UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 10 June 2005 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIRMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Linear Algebra

Define what it means for an $n \times n$ complex matrix to be unitary or Hermitian. Show that every eigenvalue of a Hermitian matrix is real. Show that every eigenvalue of a unitary matrix has absolute value 1.

Show that two eigenvectors of a Hermitian matrix that correspond to different eigenvalues are orthogonal, using the standard inner product on $\mathbf{C}^n$.

2C Groups, Rings and Modules

State Eisenstein's irreducibility criterion. Let n be an integer > 1. Prove that $1 + x + \ldots + x^{n-1}$ is irreducible in $\mathbb{Z}[x]$ if and only if n is a prime number.

3B Analysis II

Let V be the vector space of continuous real-valued functions on $[0, 1]$. Show that the function

$$||f|| = \int_0^1 |f(x)|\, dx$$

defines a norm on V.

For $n = 1, 2, \ldots$, let $f_n(x) = e^{-nx}$. Is f_n a convergent sequence in the space V with this norm? Justify your answer.

4A Complex Analysis

Let $\gamma : [0,1] \to \mathbf{C}$ be a closed path, where all paths are assumed to be piecewise continuously differentiable, and let a be a complex number not in the image of γ. Write down an expression for the winding number $n(\gamma, a)$ in terms of a contour integral. From this characterization of the winding number, prove the following properties:

(a) If γ_1 and γ_2 are closed paths not passing through zero, and if $\gamma : [0,1] \to \mathbf{C}$ is defined by $\gamma(t) = \gamma_1(t)\gamma_2(t)$ for all t, then

$$n(\gamma, 0) = n(\gamma_1, 0) + n(\gamma_2, 0).$$

(b) If $\eta : [0,1] \to \mathbf{C}$ is a closed path whose image is contained in $\{\mathrm{Re}(z) > 0\}$, then $n(\eta, 0) = 0$.

(c) If γ_1 and γ_2 are closed paths and a is a complex number, not in the image of either path, such that

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t) - a|$$

for all t, then $n(\gamma_1, a) = n(\gamma_2, a)$.

[*You may wish here to consider the path defined by* $\eta(t) = 1 - (\gamma_1(t) - \gamma_2(t))/(\gamma_1(t) - a)$.]

5H Methods

Show how the general solution of the wave equation for $y(x,t)$,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} y(x,t) - \frac{\partial^2}{\partial x^2} y(x,t) = 0\,,$$

can be expressed as

$$y(x,t) = f(ct - x) + g(ct + x)\,.$$

Show that the boundary conditions $y(0,t) = y(L,t) = 0$ relate the functions f and g and require them to be periodic with period $2L$.

Show that, with these boundary conditions,

$$\frac{1}{2}\int_0^L \left(\frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 \right) \mathrm{d}x = \int_{-L}^{L} g'(ct + x)^2 \,\mathrm{d}x\,,$$

and that this is a constant independent of t.

6G Quantum Mechanics

Define the commutator $[A\,,B]$ of two operators, A and B. In three dimensions angular momentum is defined by a vector operator $\mathbf{L}$ with components

$$L_x = y\,p_z - z\,p_y \qquad L_y = z\,p_x - x\,p_z \qquad L_z = x\,p_y - y\,p_x\,.$$

Show that $[L_x\,,L_y] = i\,\hbar\,L_z$ and use this, together with permutations, to show that $[\mathbf{L^2}\,,L_w] = 0$, where w denotes any of the directions x, y, z.

At a given time the wave function of a particle is given by

$$\psi = (x+y+z)\,\exp\left(-\sqrt{x^2+y^2+z^2}\right)\,.$$

Show that this is an eigenstate of $\mathbf{L}^2$ with eigenvalue equal to $2\hbar^2$.

7H Electromagnetism

For a static current density $\mathbf{J}(\mathbf{x})$ show that we may choose the vector potential $\mathbf{A}(\mathbf{x})$ so that

$$-\nabla^2\mathbf{A} = \mu_0\mathbf{J}\,.$$

For a loop L, centred at the origin, carrying a current I show that

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\oint_L \frac{1}{|\mathbf{x}-\mathbf{r}|}\,\mathrm{d}\mathbf{r} \sim -\frac{\mu_0 I}{4\pi}\,\frac{1}{|\mathbf{x}|^3}\oint_L \tfrac{1}{2}\,\mathbf{x}\times(\mathbf{r}\times\mathrm{d}\mathbf{r}) \quad \text{as} \quad |\mathbf{x}|\to\infty\,.$$

[*You may assume*

$$-\nabla^2\frac{1}{4\pi|\mathbf{x}|} = \delta^3(\mathbf{x})\,,$$

and for fixed vectors $\mathbf{a},\mathbf{b}$

$$\oint_L \mathbf{a}\cdot\mathrm{d}\mathbf{r} = 0, \qquad \oint_L (\mathbf{a}\cdot\mathbf{r}\;\mathbf{b}\cdot\mathrm{d}\mathbf{r} + \mathbf{b}\cdot\mathbf{r}\;\mathbf{a}\cdot\mathrm{d}\mathbf{r}) = 0\,.\Bigg]$$

8F Numerical Analysis

Define Gaussian quadrature.

Evaluate the coefficients of the Gaussian quadrature of the integral

$$\int_{-1}^{1}(1-x^2)f(x)dx$$

which uses two function evaluations.

9D Markov Chains

Prove that the simple symmetric random walk in three dimensions is transient.

[*You may wish to recall Stirling's formula:* $n! \sim (2\pi)^{\frac{1}{2}}\, n^{n+\frac{1}{2}} e^{-n}$.]

SECTION II

10B Linear Algebra

(i) Let V be a finite-dimensional real vector space with an inner product. Let $e_1, \ldots, e_n$ be a basis for V. Prove by an explicit construction that there is an orthonormal basis $f_1, \ldots, f_n$ for V such that the span of $e_1, \ldots, e_i$ is equal to the span of $f_1, \ldots, f_i$ for every $1 \leqslant i \leqslant n$.

(ii) For any real number a, consider the quadratic form

$$q_a(x, y, z) = xy + yz + zx + ax^2$$

on $\mathbf{R}^3$. For which values of a is q_a nondegenerate? When q_a is nondegenerate, compute its signature in terms of a.

11C Groups, Rings and Modules

Let R be the ring of Gaussian integers $\mathbb{Z}[i]$, where $i^2 = -1$, which you may assume to be a unique factorization domain. Prove that every prime element of R divides precisely one positive prime number in $\mathbb{Z}$. List, without proof, the prime elements of R, up to associates.

Let p be a prime number in $\mathbb{Z}$. Prove that R/pR has cardinality p^2. Prove that $R/2R$ is not a field. If $p \equiv 3 \bmod 4$, show that R/pR is a field. If $p \equiv 1 \bmod 4$, decide whether R/pR is a field or not, justifying your answer.

12A Geometry

Given a parametrized smooth embedded surface $\sigma : V \to U \subset \mathbf{R}^3$, where V is an open subset of $\mathbf{R}^2$ with coordinates (u, v), and a point $P \in U$, define what is meant by the *tangent space* at P, the *unit normal* $\mathbf{N}$ at P, and the *first fundamental form*

$$Edu^2 + 2F du\, dv + G dv^2.$$

[*You need not show that your definitions are independent of the parametrization.*]

The *second fundamental form* is defined to be

$$Ldu^2 + 2M du\, dv + N dv^2,$$

where $L = \sigma_{uu} \cdot \mathbf{N}$, $M = \sigma_{uv} \cdot \mathbf{N}$ and $N = \sigma_{vv} \cdot \mathbf{N}$. Prove that the partial derivatives of $\mathbf{N}$ (considered as a vector-valued function of u, v) are of the form $\mathbf{N}_u = a\sigma_u + b\sigma_v$, $\mathbf{N}_v = c\sigma_u + d\sigma_v$, where

$$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Explain briefly the significance of the determinant $ad - bc$.

13B Analysis II

Let $F : [-a, a] \times [x_0 - r, x_0 + r] \to \mathbf{R}$ be a continuous function. Let C be the maximum value of $|F(t, x)|$. Suppose there is a constant K such that

$$|F(t, x) - F(t, y)| \leqslant K|x - y|$$

for all $t \in [-a, a]$ and $x, y \in [x_0 - r, x_0 + r]$. Let $b < \min(a, r/C, 1/K)$. Show that there is a unique C^1 function $x : [-b, b] \to [x_0 - r, x_0 + r]$ such that

$$x(0) = x_0$$

and

$$\frac{dx}{dt} = F(t, x(t)).$$

[*Hint: First show that the differential equation with its initial condition is equivalent to the integral equation*

$$x(t) = x_0 + \int_0^t F(s, x(s))\, ds.$$]

14A Metric and Topological Spaces

Let (M, d) be a metric space, and F a non-empty closed subset of M. For $x \in M$, set

$$d(x, F) = \inf_{z \in F} d(x, z).$$

Prove that $d(x, F)$ is a continuous function of x, and that it is strictly positive for $x \notin F$.

A topological space is called *normal* if for any pair of disjoint closed subsets F_1, F_2, there exist disjoint open subsets $U_1 \supset F_1$, $U_2 \supset F_2$. By considering the function

$$d(x, F_1) - d(x, F_2),$$

or otherwise, deduce that any metric space is normal.

Suppose now that X is a normal topological space, and that F_1, F_2 are disjoint closed subsets in X. Prove that there exist open subsets $W_1 \supset F_1, W_2 \supset F_2$, whose *closures* are disjoint. In the case when $X = \mathbf{R}^2$ with the standard metric topology, $F_1 = \{(x, -1/x) : x < 0\}$ and $F_2 = \{(x, 1/x) : x > 0\}$, find explicit open subsets W_1, W_2 with the above property.

15F Complex Methods

Determine the Fourier expansion of the function $f(x) = \sin \lambda x$, where $-\pi \leqslant x \leqslant \pi$, in the two cases where λ is an integer and λ is a real non-integer.

Using the Parseval identity in the case $\lambda = \frac{1}{2}$, find an explicit expression for the sum

$$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2 - 1)^2}.$$

16H Methods

Define an isotropic tensor and show that δ_{ij}, ϵ_{ijk} are isotropic tensors.

For $\hat{\mathbf{x}}$ a unit vector and $\mathrm{d}S(\hat{\mathbf{x}})$ the area element on the unit sphere show that

$$\int \mathrm{d}S(\hat{\mathbf{x}})\, \hat{x}_{i_1} \ldots \hat{x}_{i_n}$$

is an isotropic tensor for any n. Hence show that

$$\int \mathrm{d}S(\hat{\mathbf{x}})\, \hat{x}_i \hat{x}_j = a\delta_{ij}\,, \qquad \int \mathrm{d}S(\hat{\mathbf{x}})\, \hat{x}_i \hat{x}_j \hat{x}_k = 0\,,$$
$$\int \mathrm{d}S(\hat{\mathbf{x}})\, \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l = b\big(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big)\,,$$

for some a, b which should be determined.

Explain why

$$\int_V \mathrm{d}^3x \,\big(x_1 + \sqrt{-1}\, x_2\big)^n f(|\mathbf{x}|) = 0\,, \quad n = 2, 3, 4\,,$$

where V is the region inside the unit sphere.

[*The general isotropic tensor of rank 4 has the form* $a\,\delta_{ij}\delta_{kl} + b\,\delta_{ik}\delta_{jl} + c\,\delta_{il}\delta_{jk}$.]

17G Special Relativity

Obtain the Lorentz transformations that relate the coordinates of an event measured in one inertial frame (t, x, y, z) to those in another inertial frame moving with velocity v along the x axis. Take care to state the assumptions that lead to your result.

A star is at rest in a three-dimensional coordinate frame $\mathcal{S}'$ that is moving at constant velocity v along the x axis of a second coordinate frame $\mathcal{S}$. The star emits light of frequency ν', which may considered to be a beam of photons. A light ray from the star to the origin in $\mathcal{S}'$ is a straight line that makes an angle θ' with the x' axis. Write down the components of the four-momentum of a photon in this light ray.

The star is seen by an observer at rest at the origin of $\mathcal{S}$ at time $t = t' = 0$, when the origins of the coordinate frames $\mathcal{S}$ and $\mathcal{S}'$ coincide. The light that reaches the observer moves along a line through the origin that makes an angle θ to the x axis and has frequency ν. Make use of the Lorentz transformations between the four-momenta of a photon in these two frames to determine the relation

$$\lambda = \lambda' \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \left(1 + \frac{v}{c}\cos\theta\right) .$$

where λ is the observed wavelength of the photon and λ' is the wavelength in the star's rest frame.

Comment on the form of this result in the special cases with $\cos\theta = 1$, $\cos\theta = -1$ and $\cos\theta = 0$.

[*You may assume that the energy of a photon of frequency ν is $h\nu$ and its three-momentum is a three-vector of magnitude $h\nu/c$.*]

18E Fluid Dynamics

A fluid of density ρ_1 occupies the region $z > 0$ and a second fluid of density ρ_2 occupies the region $z < 0$. State the equations and boundary conditions that are satisfied by the corresponding velocity potentials ϕ_1 and ϕ_2 and pressures p_1 and p_2 when the system is perturbed so that the interface is at $z = \zeta(x, t)$ and the motion is irrotational.

Seek a set of linearised equations and boundary conditions when the disturbances are proportional to $e^{i(kx-\omega t)}$, and derive the dispersion relation

$$\omega^2 = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} gk,$$

where g is the gravitational acceleration.

19D Statistics

Let $Y_1, \ldots, Y_n$ be observations satisfying

$$Y_i = \beta x_i + \epsilon_i, \quad 1 \leqslant i \leqslant n,$$

where $\epsilon_1, \ldots, \epsilon_n$ are independent random variables each with the $N(0, \sigma^2)$ distribution. Here $x_1, \ldots, x_n$ are known but β and σ^2 are unknown.

(i) Determine the maximum-likelihood estimates $(\widehat{\beta}, \widehat{\sigma}^2)$ of (β, σ^2).

(ii) Find the distribution of $\widehat{\beta}$.

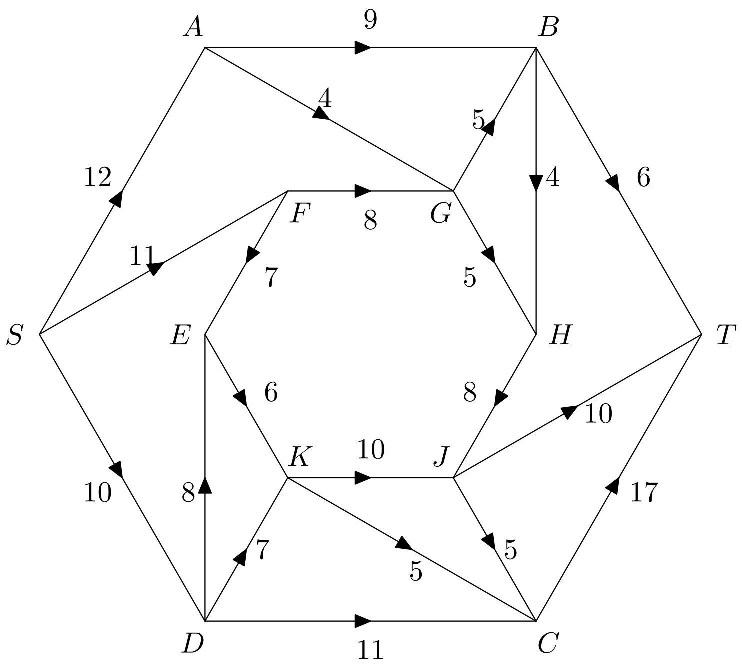
(iii) By showing that $Y_i - \widehat{\beta} x_i$ and $\widehat{\beta}$ are independent, or otherwise, determine the joint distribution of $\widehat{\beta}$ and $\widehat{\sigma}^2$.

(iv) Explain carefully how you would test the hypothesis $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$.

20D Optimization

Describe the Ford–Fulkerson algorithm for finding a maximal flow from a source to a sink in a directed network with capacity constraints on the arcs. Explain why the algorithm terminates at an optimal flow when the initial flow and the capacity constraints are rational.

Illustrate the algorithm by applying it to the problem of finding a maximal flow from S to T in the network below.



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