ are polar co-ordinates with origin at the vertex and $\theta = \pm\alpha$ describing the two planes emanating from the corner. Determine $f(\theta)$.

If $\mathcal{D}$ is the sector bounded by the lines $\theta = \pm\alpha$ and the circular arc $r = a$, show that the flow is given by

$$w = (G/\mu)r^2 f(\theta) + \sum_{n=0}^{\infty} A_n r^{\lambda_n} \cos \lambda_n \theta,$$

where λ_n and A_n are to be determined.

[*Note that* $\int \cos(ax)\cos(bx)\,dx = \{a\sin(ax)\cos(bx) - b\sin(bx)\cos(ax)\}/(a^2 - b^2)$.]

Considering the values of λ_0 and λ_1, comment briefly on the cases: (i) $2\alpha < \frac{1}{2}\pi$; (ii) $\frac{1}{2}\pi < 2\alpha < \frac{3}{2}\pi$; and (iii) $\frac{3}{2}\pi < 2\alpha < 2\pi$.

37E Waves

An elastic solid with density ρ has Lamé moduli λ and μ. Write down equations satisfied by the dilational and shear potentials ϕ and $\boldsymbol{\psi}$.

For a two-dimensional disturbance give expressions for the displacement field $\mathbf{u} = (u_x, u_y, 0)$ in terms of $\phi(x, y; t)$ and $\boldsymbol{\psi} = (0, 0, \psi(x, y; t))$.

Suppose the solid occupies the region $y < 0$ and that the surface $y = 0$ is free of traction. Find a combination of solutions for ϕ and ψ that represent a propagating surface wave (a Rayleigh wave) near $y = 0$. Show that the wave is non-dispersive and obtain an equation for the speed c. [*You may assume without proof that this equation has a unique positive root.*]

38A Numerical Analysis

Let

$$\frac{\mu}{4}u_{m-1}^{n+1} + u_m^{n+1} - \frac{\mu}{4}u_{m+1}^{n+1} = -\frac{\mu}{4}u_{m-1}^{n} + u_m^{n} + \frac{\mu}{4}u_{m+1}^{n},$$

where n is a positive integer and m ranges over all integers, be a finite-difference method for the advection equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \qquad -\infty < x < \infty, \quad t \geqslant 0.$$

Here $\mu = \frac{\Delta t}{\Delta x}$ is the Courant number.

(a) Show that the local error of the method is $O((\Delta x)^3)$.

(b) Determine the range of $\mu > 0$ for which the method is stable.

END OF PAPER