UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 6 June 2005 9 to 12

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***C*** *and* ***E*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIRMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

Find the unique positive integer a with $a \leq 19$, for which

$$17! \cdot 3^{16} \equiv a \pmod{19}.$$

Results used should be stated but need not be proved.

Solve the system of simultaneous congruences

$$\begin{aligned} x &\equiv 1 \pmod{2}, \\ x &\equiv 1 \pmod{3}, \\ x &\equiv 3 \pmod{4}, \\ x &\equiv 4 \pmod{5}. \end{aligned}$$

Explain very briefly your reasoning.

2E Numbers and Sets

Give a combinatorial definition of the binomial coefficient $\binom{n}{m}$ for any non-negative integers n, m.

Prove that $\binom{n}{m} = \binom{n}{n-m}$ for $0 \leq m \leq n$.

Prove the identities

$$\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}$$

and

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{n+m}{k}.$$

3C Dynamics

Planetary Explorers Ltd. want to put a communications satellite of mass m into geostationary orbit around the spherical planet Zog (*i.e.* with the satellite always above the same point on the surface of Zog). The mass of Zog is M, the length of its day is T and G is the gravitational constant.

Write down the equations of motion for a general orbit of the satellite and determine the radius and speed of the geostationary orbit.

Describe briefly how the orbit is modified if the satellite is released at the correct radius and on the correct trajectory for a geostationary orbit, but with a little too much speed. Comment on how the satellite's speed varies around such an orbit.

4C Dynamics

A car of mass M travelling at speed U on a smooth, horizontal road attempts an emergency stop. The car skids in a straight line with none of its wheels able to rotate.

Calculate the stopping distance and time on a dry road where the dry friction coefficient between the tyres and the road is μ.

At high speed on a wet road the grip of each of the four tyres changes from dry friction to a lubricated drag equal to $\frac{1}{4}\lambda u$ for each tyre, where λ is the drag coefficient and u the instantaneous speed of the car. However, the tyres regain their dry-weather grip when the speed falls below $\frac{1}{4}U$. Calculate the stopping distance and time under these conditions.

SECTION II

5E Numbers and Sets

What does it mean for a set to be countable? Show that $\mathbb{Q} \times \mathbb{Q}$ is countable, and $\mathbb{R}$ is not countable.

Let D be any set of non-trivial discs in a plane, any two discs being disjoint. Show that D is countable.

Give an example of a set C of non-trivial circles in a plane, any two circles being disjoint, which is not countable.

6E Numbers and Sets

Let R be a relation on the set S. What does it mean for R to be an equivalence relation on S? Show that if R is an equivalence relation on S, the set of equivalence classes forms a partition of S.

Let G be a group, and let H be a subgroup of G. Define a relation R on G by $a\ R\ b$ if $a^{-1}b \in H$. Show that R is an equivalence relation on G, and that the equivalence classes are precisely the left cosets gH of H in G. Find a bijection from H to any other coset gH. Deduce that if G is finite then the order of H divides the order of G.

Let g be an element of the finite group G. The order $o(g)$ of g is the least positive integer n for which $g^n = 1$, the identity of G. If $o(g) = n$, then G has a subgroup of order n; deduce that $g^{|G|} = 1$ for all $g \in G$.

Let m be a natural number. Show that the set of integers in $\{1, 2, \ldots, m\}$ which are prime to m is a group under multiplication modulo m. [*You may use any properties of multiplication and divisibility of integers without proof, provided you state them clearly.*]

Deduce that if a is any integer prime to m then $a^{\phi(m)} \equiv 1$ (mod m), where ϕ is the Euler totient function.

7E Numbers and Sets

State and prove the Principle of Inclusion and Exclusion.

Use the Principle to show that the Euler totient function ϕ satisfies

$$\phi(p_1^{c_1} \cdots p_r^{c_r}) = p_1^{c_1-1}(p_1 - 1) \cdots p_r^{c_r-1}(p_r - 1).$$

Deduce that if a and b are coprime integers, then $\phi(ab) = \phi(a)\phi(b)$, and more generally, that if d is any divisor of n then $\phi(d)$ divides $\phi(n)$.

Show that if $\phi(n)$ divides n then $n = 2^c 3^d$ for some non-negative integers c, d.

8E Numbers and Sets

The Fibonacci numbers are defined by the equations $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for any positive integer n. Show that the highest common factor (F_{n+1}, F_n) is 1.

Let n be a natural number. Prove by induction on k that for all positive integers k,

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n.$$

Deduce that F_n divides F_{nl} for all positive integers l. Deduce also that if $m \geq n$ then

$$(F_m, F_n) = (F_{m-n}, F_n).$$

9C Dynamics

A particle of mass m and charge q moving in a vacuum through a magnetic field $\mathbf{B}$ and subject to no other forces obeys

$$m\,\ddot{\mathbf{r}} = q\,\dot{\mathbf{r}} \times \mathbf{B},$$

where $\mathbf{r}(t)$ is the location of the particle.

For $\mathbf{B} = (0, 0, B)$ with constant B, and using cylindrical polar coordinates $\mathbf{r} = (r, \theta, z)$, or otherwise, determine the motion of the particle in the $z = 0$ plane if its initial speed is u_0 with $\dot{z} = 0$. [*Hint: Choose the origin so that* $\dot{r} = 0$ *and* $\ddot{r} = 0$ *at* $t = 0$.]

Due to a leak, a small amount of gas enters the system, causing the particle to experience a drag force $\mathbf{D} = -\mu\dot{\mathbf{r}}$, where $\mu \ll qB$. Write down the new governing equations and show that the speed of the particle decays exponentially. Sketch the path followed by the particle. [*Hint: Consider the equations for the velocity in Cartesian coordinates; you need not apply any initial conditions.*]

10C Dynamics

A keen cyclist wishes to analyse her performance on training rollers. She decides that the key components are her bicycle's rear wheel and the roller on which the wheel sits. The wheel, of radius R, has its mass M entirely at its outer edge. The roller, which is driven by the wheel without any slippage, is a solid cylinder of radius S and mass $M/2$. The angular velocities of the wheel and roller are ω and σ, respectively.

Determine I and J, the moments of inertia of the wheel and roller, respectively. Find the ratio of the angular velocities of the wheel and roller. Show that the combined total kinetic energy of the wheel and roller is $\frac{1}{2}K\omega^2$, where

$$K = \frac{5}{4}MR^2$$

is the effective combined moment of inertia of the wheel and roller.

Why should K be used instead of just I or J in the equation connecting torque with angular acceleration? The cyclist believes the torque she can produce at the back wheel is $T = Q(1 - \omega/\Omega)$ where Q and Ω are dimensional constants. Determine the angular velocity of the wheel, starting from rest, as a function of time.

In an attempt to make the ride more realistic, the cyclist adds a fan (of negligible mass) to the roller. The fan imposes a frictional torque $-\gamma\sigma^2$ on the roller, where γ is a dimensional constant. Determine the new maximum speed for the wheel.

11C Dynamics

A puck of mass m located at $\mathbf{r} = (x, y)$ slides without friction under the influence of gravity on a surface of height $z = h(x, y)$. Show that the equations of motion can be approximated by

$$\ddot{\mathbf{r}} = -g\nabla h\,,$$

where g is the gravitational acceleration and the small slope approximation $\sin\phi \approx \tan\phi$ is used.

Determine the motion of the puck when $h(x, y) = \alpha x^2$.

Sketch the surface

$$h(x, y) = h(r) = \frac{1}{r^2} - \frac{1}{r}$$

as a function of r, where $r^2 = x^2 + y^2$. Write down the equations of motion of the puck on this surface in polar coordinates $\mathbf{r} = (r, \theta)$ under the assumption that the small slope approximation can be used. Show that L, the angular momentum per unit mass about the origin, is conserved. Show also that the initial kinetic energy per unit mass of the puck is $E_0 = \frac{1}{2}L^2/r_0^2$ if the puck is released at radius r_0 with negligible radial velocity. Determine and sketch $\dot{r}^2$ as a function of r for this release condition. What condition relating L, r_0 and g must be satisfied for the orbit to be bounded?

12C Dynamics

In an experiment a ball of mass m is released from a height h_0 above a flat, horizontal plate. Assuming the gravitational acceleration g is constant and the ball falls through a vacuum, find the speed u_0 of the ball on impact.

Determine the speed u_1 at which the ball rebounds if the coefficient of restitution for the collision is γ. What fraction of the impact energy is dissipated during the collision? Determine also the maximum height h_n the ball reaches after the n^{th} bounce, and the time T_n between the n^{th} and $(n+1)^{th}$ bounce. What is the total distance travelled by the ball before it comes to rest if $\gamma < 1$?

If the experiment is repeated in an atmosphere then the ball experiences a drag force $D = -\alpha\,|u|\,u$, where α is a dimensional constant and u the instantaneous velocity of the ball. Write down and solve the modified equation for $u(t)$ before the ball first hits the plate.

END OF PAPER