UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 7 June 2005 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* **three** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A*** *and* ***D*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIRMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Algebra and Geometry

Let A be a real 3×3 symmetric matrix with eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Consider the surface S in $\mathbb{R}^3$ given by

$$x^T A x = 1.$$

Find the minimum distance between the origin and S. How many points on S realize this minimum distance? Justify your answer.

2D Algebra and Geometry

Define what it means for a group to be cyclic. If p is a prime number, show that a finite group G of order p must be cyclic. Find all homomorphisms $\varphi : C_{11} \to C_{14}$, where C_n denotes the cyclic group of order n. [*You may use Lagrange's theorem.*]

3A Vector Calculus

Let $\mathbf{A}(t,\mathbf{x})$ and $\mathbf{B}(t,\mathbf{x})$ be time-dependent, continuously differentiable vector fields on $\mathbb{R}^3$ satisfying

$$\frac{\partial \mathbf{A}}{\partial t} = \nabla \times \mathbf{B} \quad \text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{A}\,.$$

Show that for any bounded region V,

$$\frac{d}{dt}\left[\frac{1}{2}\int_V (\mathbf{A}^2 + \mathbf{B}^2)dV\right] = -\int_S (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{S}\,,$$

where S is the boundary of V.

4A Vector Calculus

Given a curve $\boldsymbol{\gamma}(s)$ in $\mathbb{R}^3$, parameterised such that $\|\boldsymbol{\gamma}'(s)\| = 1$ and with $\boldsymbol{\gamma}''(s) \neq 0$, define the tangent $\mathbf{t}(s)$, the principal normal $\mathbf{p}(s)$, the curvature $\kappa(s)$ and the binormal $\mathbf{b}(s)$.

The torsion $\tau(s)$ is defined by

$$\tau = -\mathbf{b}' \cdot \mathbf{p}\,.$$

Sketch a circular helix showing $\mathbf{t}, \mathbf{p}, \mathbf{b}$ and $\mathbf{b}'$ at a chosen point. What is the sign of the torsion for your helix? Sketch a second helix with torsion of the opposite sign.

SECTION II

5D Algebra and Geometry

Define the notion of an action of a group G on a set X. Assuming that G is finite, state and prove the Orbit-Stabilizer Theorem.

Let G be a finite group and X the set of its subgroups. Show that $g(K) = gKg^{-1}$ ($g \in G$, $K \in X$) defines an action of G on X. If H is a subgroup of G, show that the orbit of H has at most $|G|/|H|$ elements.

Suppose H is a subgroup of G and $H \neq G$. Show that there is an element of G which does not belong to any subgroup of the form gHg^{-1} for $g \in G$.

6D Algebra and Geometry

Let $\mathcal{M}$ be the group of Möbius transformations of $\mathbb{C} \cup \{\infty\}$ and let $SL(2, \mathbb{C})$ be the group of all 2×2 complex matrices with determinant 1.

Show that the map $\theta : SL(2, \mathbb{C}) \to \mathcal{M}$ given by

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}$$

is a surjective homomorphism. Find its kernel.

Show that every $T \in \mathcal{M}$ not equal to the identity is conjugate to a Möbius map S where either $Sz = \mu z$ with $\mu \neq 0, 1$, or $Sz = z \pm 1$. [*You may use results about matrices in $SL(2, \mathbb{C})$, provided they are clearly stated.*]

Show that if $T \in \mathcal{M}$, then T is the identity, or T has one, or two, fixed points. Also show that if $T \in \mathcal{M}$ has only one fixed point z_0 then $T^n z \to z_0$ as $n \to \infty$ for any $z \in \mathbb{C} \cup \{\infty\}$.

7D Algebra and Geometry

Let G be a group and let $Z(G) = \{h \in G : \ gh = hg \text{ for all } \ g \in G\}$. Show that $Z(G)$ is a normal subgroup of G.

Let H be the set of all 3×3 real matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$. Show that H is a subgroup of the group of invertible real matrices under multiplication.

Find $Z(H)$ and show that $H/Z(H)$ is isomorphic to $\mathbb{R}^2$ with vector addition.

8D Algebra and Geometry

Let A be a 3×3 real matrix such that $\det(A) = -1$, $A \neq -I$, and $A^T A = I$, where A^T is the transpose of A and I is the identity.

Show that the set E of vectors x for which $Ax = -x$ forms a 1-dimensional subspace.

Consider the plane Π through the origin which is orthogonal to E. Show that A maps Π to itself and induces a rotation of Π by angle θ, where $\cos\theta = \frac{1}{2}(\operatorname{trace}(A) + 1)$. Show that A is a reflection in Π if and only if A has trace 1. [*You may use the fact that* $\operatorname{trace}(BAB^{-1}) = \operatorname{trace}(A)$ *for any invertible matrix* B.]

Prove that $\det(A - I) = 4(\cos\theta - 1)$.

9A Vector Calculus

Let V be a bounded region of $\mathbb{R}^3$ and S be its boundary. Let ϕ be the unique solution to $\nabla^2\phi = 0$ in V, with $\phi = f(\mathbf{x})$ on S, where f is a given function. Consider any smooth function w also equal to $f(\mathbf{x})$ on S. Show, by using Green's first theorem or otherwise, that

$$\int_V |\nabla w|^2 \, dV \quad \geqslant \quad \int_V |\nabla \phi|^2 \, dV \, .$$

[*Hint: Set* $w = \phi + \delta$.]

Consider the partial differential equation

$$\frac{\partial}{\partial t} w = \nabla^2 w \, ,$$

for $w(t, \mathbf{x})$, with initial condition $w(0, \mathbf{x}) = w_0(\mathbf{x})$ in V, and boundary condition $w(t, \mathbf{x}) = f(\mathbf{x})$ on S for all $t \geqslant 0$. Show that

$$\frac{\partial}{\partial t} \int_V |\nabla w|^2 \, dV \quad \leqslant 0 \, , \qquad (*)$$

with equality holding only when $w(t, \mathbf{x}) = \phi(\mathbf{x})$.

Show that $(*)$ remains true with the boundary condition

$$\frac{\partial w}{\partial t} + \alpha(\mathbf{x}) \frac{\partial w}{\partial n} = 0$$

on S, provided $\alpha(\mathbf{x}) \geqslant 0$.

10A Vector Calculus

Write down Stokes' theorem for a vector field $\mathbf{B}(\mathbf{x})$ on $\mathbb{R}^3$.

Consider the bounded surface S defined by

$$z = x^2 + y^2, \quad \frac{1}{4} \leqslant z \leqslant 1 \, .$$

Sketch the surface and calculate the surface element $d\mathbf{S}$. For the vector field

$$\mathbf{B} = (-y^3, x^3, z^3) \, ,$$

calculate $I = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S}$ directly.

Show using Stokes' theorem that I may be rewritten as a line integral and verify this yields the same result.

11A Vector Calculus

Explain, with justification, the significance of the eigenvalues of the Hessian in classifying the critical points of a function $f : \mathbb{R}^n \to \mathbb{R}$. In what circumstances are the eigenvalues inconclusive in establishing the character of a critical point?

Consider the function on $\mathbb{R}^2$,

$$f(x, y) = xye^{-\alpha(x^2+y^2)} .$$

Find and classify all of its critical points, for all real α. How do the locations of the critical points change as $\alpha \to 0$?

12A Vector Calculus

Express the integral

$$I = \int_0^\infty dx \int_0^1 dy \int_0^x dz\ xe^{-Ax/y - Bxy - Cyz}$$

in terms of the new variables $\alpha = x/y$, $\beta = xy$, and $\gamma = yz$. Hence show that

$$I = \frac{1}{2A(A+B)(A+B+C)} .$$

You may assume A, B and C are positive. [*Hint: Remember to calculate the limits of the integral.*]

END OF PAPER