UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Thursday 3 June 2004 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

The number of marks for each question is the same.

Additional credit will be given for a substantially complete answer.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **2F, 11F** *should be in one bundle and* **1I, 12I** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1I Markov Chains

(i) Give the definition of the *time-reversal* of a discrete-time Markov chain (X_n). Define a *reversible* Markov chain and check that every probability distribution satisfying the detailed balance equations is invariant.

(ii) Customers arrive in a hairdresser's shop according to a Poisson process of rate $\lambda > 0$. The shop has s hairstylists and N waiting places; each stylist is working (on a single customer) provided that there is a customer to serve, and any customer arriving when the shop is full (i.e. the numbers of customers present is $N + s$) is not admitted and never returns. Every admitted customer waits in the queue and then is served, in the first-come-first-served order (say), the service taking an exponential time of rate $\mu > 0$; the service times of admitted customers are independent. After completing his/her service, the customer leaves the shop and never returns.

Set up a Markov chain model for the number X_t of customers in the shop at time $t \geq 0$. Assuming $\lambda < s\mu$, calculate the equilibrium distribution π of this chain and explain why it is unique. Show that (X_t) in equilibrium is time-reversible, i.e. $\forall\, T > 0, (X_t, 0 \leq t \leq T)$ has the same distribution as $(Y_t, 0 \leq t \leq T)$ where $Y_t = X_{T-t}$, and $X_0 \sim \pi$.

2F Functional Analysis

(i) Let H be an infinite-dimensional Hilbert space. Show that H has a (countable) orthonormal basis if and only if H has a countable dense subset. [*You may assume familiarity with the Gram-Schmidt process.*]

State and prove Bessel's inequality.

(ii) State Parseval's equation. Using this, prove that if H has a countable dense subset then there is a surjective isometry from H to l^2.

Explain carefully why the functions $e^{in\theta}$, $n \in \mathbb{Z}$, form an orthonormal basis for $L^2(\mathbb{T})$.

3C Electromagnetism

(i) State Maxwell's equations and show that the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ can be expressed in terms of a scalar potential ϕ and a vector potential $\mathbf{A}$. Hence derive the inhomogeneous wave equations that are satisfied by ϕ and $\mathbf{A}$ respectively.

(ii) The plane $x = 0$ separates a vacuum in the half-space $x < 0$ from a perfectly conducting medium occupying the half-space $x > 0$. Derive the boundary conditions on $\mathbf{E}$ and $\mathbf{B}$ at $x = 0$.

A plane electromagnetic wave with a magnetic field $\mathbf{B} = B(t, x, z)\hat{\mathbf{y}}$, travelling in the xz-plane at an angle θ to the x-direction, is incident on the interface at $x = 0$. If the wave has frequency ω show that the total magnetic field is given by

$$\mathbf{B} = B_0 \cos\left(\tfrac{\omega x}{c}\cos\theta\right)\exp\left[i\left(\tfrac{\omega z}{c}\sin\theta - \omega t\right)\right]\hat{\mathbf{y}},$$

where B_0 is a constant. Hence find the corresponding electric field $\mathbf{E}$, and obtain the surface charge density and the surface current at the interface.

4B Dynamics of Differential Equations

(i) Describe the use of the *stroboscopic method* for obtaining approximate solutions to the second order equation

$$\ddot{x} + x = \epsilon f(x, \dot{x}, t)$$

when $|\epsilon| \ll 1$. In particular, by writing $x = R\cos(t + \phi)$, $\dot{x} = -R\sin(t + \phi)$, obtain expressions in terms of f for the rate of change of R and ϕ. Evaluate these expressions when $f = x^2 \cos t$.

(ii) In planetary orbit theory a crude model of an orbit subject to perturbation from a distant body is given by the equation

$$\frac{d^2u}{d\theta^2} + u = \lambda - \delta^2 u^{-2} - 2\delta^3 u^{-3}\cos\theta,$$

where $0 < \delta \ll 1$, (u^{-1}, θ) are polar coordinates in the plane, and λ is a positive constant.

(a) Show that when $\delta = 0$ all bounded orbits are closed.

(b) Now suppose $\delta \neq 0$, and look for almost circular orbits with $u = \lambda + \delta w(\theta) + a\delta^2$, where a is a constant. By writing $w = R(\theta)\cos(\theta + \phi(\theta))$, and by making a suitable choice of the constant a, use the stroboscopic method to find equations for $dw/d\theta$ and $d\phi/d\theta$. By writing $z = R\exp(i\phi)$ and considering $dz/d\theta$, or otherwise, determine $R(\theta)$ and $\phi(\theta)$ in the case $R(0) = R_0$, $\phi(0) = 0$. Hence describe the orbits of the system.

5G Representation Theory

Compute the character table for the group A_5 of even permutations of five elements. You may wish to follow the steps below.

(a) List the conjugacy classes in A_5 and their orders.

(b) A_5 acts on $\mathbb{C}^5$ by permuting the standard basis vectors. Show that $\mathbb{C}^5$ splits as $\mathbb{C} \oplus V$, where $\mathbb{C}$ is the trivial 1-dimensional representation and V is irreducible.

(c) By using the formula for the character of the symmetric square S^2V,

$$\chi_{S^2V}(g) = \frac{1}{2}\left[\chi_V(g)^2 + \chi_V(g^2)\right],$$

decompose S^2V to produce a 5-dimensional, irreducible representation, and find its character.

(d) Show that the exterior square $\Lambda^2 V$ decomposes into two distinct irreducibles and compute their characters, to complete the character table of A_5.

[*Hint: You can save yourself some computational effort if you can explain why the automorphism of A_5, defined by conjugation by a transposition in S_5, must swap the two summands of $\Lambda^2 V$.*]

6H Galois Theory

Let K be a field, and G a finite subgroup of K^*. Show that G is cyclic.

Define the cyclotomic polynomials Φ_m, and show from your definition that

$$X^m - 1 = \prod_{d|m} \Phi_d(X).$$

Deduce that Φ_m is a polynomial with integer coefficients.

Let p be a prime with $(m,p)=1$. Let $\Phi_m \equiv f_1 \ldots f_r \pmod p$, where $f_i \in \mathbb{F}_p[X]$ are irreducible. Show that for each i the degree of f_i is equal to the order of p in the group $(\mathbb{Z}/m\mathbb{Z})^*$.

Use this to write down an irreducible polynomial of degree 10 over $\mathbb{F}_2$.

7G Algebraic Topology

A finite simplicial complex K is the union of subcomplexes L and M. Describe the Mayer-Vietoris exact sequence that relates the homology groups of K to those of L, M and $L \cap M$. Define all the homomorphisms in the sequence, proving that they are *well-defined* (a proof of exactness is *not* required).

A surface X is constructed by identifying together (by means of a homeomorphism) the boundaries of two Möbius strips Y and Z. Assuming relevant triangulations exist, determine the homology groups of X.

8G Hilbert Spaces

Let H be a Hilbert space. An operator T in $L(H)$ is *normal* if $TT^* = T^*T$. Suppose that T is normal and that $\sigma(T) \subseteq \mathbb{R}$. Let $U = (T + iI)(T - iI)^{-1}$.

(a) Suppose that A is invertible and $AT = TA$. Show that $A^{-1}T = TA^{-1}$.

(b) Show that U is normal, and that $\sigma(U) \subseteq \{\lambda : |\lambda| = 1\}$.

(c) Show that U^{-1} is normal.

(d) Show that U is unitary.

(e) Show that T is Hermitian.

[*You may use the fact that, if S is normal, the spectral radius of S is equal to $\|S\|$.*]

9H Riemann Surfaces

(a) Let $f : R \to S$ be a non-constant holomorphic map between compact connected Riemann surfaces R and S.

Define the *branching order* $v_f(p)$ at a point $p \in R$ and show that it is well-defined. Show further that if h is a holomorphic map on S then $v_{h\circ f}(p) = v_h(f(p))\, v_f(p)$.

Define the *degree* of f and state the Riemann–Hurwitz formula. Show that if R has Euler characteristic 0 then either S is the 2-sphere or $v_p(f) = 1$ for all $p \in R$.

(b) Let P and Q be complex polynomials of degree $m \geq 2$ with no common roots. Explain briefly how the rational function $P(z)/Q(z)$ induces a holomorphic map F from the 2-sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$ to itself. What is the degree of F? Show that there is at least one and at most $2m - 2$ points $w \in S^2$ such that the number of *distinct* solutions $z \in S^2$ of the equation $F(z) = w$ is strictly less than $\deg F$.

10H Algebraic Curves

(i) Let $f : X \to Y$ be a morphism of smooth projective curves. Define the divisor $f^*(D)$ if D is a divisor on Y, and state the "finiteness theorem".

(ii) Suppose $f : X \to \mathbb{P}^1$ is a morphism of degree 2, that X is smooth projective, and that $X \neq \mathbb{P}^1$. Let $P, Q \in X$ be distinct ramification points for f. Show that, as elements of $cl(X)$, we have $[P] \neq [Q]$, but $2[P] = 2[Q]$.

11F Logic, Computation and Set Theory

(i) State and prove the Compactness Theorem for first-order predicate logic.

State and prove the Upward Löwenheim-Skolem Theorem.

[*You may use the Completeness Theorem for first-order predicate logic.*]

(ii) For each of the following theories, either give axioms (in the language of posets) for the theory or prove carefully that the theory is not axiomatisable.

(a) The theory of posets having no maximal element.

(b) The theory of posets having a unique maximal element.

(c) The theory of posets having infinitely many maximal elements.

(d) The theory of posets having finitely many maximal elements.

(e) The theory of countable posets having a unique maximal element.

12I Probability and Measure

(a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\theta : \Omega \to \Omega$ be measurable. What is meant by saying that θ is *measure-preserving*? Define an *invariant event* and an *invariant random variable*, and explain what is meant by saying that θ is *ergodic*.

(b) Let m be a probability measure on $(\mathbb{R}, \mathcal{B})$. Let

$$\Omega = \mathbb{R}^{\mathbb{N}} = \{x = (x_1, x_2, \ldots) : x_i \in \mathbb{R} \text{ for } i \geqslant 1\},$$

let $\mathcal{F}$ be the smallest σ-field of Ω with respect to which the coordinate maps $X_n(x) = x_n$, for $x \in \Omega$, $n \geqslant 1$, are measurable, and let $\mathbb{P}$ be the unique probability measure on $(\Omega, \mathcal{F})$ satisfying

$$\mathbb{P}(X_i \in A_i \text{ for } 1 \leqslant i \leqslant n) = \prod_{i=1}^{n} m(A_i)$$

for all $A_i \in \mathcal{B}$, $n \geqslant 1$. Define $\theta : \Omega \to \Omega$ by $\theta(x) = (x_2, x_3, \ldots)$ for $x = (x_1, x_2, \ldots)$.

(i) Show that θ is measurable and measure-preserving.

(ii) Define the tail σ-field $\mathcal{T}$ of the coordinate maps $X_1, X_2, \ldots$, and show that the invariant σ-field $\mathcal{I}$ of θ satisfies $\mathcal{I} \subseteq \mathcal{T}$. Deduce that θ is ergodic. [*Any general result used must be stated clearly but the proof may be omitted.*]

(c) State Birkhoff's ergodic theorem and explain how to deduce that, given independent identically-distributed integrable random variables $Y_1, Y_2, \ldots$, there exists $\nu \in \mathbb{R}$ such that

$$\frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n) \to \nu \quad \text{a.e. as} \quad n \to \infty.$$

13I Applied Probability

Let $(X_t,\, t\geq 0)$ be a renewal process with holding times $(S_n,\, n=1,2,\dots)$ and $(Y_t,\, t\geq 0)$ be a renewal-reward process over (X_t) with a sequence of rewards $(W_n, n=1,2,\dots)$. Under assumptions on (S_n) and (W_n) which you should state clearly, prove that the ratios

$$X_t/t \quad \text{and} \quad Y_t/t$$

converge as $t\to\infty$. You should specify the form of convergence guaranteed by your assumptions. The law of large numbers, in the appropriate form, for sums $S_1+\ldots+S_n$ and $W_1+\ldots+W_n$ can be used without proof.

In a mountain resort, when you rent skiing equipment you are given two options. (1) You buy an insurance waiver that costs $C/4$ where C is the daily equipment rent. Under this option, the shop will immediately replace, at no cost to you, any piece of equipment you break during the day, no matter how many breaks you had. (2) If you don't buy the waiver, you'll pay $3C$ in the case of any break.

To find out which option is better for me, I decided to set up two models of renewal-reward process (Y_t). In the first model, (Option 1), all of the holding times S_n are equal to 6. In the second model, given that there is no break on day n (an event of probability $4/5$), we have $S_n=6, W_n=C$, but given that there is a break on day n, we have that S_n is uniformly distributed on $(0,6)$, and $W_n=4C$. (In the second model, I would not continue skiing after a break, whereas in the first I would.)

Calculate in each of these models the limit

$$\lim_{t\to\infty} Y_t/t$$

representing the long-term average cost of a unit of my skiing time.

14I Optimization and Control

The strength of the economy evolves according to the equation

$$\ddot{x}_t = -\alpha^2 x_t + u_t\,,$$

where $x_0=\dot{x}_0=0$ and u_t is the effort that the government puts into reform at time t, $t\geq 0$. The government wishes to maximize its chance of re-election at a given future time T, where this chance is some monotone increasing function of

$$x_T - \frac{1}{2}\int_0^T u_t^2 dt\,.$$

Use Pontryagin's maximum principle to determine the government's optimal reform policy, and show that the optimal trajectory of x_t is

$$x_t = \frac{t}{2}\alpha^{-2}\cos(\alpha(T-t)) - \frac{1}{2}\alpha^{-3}\cos(\alpha T)\sin(\alpha t)\,.$$

15J Principles of Statistics

(i) What is a *sufficient statistic*? What is a *minimal* sufficient statistic? Explain the terms *nuisance parameter* and *ancillary statistic*.

(ii) Let $U_1, \dots, U_n$ be independent random variables with common uniform$([0,1])$ distribution, and suppose you observe $X_i \equiv aU_i^{-\beta}$, $i = 1, \dots, n$, where the positive parameters a, β are unknown. Write down the joint density of $X_1, \dots, X_n$ and prove that the statistic

$$(m, p) \equiv (\min_{1 \leqslant j \leqslant n} \{X_j\}, \prod_{j=1}^{n} X_j)$$

is minimal sufficient for (a, β). Find the maximum-likelihood estimator $(\hat{a}, \hat{\beta})$ of (a, β).

Regarding β as the parameter of interest and a as the nuisance parameter, is m ancillary? Find the mean and variance of $\hat{\beta}$. Hence find an unbiased estimator of β.

16J Stochastic Financial Models

(i) Consider a single-period binomial model of a riskless asset (asset 0), worth 1 at time 0 and $1 + r$ at time 1, and a risky asset (asset 1), worth 1 at time 0 and worth u at time 1 if the period was good, otherwise worth d. Assuming that

$$d < 1 + r < u \qquad (*)$$

show how any contingent claim Y to be paid at time 1 can be priced and exactly replicated. Briefly explain the significance of the condition $(*)$, and indicate how the analysis of the single-period model extends to many periods.

(ii) Now suppose that $u = 5/3$, $d = 2/3$, $r = 1/3$, and that the risky asset is worth $S_0 = 864 = 2^5 \times 3^3$ at time zero. Show that the time-0 value of an American put option with strike $K = S_0$ and expiry at time $t = 3$ is equal to 79, and find the optimal exercise policy.

17B Nonlinear Dynamical Systems

(i) Consider a system in $\mathbb{R}^2$ that is *almost Hamiltonian*:

$$\dot{x} = \frac{\partial H}{\partial y} + \epsilon g_1(x,y), \quad \dot{y} = -\frac{\partial H}{\partial x} + \epsilon g_2(x,y) \ ,$$

where $H = H(x,y)$ and $|\epsilon| \ll 1$. Show that if the system has a periodic orbit $\mathcal{C}$ then $\oint_{\mathcal{C}} g_2 dx - g_1 dy = 0$, and explain how to evaluate this orbit approximately for small ϵ. Illustrate your method by means of the system

$$\dot{x} = y + \epsilon x(1 - x^2), \quad \dot{y} = -x.$$

(ii) Consider the system

$$\dot{x} = y, \quad \dot{y} = x - x^3 + \epsilon y(1 - \alpha x^2).$$

(a) Show that when $\epsilon = 0$ the system is Hamiltonian, and find the Hamiltonian. Sketch the trajectories in the case $\epsilon = 0$. Identify the value H_c of H for which there is a homoclinic orbit.

(b) Suppose $\epsilon > 0$. Show that the small change ΔH in H around an orbit of the Hamiltonian system can be expressed to leading order as an integral of the form

$$\int_{x_1}^{x_2} \mathcal{F}(x,H) dx,$$

where x_1, x_2 are the extrema of the x-coordinates of the orbits of the Hamiltonian system, distinguishing between the cases $H < H_c$, $H > H_c$.

(c) Find the value of α, correct to leading order in ϵ, at which the system has a homoclinic orbit.

(d) By examining the eigenvalues of the Jacobian at the origin, determine the stability of the homoclinic orbit, being careful to state clearly any standard results that you use.

18D Partial Differential Equations

(i) Find $w : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $w(t,\cdot)$ is a Schwartz function of ξ for each t and solves

$$w_t(t,\xi) + (1+\xi^2)w(t,\xi) = g(\xi)\,,$$
$$w(0,\xi) = w_0(\xi)\,,$$

where g and w_0 are given Schwartz functions and w_t denotes $\partial_t w$. If $\mathcal{F}$ represents the Fourier transform operator in the ξ variables only and $\mathcal{F}^{-1}$ represents its inverse, show that the solution w satisfies

$$\partial_t(\mathcal{F}^{-1})w(t,x) = \mathcal{F}^{-1}(\partial_t w)(t,x)$$

and calculate $\lim\limits_{t\to\infty} w(t,\cdot)$ in Schwartz space.

(ii) Using the results of Part (i), or otherwise, show that there exists a solution of the initial value problem

$$u_t(t,x) - u_{xx}(t,x) + u(t,x) = f(x)$$

$$u(0,x) = u_0\,,$$

with f and u_0 given Schwartz functions, such that

$$\|u(t,\cdot) - \phi\|_{L^\infty(\mathbb{R})} \longrightarrow 0$$

as $t \to \infty$ in Schwartz space, where ϕ is the solution of

$$-\phi^{''} + \phi = f.$$

19D Methods of Mathematical Physics

The function $w(z)$ satisfies the third-order differential equation

$$\frac{d^3w}{dz^3} - zw = 0$$

subject to the conditions $w(z) \to 0$ as $z \to \pm i\infty$ and $w(0) = 1$. Obtain an integral representation for $w(z)$ of the form

$$w(z) = \int_\gamma e^{zt} f(t)dt\,,$$

and determine the function $f(t)$ and the contour γ.

Using the change of variable $t = z^{1/3}\tau$, or otherwise, compute the leading term in the asymptotic expansion of $w(z)$ as $z \to +\infty$.

20D Numerical Analysis

(i) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant 1, \quad t \geqslant 0\,,$$

with the initial condition $u(x,0) = \phi(x), 0 \leqslant x \leqslant 1$, and with zero boundary conditions at $x = 0$ and $x = 1$, can be solved by the method

$$u_m^{n+1} = u_m^n + \mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n), \quad m = 1,2,\ldots,M, \quad n \geqslant 0\,,$$

where $\Delta x = 1/(M+1), \mu = \Delta t/(\Delta x)^2$, and $u_m^n \approx u(m\Delta x, n\Delta t)$. Prove that $\mu \leqslant \frac{1}{2}$ implies convergence.

(ii) By discretizing the same equation and employing the same notation as in Part (i), determine conditions on $\mu > 0$ such that the method

$$\begin{aligned}&\left(\frac{1}{12} - \frac{1}{2}\mu\right)u_{m-1}^{n+1} + \left(\frac{5}{6} + \mu\right)u_m^{n+1} + \left(\frac{1}{12} - \frac{1}{2}\mu\right)u_{m+1}^{n+1} = \\ &\left(\frac{1}{12} + \frac{1}{2}\mu\right)u_{m-1}^{n} + \left(\frac{5}{6} - \mu\right)u_m^{n} + \left(\frac{1}{12} + \frac{1}{2}\mu\right)u_{m+1}^{n}\end{aligned}$$

is stable.

21E Foundations of Quantum Mechanics

(i) A quantum mechanical system consists of two identical non-interacting particles with associated single-particle wave functions $\psi_i(x)$ and energies $E_i,\ i = 1,2,\ldots$, where $E_1 < E_2 < \ldots$. Show how the states for the two lowest energy levels of the system are constructed and discuss their degeneracy when the particles have (a) spin 0, (b) spin 1/2.

(ii) The Pauli matrices are defined to be

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

State how the spin operators s_1, s_2, s_3 may be expressed in terms of the Pauli matrices, and show that they describe states with total angular momentum $\frac{1}{2}\hbar$.

An electron is at rest in the presence of a magnetic field $\mathbf{B} = (B,0,0)$, and experiences an interaction potential $-\mu\boldsymbol{\sigma}\cdot\mathbf{B}$. At $t = 0$ the state of the electron is the eigenstate of s_3 with eigenvalue $\frac{1}{2}\hbar$. Calculate the probability that at later time t the electron will be measured to be in the eigenstate of s_3 with eigenvalue $\frac{1}{2}\hbar$.

22E Statistical Physics

Describe briefly why a low density gas can be investigated using classical statistical mechanics.

Explain why, for a gas of N structureless atoms, the measure on phase space is

$$\frac{1}{N!}\frac{d^{3N}q\, d^{3N}p}{(2\pi\hbar)^{3N}} ,$$

and the probability density in phase space is proportional to

$$e^{-E(q,p)/T} ,$$

where T is the temperature in energy units.

Derive the Maxwell probability distribution for atomic speeds v,

$$\rho(v) \;=\; \left(\frac{m}{2\pi T}\right)^{3/2} \; 4\pi v^2 e^{-mv^2/2T} \;.$$

Why is this valid even if the atoms interact?

Find the mean value $\bar{v}$ of the speed of the atoms.

Is $\frac{1}{2}m(\bar{v})^2$ the mean kinetic energy of the atoms?

23E Applications of Quantum Mechanics

For a periodic potential $V(\mathbf{r}) = V(\mathbf{r}+\boldsymbol{\ell})$, where $\boldsymbol{\ell}$ is a lattice vector, show that we may write

$$V(\mathbf{r}) = \sum_{\mathbf{g}} a_{\mathbf{g}}\, e^{i\mathbf{g}\cdot\mathbf{r}} , \qquad {a_{\mathbf{g}}}^* = a_{-\mathbf{g}} ,$$

where the set of $\mathbf{g}$ should be defined.

Show how to construct general wave functions satisfying $\psi(\mathbf{r}+\boldsymbol{\ell}) = e^{i\mathbf{k}\cdot\boldsymbol{\ell}}\psi(\mathbf{r})$ in terms of free plane-wave wave-functions.

Show that the nearly free electron model gives an energy gap $2|a_{\mathbf{g}}|$ when $\mathbf{k} = \frac{1}{2}\mathbf{g}$.

Explain why, for a periodic potential, the allowed energies form bands $E_n(\mathbf{k})$ where $\mathbf{k}$ may be restricted to a single Brillouin zone. Show that $E_n(\mathbf{k}) = E_n(\mathbf{k}+\mathbf{g})$ if $\mathbf{k}$ and $\mathbf{k}+\mathbf{g}$ belong to the Brillouin zone.

How are bands related to whether a material is a conductor or an insulator?

24A Fluid Dynamics II

Using the Milne-Thompson circle theorem, or otherwise, write down the complex potential w describing inviscid incompressible two-dimensional flow past a circular cylinder of radius a centred on the origin, with circulation κ and uniform velocity (U, V) in the far field.

Hence, or otherwise, find an expression for the velocity field if the cylinder is replaced by a flat plate of length $4a$, centred on the origin and aligned with the x-axis. Evaluate the velocity field on the two sides of the plate and confirm that the normal velocity is zero.

Explain the significance of the Kutta condition, and determine the value of the circulation that satisfies the Kutta condition when $U > 0$.

With this value of the circulation, calculate the difference in pressure between the upper and lower sides of the plate at position x $(-2a \leq x \leq 2a)$. Comment briefly on the value of the pressure at the leading edge and the force that this would produce if the plate had a small non-zero thickness.

Determine the force on the plate, explaining carefully the direction in which it acts.

[*The Blasius formula* $F_x - iF_y = \dfrac{i\rho}{2}\displaystyle\oint_C \left(\frac{dw}{dz}\right)^2 dz$, *where* C *is a closed contour lying just outside the body, may be used without proof.*]

25A Waves in Fluid and Solid Media

The dispersion relation for sound waves of frequency ω in a stationary, homogeneous gas is $\omega = c|\mathbf{k}|$, where c is the speed of sound and $\mathbf{k}$ is the wavevector. Derive the dispersion relation for sound waves of frequency ω in a uniform flow with velocity $\mathbf{U}$.

For a slowly-varying medium with a local dispersion relation $\omega = \Omega(\mathbf{k}; \mathbf{x}, t)$, derive the ray-tracing equations

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i}, \qquad \frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}, \qquad \frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t}.$$

The meaning of the notation d/dt should be carefully explained.

Suppose that two-dimensional sound waves with initial wavevector (k_0, l_0) are generated at the origin in a gas occupying the half-space $y > 0$. The gas has a mean velocity $(\gamma y, 0)$, where $0 < \gamma \ll (k_0^2 + l_0^2)^{\frac{1}{2}}$. Show that

(a) if $k_0 > 0$ and $l_0 > 0$ then an initially upward propagating wavepacket returns to the level $y = 0$ within a finite time, after having reached a maximum height that should be identified;

(b) if $k_0 < 0$ and $l_0 > 0$ then an initially upward propagating wavepacket continues to propagate upwards for all time.

For the case of a fixed frequency disturbance comment *briefly* on whether or not there is a quiet zone.