UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Friday 4 June 2003 9 to 12

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than **FOUR** *questions.*

The number of marks for each question is the same.

Additional credit will be given for a substantially complete answer.

Write on **one side** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **3F, 8F** *should be in one bundle and* **11J, 13J** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1I Markov Chains

(a) Give three definitions of a continuous-time Markov chain with a given Q-matrix on a finite state space: (i) in terms of holding times and jump probabilities, (ii) in terms of transition probabilities over small time intervals, and (iii) in terms of finite-dimensional distributions.

(b) A flea jumps clockwise on the vertices of a triangle; the holding times are independent exponential random variables of rate one. Find the eigenvalues of the corresponding Q-matrix and express transition probabilities $p_{xy}(t)$, $t \geq 0$, $x, y = A, B, C$, in terms of these roots. Deduce the formulas for the sums

$$S_0(t) = \sum_{n=0}^{\infty} \frac{t^{3n}}{(3n)!}\,, \quad S_1(t) = \sum_{n=0}^{\infty} \frac{t^{3n+1}}{(3n+1)!}\,, \quad S_2(t) = \sum_{n=0}^{\infty} \frac{t^{3n+2}}{(3n+2)!}\,,$$

in terms of the functions $e^t, e^{-t/2}, \cos(\sqrt{3}t/2)$ and $\sin(\sqrt{3}t/2)$.

Find the limits

$$\lim_{t\to\infty} e^{-t} S_j(t), \quad j = 0, 1, 2\,.$$

What is the connection between the decompositions $e^t = S_0(t) + S_1(t) + S_2(t)$ and $e^t = \cosh t + \sinh t$?

2B Principles of Dynamics

Consider a system of coordinates rotating with angular velocity $\boldsymbol{\omega}$ relative to an inertial coordinate system.

Show that if a vector $\mathbf{v}$ is changing at a rate $d\mathbf{v}/dt$ in the inertial system, then it is changing at a rate

$$\left.\frac{d\mathbf{v}}{dt}\right|_{\mathrm{rot}} = \frac{d\mathbf{v}}{dt} - \boldsymbol{\omega} \wedge \mathbf{v}$$

with respect to the rotating system.

A solid body rotates with angular velocity $\boldsymbol{\omega}$ in the absence of external torque. Consider the rotating coordinate system aligned with the principal axes of the body.

(a) Show that in this system the motion is described by the Euler equations

$$I_1 \left.\frac{d\omega_1}{dt}\right|_{\mathrm{rot}} = \omega_2\omega_3(I_2 - I_3) \quad , \quad I_2 \left.\frac{d\omega_2}{dt}\right|_{\mathrm{rot}} = \omega_3\omega_1(I_3 - I_1) \quad , \quad I_3 \left.\frac{d\omega_3}{dt}\right|_{\mathrm{rot}} = \omega_1\omega_2(I_1 - I_2) \ ,$$

where $(\omega_1, \omega_2, \omega_3)$ are the components of the angular velocity in the rotating system and $I_{1,2,3}$ are the principal moments of inertia.

(b) Consider a body with three unequal moments of inertia, $I_3 < I_2 < I_1$. Show that rotation about the 1 and 3 axes is stable to small perturbations, but rotation about the 2 axis is unstable.

(c) Use the Euler equations to show that the kinetic energy, T, and the magnitude of the angular momentum, L, are constants of the motion. Show further that

$$2TI_3 \leq L^2 \leq 2TI_1 \ .$$

3F Functional Analysis

State and prove the Dominated Convergence Theorem. [*You may assume the Monotone Convergence Theorem.*]

Let a and p be real numbers, with $a > 0$. Prove carefully that

$$\int_0^\infty e^{-ax} \sin px \, dx \ = \ \frac{p}{a^2 + p^2} \, .$$

[*Any standard results that you use should be stated precisely.*]

4G Groups, Rings and Fields

(a) Let t be the maximal power of the prime p dividing the order of the finite group G, and let $N(p^t)$ denote the number of subgroups of G of order p^t. State clearly the numerical restrictions on $N(p^t)$ given by the Sylow theorems.

If H and K are subgroups of G of orders r and s respectively, and their intersection $H \cap K$ has order t, show the set $HK = \{hk : h \in H, k \in K\}$ contains rs/t elements.

(b) The finite group G has 48 elements. By computing the possible values of $N(16)$, show that G cannot be simple.

5C Electromagnetism

Consider a frame S' moving with velocity $\mathbf{v}$ relative to the laboratory frame S where $|\mathbf{v}|^2 \ll c^2$. The electric and magnetic fields in S are $\mathbf{E}$ and $\mathbf{B}$, while those measured in S' are $\mathbf{E}'$ and $\mathbf{B}'$. Given that $\mathbf{B}' = \mathbf{B}$, show that

$$\oint_\Gamma \mathbf{E}' \cdot d\mathbf{l} = \oint_\Gamma (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot d\mathbf{l},$$

for any closed circuit Γ and hence that $\mathbf{E}' = \mathbf{E} + \mathbf{v} \wedge \mathbf{B}$.

Now consider a fluid with electrical conductivity σ and moving with velocity $\mathbf{v}(\mathbf{r})$. Use Ohm's law in the moving frame to relate the current density $\mathbf{j}$ to the electric field $\mathbf{E}$ in the laboratory frame, and show that if $\mathbf{j}$ remains finite in the limit $\sigma \to \infty$ then

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \mathbf{B}).$$

The *magnetic helicity* H in a volume V is given by $\int_V \mathbf{A} \cdot \mathbf{B} \, d\tau$ where $\mathbf{A}$ is the vector potential. Show that if the normal components of $\mathbf{v}$ and $\mathbf{B}$ both vanish on the surface bounding V then $dH/dt = 0$.

6B Nonlinear Dynamical Systems

(a) Consider the map $G_1(x) = f(x+a)$, defined on $0 \leqslant x < 1$, where $f(x) = x\,[\text{mod } 1]$, $0 \leqslant f < 1$, and the constant a satisfies $0 \leqslant a < 1$. Give, with reasons, the values of a (if any) for which the map has (i) a fixed point, (ii) a cycle of least period n, (iii) an aperiodic orbit. Does the map exhibit sensitive dependence on initial conditions?

Show (graphically if you wish) that if the map has an n-cycle then it has an infinite number of such cycles. Is this still true if G_1 is replaced by $f(cx+a)$, $0 < c < 1$?

(b) Consider the map

$$G_2(x) = f(x + a + \frac{b}{2\pi}\sin 2\pi x),$$

where $f(x)$ and a are defined as in Part (a), and $b > 0$ is a parameter.

Find the regions of the (a, b) plane for which the map has (i) no fixed points, (ii) exactly two fixed points.

Now consider the possible existence of a 2-cycle of the map G_2 when $b \ll 1$, and suppose the elements of the cycle are X, Y with $X < \frac{1}{2}$. By expanding X, Y, a in powers of b, so that $X = X_0 + bX_1 + b^2X_2 + O(b^3)$, and similarly for Y and a, show that

$$a = \frac{1}{2} + \frac{b^2}{8\pi}\sin 4\pi X_0 + O(b^3).$$

Use this result to sketch the region of the (a, b) plane in which 2-cycles exist. How many distinct cycles are there for each value of a in this region?

7G Geometry of Surfaces

Write an essay on the Gauss–Bonnet theorem and its proof.

8F Logic, Computation and Set Theory

Write an essay on recursive functions. Your essay should include a sketch of why every computable function is recursive, and an explanation of the existence of a universal recursive function, as well as brief discussions of the Halting Problem and of the relationship between recursive sets and recursively enumerable sets.

[*You may assume that every recursive function is computable. You do* **not** *need to give proofs that particular functions to do with prime-power decompositions are recursive.*]

9F Graph Theory

Write an essay on trees. You should include a proof of Cayley's result on the number of labelled trees of order n.

Let G be a graph of order $n \geq 2$. Which of the following statements are equivalent to the statement that G is a tree? Give a proof or counterexample in each case.

(a) G is acyclic and $e(G) \geq n-1$.

(b) G is connected and $e(G) \leq n-1$.

(c) G is connected, triangle-free and has at least two leaves.

(d) G has the same degree sequence as T, for some tree T.

10H Number Theory

Write an essay on pseudoprimes and their role in primality testing. You should discuss pseudoprimes, Carmichael numbers, and Euler and strong pseudoprimes. Where appropriate, your essay should include small examples to illustrate your statements.

11J Algorithms and Networks

(i) Consider an unrestricted geometric programming problem

$$\min g(t), \quad t = (t_1, \ldots, t_m) > 0\,, \tag{$*$}$$

where $g(t)$ is given by

$$g(t) = \sum_{i=1}^{n} c_i t_1^{a_{i1}} \ldots t_m^{a_{im}}$$

with $n \geq m$ and positive coefficients $c_1 \ldots, c_n$. State the dual problem of $(*)$ and show that if $\lambda^* = (\lambda_1^*, \ldots, \lambda_n^*)$ is a dual optimum then any positive solution to the system

$$t_1^{a_{i1}} \ldots t_m^{a_{im}} = \frac{1}{c_i}\lambda_i^* v(\lambda^*), \quad i = 1, \ldots, n\,,$$

gives an optimum for primal problem $(*)$. Here $v(\lambda)$ is the dual objective function.

(ii) An amount of ore has to be moved from a pit in an open rectangular skip which is to be ordered from a supplier.

The skip cost is £36 per 1m^2 for the bottom and two side walls and £18 per 1m^2 for the front and the back walls. The cost of loading ore into the skip is £3 per 1m^3, the cost of lifting is £2 per 1m^3, and the cost of unloading is £1 per 1m^3. The cost of moving an empty skip is negligible.

Write down an unconstrained geometric programming problem for the optimal size (length, width, height) of skip minimizing the cost of moving 48m^3 of ore. By considering the dual problem, or otherwise, find the optimal cost and the optimal size of the skip.

12J Stochastic Financial Models

What is *Brownian motion* $(B_t)_{t \geqslant 0}$? Briefly explain how Brownian motion can be considered as a limit of simple random walks. State the *Reflection Principle* for Brownian motion, and use it to derive the distribution of the first passage time $\tau_a \equiv \inf\{t : B_t = a\}$ to some level $a > 0$.

Suppose that $X_t = B_t + ct$, where $c > 0$ is constant. Stating clearly any results to which you appeal, derive the distribution of the first-passage time $\tau_a^{(c)} \equiv \inf\{t : X_t = a\}$ to $a > 0$.

Now let $\sigma_a \equiv \sup\{t : X_t = a\}$, where $a > 0$. Find the density of σ_a.

13J Principles of Statistics

Suppose that $\theta \in \mathbb{R}^d$ is the parameter of a non-degenerate exponential family. Derive the asymptotic distribution of the maximum-likelihood estimator $\hat{\theta}_n$ of θ based on a sample of size n. [*You may assume that the density is infinitely differentiable with respect to the parameter, and that differentiation with respect to the parameter commutes with integration.*]

14I Computational Statistics and Statistical Modelling

Suppose that $Y_1, \ldots, Y_n$ are independent observations, with Y_i having probability density function of the following form

$$f(y_i|\theta_i, \phi) = \exp\left[\frac{y_i\theta_i - b(\theta_i)}{\phi} + c(y_i, \phi)\right]$$

where $\mathbb{E}(Y_i) = \mu_i$ and $g(\mu_i) = \beta^T x_i$. You should assume that $g(\ )$ is a known function, and β, ϕ are unknown parameters, with $\phi > 0$, and also $x_1, \ldots, x_n$ are given linearly independent covariate vectors. Show that

$$\frac{\partial \ell}{\partial \beta} = \sum \frac{(y_i - \beta_i)}{g'(\mu_i) V_i} x_i,$$

where ℓ is the log-likelihood and $V_i = \mathrm{var}\,(Y_i) = \phi b''(\theta_i)$.

Discuss carefully the (slightly edited) R output given below, and briefly suggest another possible method of analysis using the function glm ( ).

```
> s <- scan()
1: 33 63 157 38 108 159
7:
Read 6 items
> r <- scan()
1: 3271 7256 5065 2486 8877 3520
7:
Read 6 items
> gender <- scan(,"")
1: b b b g g g
7:
Read 6 items
> age <- scan(,"")
1: 13&under 14-18 19&over
4: 13&under 14-18 19&over
7:
Read 6 items
> gender <- factor(gender) ; age <- factor(age)
>  summary(glm(s/r ~ gender + age,binomial, weights=r))


       Coefficients:
```

Question continues on next page.

```
            Estimate  Std.Error  z-value Pr(>|z|)
(Intercept) -4.56479    0.12783 -35.710  < 2e-16
genderg      0.38028    0.08689   4.377 1.21e-05
age14-18    -0.19797    0.14241  -1.390    0.164
age19&over   1.12790    0.13252   8.511  < 2e-16

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 221.797542  on 5  degrees of freedom
Residual deviance:   0.098749  on 2  degrees of freedom

Number of Fisher Scoring iterations: 3
```

15E Foundations of Quantum Mechanics

The states of the hydrogen atom are denoted by $|nlm\rangle$ with $l < n, -l \leq m \leq l$ and associated energy eigenvalue E_n, where

$$E_n \;=\; -\frac{e^2}{8\pi\epsilon_0 a_0 n^2}\ .$$

A hydrogen atom is placed in a weak electric field with interaction Hamiltonian

$$H_1 \;=\; -e\mathcal{E}z\ .$$

a) Derive the necessary perturbation theory to show that to $O(\mathcal{E}^2)$ the change in the energy associated with the state $|100\rangle$ is given by

$$\Delta E_1 \;=\; e^2\mathcal{E}^2 \sum_{n=2}^{\infty}\sum_{l=0}^{n-1}\sum_{m=-l}^{l} \frac{\Big|\langle 100|z|nlm\rangle\Big|^2}{E_1 - E_n}\ . \qquad (*)$$

The wavefunction of the ground state $|100\rangle$ is

$$\psi_{n=1}(\mathbf{r}) \;=\; \frac{1}{(\pi a_0^3)^{1/2}}\, e^{-r/a_0}\ .$$

By replacing E_n, $\forall\ n > 1$, in the denominator of $(*)$ by E_2 show that

$$|\Delta E_1| \;<\; \frac{32\pi}{3}\epsilon_0\mathcal{E}^2 a_0^3\ .$$

b) Find a matrix whose eigenvalues are the perturbed energies to $O(\mathcal{E})$ for the states $|200\rangle$ and $|210\rangle$. Hence, determine these perturbed energies to $O(\mathcal{E})$ in terms of the matrix elements of z between these states.

[*Hint:*

$$\begin{aligned}\langle nlm|z|nlm\rangle &= 0 && \forall\ n,l,m\\ \langle nlm|z|nl'm'\rangle &= 0 && \forall\ n,l,l',m,m', \quad m\neq m'\end{aligned}$$

]

16E Quantum Physics

Explain the operation of the *np* junction. Your account should include a discussion of the following topics:

(a) the rôle of doping and the fermi-energy;

(b) the rôle of majority and minority carriers;

(c) the contact potential;

(d) the relationship $I(V)$ between the current I flowing through the junction and the external voltage V applied across the junction;

(e) the property of rectification.

17C General Relativity

Starting from the Ricci identity

$$V_{a;b;c} - V_{a;c;b} = V_e R^e{}_{abc},$$

give an expression for the curvature tensor $R^e{}_{abc}$ of the Levi-Civita connection in terms of the Christoffel symbols and their partial derivatives. Using local inertial coordinates, or otherwise, establish that

$$R^e{}_{abc} + R^e{}_{bca} + R^e{}_{cab} = 0. \qquad (*)$$

A vector field with components V^a satisfies

$$V_{a;b} + V_{b;a} = 0. \qquad (**)$$

Show, using equation $(*)$ that

$$V_{a;b;c} = V_e R^e{}_{cba},$$

and hence that

$$V_{a;b}{}^{;b} + R_a{}^c V_c = 0,$$

where R_{ab} is the Ricci tensor. Show that equation $(**)$ may be written as

$$(\partial_c g_{ab})V^c + g_{cb}\partial_a V^c + g_{ac}\partial_b V^c = 0. \qquad (***)$$

If the metric is taken to be the Schwarzschild metric

$$ds^2 = -\left(1 - \tfrac{2M}{r}\right) dt^2 + \left(1 - \tfrac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

show that $V^a = \delta^a{}_0$ is a solution of $(***)$. Calculate $V^a{}_{;a}$.

Electromagnetism can be described by a vector potential A_a and a Maxwell field tensor F_{ab} satisfying

$$F_{ab} = A_{b;a} - A_{a;b} \qquad \text{and} \qquad F_{ab}{}^{;b} = 0. \qquad (****)$$

The divergence of A_a is arbitrary and we may choose $A_a{}^{;a} = 0$. With this choice show that in a general spacetime

$$A_{a;b}{}^{;b} - R_a{}^c A_c = 0.$$

Hence show that in the Schwarzschild spacetime a tensor field whose only non-trivial components are $F_{tr} = -F_{rt} = Q/r^2$, where Q is a constant, satisfies the field equations $(****)$.

18C Statistical Physics and Cosmology

(a) Consider an ideal gas of Fermi particles obeying the Pauli exclusion principle with a set of one-particle energy eigenstates E_i. Given the probability $p_i(n_i)$ at temperature T that there are n_i particles in the eigenstate E_i:

$$p_i(n_i) = \frac{e^{(\mu - E_i)n_i/kT}}{Z_i}\,,$$

determine the appropriate normalization factor Z_i. Use this to find the average number $\bar{n}_i$ of Fermi particles in the eigenstate E_i.

Explain briefly why in generalizing these discrete eigenstates to a continuum in momentum space (in the range p to $p + dp$) we must multiply by the density of states

$$g(p)dp = \frac{4\pi g_s V}{h^3} p^2 dp\,,$$

where g_s is the degeneracy of the eigenstates and V is the volume.

(b) With the energy expressed as a momentum integral

$$E = \int_0^\infty E(p)\bar{n}(p)dp\,,$$

consider the effect of changing the volume V so slowly that the occupation numbers do not change (i.e. particle number N and entropy S remain fixed). Show that the momentum varies as $dp/dV = -p/3V$ and so deduce from the first law expression

$$\left(\frac{\partial E}{\partial V}\right)_{N,S} = -P$$

that the pressure is given by

$$P = \frac{1}{3V}\int_0^\infty pE'(p)\bar{n}(p)dp\,.$$

Show that in the non-relativistic limit $P = \frac{2}{3}U/V$ where U is the internal energy, while for ultrarelativistic particles $P = \frac{1}{3}E/V$.

(c) Now consider a Fermi gas in the limit $T \to 0$ with all momentum eigenstates filled up to the Fermi momentum $p_{\rm F}$. Explain why the number density can be written as

$$n = \frac{4\pi g_s}{h^3}\int_0^{p_{\rm F}} p^2 dp \;\propto\; p_{\rm F}^3\,.$$

From similar expressions for the energy, deduce in both the non-relativistic and ultra-relativistic limits that the pressure may be written as

$$P \;\propto\; n^\gamma\,,$$

where γ should be specified in each case.

(d) Examine the stability of an object of radius R consisting of such a Fermi degenerate gas by comparing the gravitational binding energy with the total kinetic energy. Briefly point out the relevance of these results to white dwarfs and neutron stars.

19A Transport Processes

(a) Solute diffuses and is advected in a moving fluid. Derive the transport equation and deduce that the solute concentration $C(\mathbf{x},t)$ satisfies the advection–diffusion equation

$$C_t + \boldsymbol{\nabla}\cdot(\mathbf{u}C) = \boldsymbol{\nabla}\cdot(D\boldsymbol{\nabla}C),$$

where $\mathbf{u}$ is the velocity field and D the diffusivity. Write down the form this equation takes when $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$, both $\mathbf{u}$ and $\boldsymbol{\nabla}C$ are unidirectional, in the x-direction, and D is a constant.

(b) A solution occupies the region $x \geqslant 0$, bounded by a semi-permeable membrane at $x = 0$ across which fluid passes (by osmosis) with velocity

$$u = -k\left(C_1 - C(0,t)\right),$$

where k is a positive constant, C_1 is a fixed uniform solute concentration in the region $x < 0$, and $C(x,t)$ is the solute concentration in the fluid. The membrane does not allow solute to pass across $x = 0$, and the concentration at $x = L$ is a fixed value C_L (where $C_1 > C_L > 0$).

Write down the differential equation and boundary conditions to be satisfied by C in a steady state. Make the equations non-dimensional by using the substitutions

$$X = \frac{xkC_1}{D}, \quad \theta(X) = \frac{C(x)}{C_1}, \quad \theta_L = \frac{C_L}{C_1},$$

and show that the concentration distribution is given by

$$\theta(X) = \theta_L \exp\left[(1-\theta_0)(\Lambda - X)\right],$$

where Λ and θ_0 should be defined, and θ_0 is given by the transcendental equation

$$\theta_0 = \theta_L e^{\Lambda - \Lambda\theta_0}. \qquad (*)$$

What is the dimensional fluid velocity u, in terms of θ_0?

(c) Show that if, instead of taking a finite value of L, you had tried to take L infinite, then you would have been unable to solve for θ unless $\theta_L = 0$, but in that case there would be no way of determining θ_0.

(d) Find asymptotic expansions for θ_0 from equation $(*)$ in the following limits:

(i) For $\theta_L \to 0$, Λ fixed, expand θ_0 as a power series in θ_L, and equate coefficients to show that

$$\theta_0 \sim e^{\Lambda}\theta_L - \Lambda e^{2\Lambda}\theta_L^2 + O\left(\theta_L^3\right).$$

(ii) For $\Lambda \to \infty$, θ_L fixed, take logarithms, expand θ_0 as a power series in $1/\Lambda$, and show that

$$\theta_0 \sim 1 + \frac{\log\theta_L}{\Lambda} + O\left(\frac{1}{\Lambda^2}\right).$$

What is the limiting value of θ_0 in the limits (i) and (ii)?

Question continues on next page.

(e) Both the expansions in (d) break down when $\theta_L = O(e^{-\Lambda})$. To investigate the double limit $\Lambda \to \infty$, $\theta_L \to 0$, show that $(*)$ can be written as

$$\lambda = \phi e^{\phi}$$

where $\phi = \Lambda\theta_0$ and λ is to be determined. Show that $\phi \sim \lambda - \lambda^2 + \ldots$ for $\lambda \ll 1$, and $\phi \sim \log\lambda - \log\log\lambda + \ldots$ for $\lambda \gg 1$.

Briefly discuss the implication of your results for the problem raised in (c) above.

20A Theoretical Geophysics

In a reference frame rotating about a vertical axis with angular velocity $f/2$, the horizontal components of the momentum equation for a shallow layer of inviscid, incompressible, fluid of uniform density ρ are

$$\begin{aligned}\frac{Du}{Dt} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x}\\ \frac{Dv}{Dt} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y},\end{aligned}$$

where u and v are independent of the vertical coordinate z, and p is given by hydrostatic balance. State the nonlinear equations for conservation of mass and of potential vorticity for such a flow in a layer occupying $0 < z < h(x,y,t)$. Find the pressure p.

By linearising the equations about a state of rest and uniform thickness H, show that small disturbances $\eta = h - H$, where $\eta \ll H$, to the height of the free surface obey

$$\frac{\partial^2\eta}{\partial t^2} - gH\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right) + f^2\eta = f^2\eta_0 - fH\zeta_0,$$

where η_0 and ζ_0 are the values of η and the vorticity ζ at $t = 0$.

Obtain the dispersion relation for homogeneous solutions of the form $\eta \propto \exp[i(kx - \omega t)]$ and calculate the group velocity of these Poincaré waves. Comment on the form of these results when $ak \ll 1$ and $ak \gg 1$, where the lengthscale a should be identified.

Explain what is meant by *geostrophic balance*. Find the long-time geostrophically balanced solution, η_∞ and (u_∞, v_∞), that results from initial conditions $\eta_0 = A\,\mathrm{sgn}(x)$ and $(u,v) = \mathbf{0}$. Explain briefly, without detailed calculation, how the evolution from the initial conditions to geostrophic balance could be found.

21A Mathematical Methods

State Watson's lemma, describing the asymptotic behaviour of the integral

$$I(\lambda) = \int_0^A e^{-\lambda t} f(t)\, dt, \qquad A > 0,$$

as $\lambda \to \infty$, given that $f(t)$ has the asymptotic expansion

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^{n\beta}$$

as $t \to 0_+$, where $\beta > 0$.

Consider the integral

$$J(\lambda) = \int_a^b e^{\lambda\phi(t)} F(t) dt,$$

where $\lambda \gg 1$ and $\phi(t)$ has a unique maximum in the interval $[a, b]$ at c, with $a < c < b$, such that

$$\phi'(c) = 0, \quad \phi''(c) < 0.$$

By using the change of variable from t to ζ, defined by

$$\phi(t) - \phi(c) = -\zeta^2,$$

deduce an asymptotic expansion for $J(\lambda)$ as $\lambda \to \infty$. Show that the leading-order term gives

$$J(\lambda) \sim e^{\lambda\phi(c)} F(c) \Big(\frac{2\pi}{\lambda|\phi''(c)|} \Big)^{\frac{1}{2}}.$$

The gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^{\infty} e^{(x-1)\log t - t}\, dt.$$

By means of the substitution $t = (x-1)s$, or otherwise, deduce that

$$\Gamma(x+1) \sim x^{(x+\frac{1}{2})} e^{-x} \sqrt{2\pi} \Big(1 + \frac{1}{12x} + \ldots \Big)$$

as $x \to \infty$.

22D Numerical Analysis

Write an essay on the method of conjugate gradients. You should define the method, list its main properties and sketch the relevant proof. You should also prove that (in exact arithmetic) the method terminates in a finite number of steps, briefly mention the connection with Krylov subspaces, and describe the approach of preconditioned conjugate gradients.

23B Nonlinear Waves

Let $\psi(k;x,t)$ satisfy the linear integral equation

$$\psi(k;x,t) + ie^{i(kx+k^3t)} \int_L \frac{\psi(l;x,t)}{l+k} d\lambda(l) = e^{i(kx+k^3t)},$$

where the measure $d\lambda(k)$ and the contour L are such that $\psi(k;x,t)$ exists and is unique. Let $q(x,t)$ be defined in terms of $\psi(k;x,t)$ by

$$q(x,t) = -\frac{\partial}{\partial x} \int_L \psi(k;x,t) d\lambda(k).$$

(a) Show that

$$(M\psi) + ie^{i(kx+k^3t)} \int_L \frac{(M\psi)(l;x,t)}{l+k} d\lambda(l) = 0,$$

where

$$M\psi \equiv \frac{\partial^2 \psi}{\partial x^2} - ik\frac{\partial \psi}{\partial x} + q\psi.$$

(b) Show that

$$(N\psi) + ie^{i(kx+k^3t)} \int_L \frac{(N\psi)(l;x,t)}{l+k} d\lambda(l) = 3ke^{i(kx+k^3t)} \int_L \frac{(M\psi)(l;x,t)}{l+k} d\lambda(l),$$

where

$$N\psi \equiv \frac{\partial \psi}{\partial t} + \frac{\partial^3 \psi}{\partial x^3} + 3q\frac{\partial \psi}{\partial x}.$$

(c) By recalling that the KdV equation

$$\frac{\partial q}{\partial t} + \frac{\partial^3 q}{\partial x^3} + 6q\frac{\partial q}{\partial x} = 0$$

admits the Lax pair

$$M\psi = 0, \quad N\psi = 0,$$

write down an expression for $d\lambda(l)$ which gives rise to the one-soliton solution of the KdV equation. Write down an expression for $\psi(k;x,t)$ and for $q(x,t)$.