UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Monday 31 May 2004 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

Each question is divided into Part **(i)** *and Part* **(ii)***, which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* **SIX** *questions.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)***.*

Additional credit will be given for a substantially complete answer to either Part.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, ..., J** *according to the letter affixed to each question. (For example,* **3F, 7F** *should be in one bundle and* **1J, 13J** *in another bundle.)*

Attach a completed cover sheet to each bundle listing the Parts of questions attempted.

Complete a master cover sheet listing separately **all** *Parts of* **all** *questions attempted.*

It is essential that every cover sheet bear the candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1I Markov Chains

(i) Give the definitions of a *recurrent* and a *null recurrent* irreducible Markov chain.

Let (X_n) be a recurrent Markov chain with state space I and irreducible transition matrix $P = (p_{ij})$. Prove that the vectors $\gamma^k = (\gamma_j^k,\, j \in I)$, $k \in I$, with entries $\gamma_k^k = 1$ and

$$\gamma_i^k = \mathbb{E}_k(\#\text{ of visits to } i \text{ before returning to } k), \quad i \neq k\,,$$

are P-invariant:

$$\gamma_j^k = \sum_{i \in I} \gamma_i^k p_{ij}\,.$$

(ii) Let (W_n) be the birth and death process on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with the following transition probabilities:

$$p_{i,i+1} = p_{i,i-1} = \frac{1}{2},\ i \geq 1$$
$$p_{01} = 1\,.$$

By relating (W_n) to the symmetric simple random walk (Y_n) on $\mathbb{Z}$, or otherwise, prove that (W_n) is a recurrent Markov chain. By considering invariant measures, or otherwise, prove that (W_n) is null recurrent.

Calculate the vectors $\gamma^k = (\gamma_i^k, i \in \mathbb{Z}_+)$ for the chain (W_n), $k \in \mathbb{Z}_+$.

Finally, let $W_0 = 0$ and let N be the number of visits to 1 before returning to 0. Show that $\mathbb{P}_0(N = n) = (1/2)^n$, $n \geq 1$.

[*You may use properties of the random walk* (Y_n) *or general facts about Markov chains without proof but should clearly state them.*]

2B Principles of Dynamics

(i) In Hamiltonian mechanics the action is written

$$S = \int dt \Big(p^a \dot{q}^a - H(q^a, p^a, t) \Big) . \tag{1}$$

Starting from Maupertius' principle $\delta S = 0$, derive Hamilton's equations

$$\dot{q}^a = \frac{\partial H}{\partial p^a}, \quad \dot{p}^a = -\frac{\partial H}{\partial q^a} .$$

Show that H is a constant of the motion if $\partial H / \partial t = 0$. When is p^a a constant of the motion?

(ii) Consider the action S given in Part (i), evaluated on a classical path, as a function of the final coordinates q_f^a and final time t_f, with the initial coordinates and the initial time held fixed. Show that $S(q_f^a, t_f)$ obeys

$$\frac{\partial S}{\partial q_f^a} = p_f^a, \quad \frac{\partial S}{\partial t_f} = -H(q_f^a, p_f^a, t_f) . \tag{2}$$

Now consider a simple harmonic oscillator with $H = \frac{1}{2}(p^2 + q^2)$. Setting the initial time and the initial coordinate to zero, find the classical solution for p and q with final coordinate $q = q_f$ at time $t = t_f$. Hence calculate $S(t_f, q_f)$, and explicitly verify (2) in this case.

3F Functional Analysis

(i) Let H be a Hilbert space, and let M be a non-zero closed vector subspace of H. For $x \in H$, show that there is a unique closest point $P_M(x)$ to x in M.

(ii) (a) Let $x \in H$. Show that $x - P_M(x) \in M^\perp$. Show also that if $y \in M$ and $x - y \in M^\perp$ then $y = P_M(x)$.

(b) Deduce that $H = M \bigoplus M^\perp$.

(c) Show that the map P_M from H to M is a continuous linear map, with $||P_M|| = 1$.

(d) Show that P_M is the projection onto M along $M^\perp$.

Now suppose that A is a subspace of H that is not necessarily closed. Explain why $A^\perp = \{0\}$ implies that A is dense in H.

Give an example of a subspace of l^2 that is dense in l^2 but is not equal to l^2.

4G Groups, Rings and Fields

(i) Let R be a commutative ring. Define the terms *prime ideal* and *maximal ideal*, and show that if an ideal M in R is maximal then M is also prime.

(ii) Let P be a non-trivial prime ideal in the commutative ring R ('non-trivial' meaning that $P \neq \{0\}$ and $P \neq R$). If P has finite index as a subgroup of R, show that P is also maximal. Give an example to show that this may fail, if the assumption of finite index is omitted. Finally, show that if R is a principal ideal domain, then every non-trivial prime ideal in R is maximal.

5C Electromagnetism

(i) Show that the work done in assembling a localised charge distribution $\rho(\mathbf{r})$ in a region V with an associated potential $\phi(\mathbf{r})$ is

$$W = \tfrac{1}{2} \int_V \rho(\mathbf{r})\phi(\mathbf{r})\, d\tau,$$

and that this can be written as an integral over all space

$$W = \tfrac{1}{2}\epsilon_0 \int |\mathbf{E}|^2\, d\tau,$$

where the electric field $\mathbf{E} = -\nabla\phi$.

(ii) What is the force per unit area on an infinite plane conducting sheet with a charge density σ per unit area (a) if it is isolated in space and (b) if the electric field vanishes on one side of the sheet?

An infinite cylindrical capacitor consists of two concentric cylindrical conductors with radii a, b ($a < b$), carrying charges $\pm q$ per unit length respectively. Calculate the capacitance per unit length and the energy per unit length. Next determine the total force on each conductor, and calculate the rate of change of energy of the inner and outer conductors if they are moved radially inwards and outwards respectively with speed v. What is the corresponding rate of change of the capacitance?

6B Nonlinear Dynamical Systems

(i) State Liapunov's First Theorem and La Salle's Invariance Principle. Use these results to show that the system

$$\ddot{x} + k\dot{x} + \sin x = 0, \quad k > 0$$

has an asymptotically stable fixed point at the origin.

(ii) Define the *basin of attraction* of an invariant set of a dynamical system.

Consider the equations

$$\dot{x} = -x + \beta x y^2 + x^3, \quad \dot{y} = -y + \beta y x^2 + y^3, \quad \beta > 2.$$

(a) Find the fixed points of the system and determine their type.

(b) Show that the basin of attraction of the origin includes the union over α of the regions

$$x^2 + \alpha^2 y^2 < \frac{4\alpha^2(1+\alpha^2)(\beta-1)}{\beta^2(1+\alpha^2)^2 - 4\alpha^2}.$$

Sketch these regions for $\alpha^2 = 1, 1/2, 2$ in the case $\beta = 3$.

7F Logic, Computation and Set Theory

(i) State and prove the Knaster-Tarski Fixed-Point Theorem.

(ii) A subset S of a poset X is called an *up-set* if whenever $x, y \in X$ satisfy $x \in S$ and $x \leqslant y$ then also $y \in S$. Show that the set of up-sets of X (ordered by inclusion) is a complete poset.

Let X and Y be totally ordered sets, such that X is isomorphic to an up-set in Y and Y is isomorphic to the complement of an up-set in X. Prove that X is isomorphic to Y. Indicate clearly where in your argument you have made use of the fact that X and Y are total orders, rather than just partial orders.

[Recall that posets X and Y are called *isomorphic* if there exists a bijection f from X to Y such that, for any $x, y \in X$, we have $f(x) \leqslant f(y)$ if and only if $x \leqslant y$.]

8F Graph Theory

(i) Let G be a connected graph of order $n \geq 3$ such that for any two vertices x and y,

$$d(x) + d(y) \geq k.$$

Show that if $k < n$ then G has a path of length k, and if $k = n$ then G is Hamiltonian.

(ii) State and prove Hall's theorem.

[*If you use any form of Menger's theorem, you must state it clearly.*]

Let G be a graph with directed edges. For $S \subset V(G)$, let

$$\Gamma_+(S) = \{y \in V(G) : xy \in E(G) \text{ for some } x \in S\}\,.$$

Find a necessary and sufficient condition, in terms of the sizes of the sets $\Gamma_+(S)$, for the existence of a set $F \subset E(G)$ such that at every vertex there is exactly one incoming edge and exactly one outgoing edge belonging to F.

9H Number Theory

(i) State the law of quadratic reciprocity. For $p \neq 5$ an odd prime, evaluate the Legendre symbol

$$\left(\frac{5}{p}\right).$$

(ii) (a) Let $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ be distinct odd primes. Show that there exists an integer x that is a quadratic residue modulo each of $p_1, \ldots, p_m$ and a quadratic non-residue modulo each of $q_1, \ldots, q_n$.

(b) Let p be an odd prime. Show that

$$\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) = 0\,.$$

(c) Let p be an odd prime. Using (b) or otherwise, evaluate

$$\sum_{a=1}^{p-2} \left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right).$$

[*Hint for (c): Use the equality* $\left(\frac{x^2 y}{p}\right) = \left(\frac{y}{p}\right)$, *valid when* p *does not divide* x.]

10H Coding and Cryptography

(i) What is a linear code? What does it mean to say that a linear code has length n and minimum weight d? When is a linear code perfect? Show that, if $n = 2^r - 1$, there exists a perfect linear code of length n and minimum weight 3.

(ii) Describe the construction of a Reed-Muller code. Establish its information rate and minimum weight.

11J Stochastic Financial Models

(i) What does it mean to say that U is a *utility function*? What is a utility function with constant absolute risk aversion (CARA)?

Let $S_t \equiv (S_t^1, \ldots, S_t^d)^T$ denote the prices at time $t = 0, 1$ of d risky assets, and suppose that there is also a riskless zeroth asset, whose price at time 0 is 1, and whose price at time 1 is $1+r$. Suppose that S_1 has a multivariate Gaussian distribution, with mean μ_1 and non-singular covariance V. An agent chooses at time 0 a portfolio $\theta = (\theta^1, \ldots, \theta^d)^T$ of holdings of the d risky assets, at total cost $\theta \cdot S_0$, and at time 1 realises his gain $X = \theta \cdot (\, S_1 - (1+r)S_0\,)$. Given that he wishes the mean of X to be equal to m, find the smallest value that the variance v of X can be. What is the portfolio that achieves this smallest variance? Hence sketch the region in the (v, m) plane of pairs (v, m) that can be achieved by some choice of θ, and indicate the mean-variance efficient frontier.

(ii) Suppose that the agent has a CARA utility with coefficient γ of absolute risk aversion. What portfolio will he choose in order to maximise $EU(X)$? What then is the mean of X?

Regulation requires that the agent's choice of portfolio θ has to satisfy the value-at-risk (VaR) constraint

$$m \geqslant -L + a\sqrt{v},$$

where $L > 0$ and $a > 0$ are determined by the regulatory authority. Show that this constraint has no effect on the agent's decision if $\kappa \equiv \sqrt{\mu \cdot V^{-1}\mu} \geqslant a$. If $\kappa < a$, will this constraint necessarily affect the agent's choice of portfolio?

12J Principles of Statistics

(i) What does it mean to say that a family $\{f(\cdot|\theta) : \theta \in \Theta\}$ of densities is an *exponential family?*

Consider the family of densities on $(0, \infty)$ parametrised by the positive parameters a, b and defined by

$$f(x|a,b) = \frac{a \exp(-(a-bx)^2/2x)}{\sqrt{2\pi x^3}} \qquad (x > 0).$$

Prove that this family is an exponential family, and identify the natural parameters and the reference measure.

(ii) Let $(X_1, \ldots, X_n)$ be a sample drawn from the above distribution. Find the maximum-likelihood estimators of the parameters (a, b). Find the Fisher information matrix of the family (in terms of the natural parameters). Briefly explain the significance of the Fisher information matrix in relation to unbiased estimation. Compute the mean of X_1 and of X_1^{-1}.

13I Computational Statistics and Statistical Modelling

(i) Assume that the n-dimensional vector Y may be written as $Y = X\beta + \epsilon$, where X is a given $n \times p$ matrix of rank p, β is an unknown vector, and

$$\epsilon \sim N_n(0, \sigma^2 I).$$

Let $Q(\beta) = (Y - X\beta)^T(Y - X\beta)$. Find $\hat{\beta}$, the least-squares estimator of β, and state without proof the joint distribution of $\hat{\beta}$ and $Q(\hat{\beta})$.

(ii) Now suppose that we have observations $(Y_{ij}, 1 \leqslant i \leqslant I,\ 1 \leqslant j \leqslant J)$ and consider the model

$$\Omega : Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}\ ,$$

where $(\alpha_i), (\beta_j)$ are fixed parameters with $\Sigma\alpha_i = 0$, $\Sigma\beta_j = 0$, and (ϵ_{ij}) may be assumed independent normal variables, with $\epsilon_{ij} \sim N(0, \sigma^2)$, where σ^2 is unknown.

(a) Find $(\hat{\alpha}_i)$, $(\hat{\beta}_j)$, the least-squares estimators of (α_i), (β_j).

(b) Find the least-squares estimators of (α_i) under the hypothesis $H_0 : \beta_j = 0$ for all j.

(c) Quoting any general theorems required, explain carefully how to test H_0, assuming Ω is true.

(d) What would be the effect of fitting the model $\Omega_1 : Y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij}$, where now (α_i), (β_j), (γ_{ij}) are all fixed unknown parameters, and (ϵ_{ij}) has the distribution given above?

14E Quantum Physics

(i) Each particle in a system of N identical fermions has a set of energy levels E_i with degeneracy g_i, where $i = 1, 2, \ldots$. Derive the expression

$$\bar{N}_i = \frac{g_i}{e^{\beta(E_i-\mu)} + 1} ,$$

for the mean number of particles $\bar{N}_i$ with energy E_i. Explain the physical significance of the parameters β and μ.

(ii) The spatial eigenfunctions of energy for an electron of mass m moving in two dimensions and confined to a square box of side L are

$$\psi_{n_1 n_2}(\mathbf{x}) = \frac{2}{L} \sin\left(\frac{n_1 \pi x}{L}\right) \sin\left(\frac{n_2 \pi y}{L}\right) ,$$

where $n_i = 1, 2, \ldots$ $(i = 1, 2)$. Calculate the associated energies.

Hence show that when L is large the number of states in energy range $E \to E + dE$ is

$$\frac{mL^2}{2\pi\hbar^2} dE .$$

How is this formula modified when electron spin is taken into account?

The box is filled with N electrons in equilibrium at temperature T. Show that the chemical potential μ is given by

$$\mu = \frac{1}{\beta} \log\left(e^{\beta\pi\hbar^2\rho/m} - 1\right) ,$$

where ρ is the number of particles per unit area in the box.

What is the value of μ in the limit $T \to 0$?

Calculate the total energy of the lowest state of the system of particles as a function of N and L.

15C General Relativity

(i) What is an affine parameter λ of a timelike or null geodesic? Prove that for a timelike geodesic one may take λ to be proper time τ. The metric

$$ds^2 = -dt^2 + a^2(t)\, d\mathbf{x}^2,$$

with $\dot{a}(t) > 0$ represents an expanding universe. Calculate the Christoffel symbols.

(ii) Obtain the law of spatial momentum conservation for a particle of rest mass m in the form

$$ma^2 \frac{d\mathbf{x}}{d\tau} = \mathbf{p} = \text{constant}.$$

Assuming that the energy $E = m\, dt/d\tau$, derive an expression for E in terms of m, $\mathbf{p}$ and $a(t)$ and show that the energy is not conserved but rather that it decreases with time. In particular, show that if the particle is moving extremely relativistically then the energy decreases as $a^{-1}(t)$, and if it is moving non-relativistically then the kinetic energy, $E - m$, decreases as $a^{-2}(t)$.

Show that the frequency ω_e of a photon emitted at time t_e will be observed at time t_o to have frequency

$$\omega_o = \omega_e \frac{a(t_e)}{a(t_o)}.$$

16C Statistical Physics and Cosmology

(i) Consider a homogeneous and isotropic universe with mass density $\rho(t)$, pressure $P(t)$ and scale factor $a(t)$. As the universe expands its energy E decreases according to the thermodynamic relation $dE = -PdV$ where V is the volume. Deduce the fluid conservation law

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right).$$

Apply the conservation of total energy (kinetic plus gravitational potential) to a test particle on the edge of a spherical region in this universe to obtain the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2},$$

where k is a constant. State clearly any assumptions you have made.

(ii) Our universe is believed to be flat ($k = 0$) and filled with two major components: pressure-free matter ($P_{\mathrm{M}} = 0$) and dark energy with equation of state $P_{\mathrm{Q}} = -\rho_{\mathrm{Q}}c^2$ where the mass densities today ($t = t_0$) are given respectively by $\rho_{\mathrm{M}0}$ and $\rho_{\mathrm{Q}0}$. Assume that each component independently satisfies the fluid conservation equation to show that the total mass density can be expressed as

$$\rho(t) = \frac{\rho_{\mathrm{M}0}}{a^3} + \rho_{\mathrm{Q}0},$$

where we have set $a(t_0) = 1$.

Now consider the substitution $b = a^{3/2}$ in the Friedmann equation to show that the solution for the scale factor can be written in the form

$$a(t) = \alpha(\sinh \beta t)^{2/3},$$

where α and β are constants. Setting $a(t_0) = 1$, specify α and β in terms of $\rho_{\mathrm{M}0}$, $\rho_{\mathrm{Q}0}$ and t_0. Show that the scale factor $a(t)$ has the expected behaviour for an Einstein-de Sitter universe at early times ($t \to 0$) and that the universe accelerates at late times ($t \to \infty$).

[*Hint: Recall that* $\int dx/\sqrt{x^2+1} = \sinh^{-1} x$.]

17A Theoretical Geophysics

(i) What is the polarisation $\mathbf{P}$ and slowness $\mathbf{s}$ of the time-harmonic plane elastic wave $\mathbf{u} = \mathbf{A}\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$?

Use the equation of motion for an isotropic homogenous elastic medium,

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{u}),$$

to show that $\mathbf{s}\cdot\mathbf{s}$ takes one of two values and obtain the corresponding conditions on $\mathbf{P}$. If $\mathbf{s}$ is complex show that $\mathrm{Re}(\mathbf{s})\cdot\mathrm{Im}(\mathbf{s}) = 0$.

(ii) A homogeneous elastic layer of uniform thickness h, S-wave speed β_1 and shear modulus μ_1 has a stress-free surface $z = 0$ and overlies a lower layer of infinite depth, S-wave speed β_2 $(> \beta_1)$ and shear modulus μ_2. Show that the horizontal phase speed c of trapped Love waves satisfies $\beta_1 < c < \beta_2$. Show further that

$$\tan\left[\left(\frac{c^2}{\beta_1^2} - 1\right)^{1/2} kh\right] = \frac{\mu_2}{\mu_1}\left(\frac{1 - c^2/\beta_2^2}{c^2/\beta_1^2 - 1}\right)^{1/2} \qquad (1)$$

where k is the horizontal wavenumber.

Assuming that (1) can be solved to give $c(k)$, explain how to obtain the propagation speed of a pulse of Love waves with wavenumber k.

18A Transport Processes

(i) In an experiment, a finite amount M of marker gas of diffusivity D is released at time $t = 0$ into an infinite tube in the neighbourhood of the origin $x = 0$. Starting from the one-dimensional diffusion equation for the concentration $C(x,t)$ of marker gas,

$$C_t = DC_{xx},$$

use dimensional analysis to show that

$$C = \frac{M}{(Dt)^{1/2}} f(\xi)$$

for some dimensionless function f of the similarity variable $\xi = x/(Dt)^{1/2}$.

Write down the equation and boundary conditions satisfied by $f(\xi)$.

(ii) Consider the experiment of Part (i). Find $f(\xi)$ and sketch your answer in the form of a plot of C against x at a few different times t.

Calculate $C(x,t)$ for a second experiment in which the concentration of marker gas at $x = 0$ is instead raised to the value C_0 at $t = 0$ and maintained at that value thereafter. Show that the total amount of marker gas released in this case becomes greater than M after a time

$$t = \frac{\pi}{16D}\left(\frac{M}{C_0}\right)^2.$$

Show further that, at much larger times than this, the concentration in the first experiment still remains greater than that in the second experiment for positions x with $|x| > 4C_0Dt/M$.

[*Hint:* $\operatorname{erfc}(z) \equiv \dfrac{2}{\sqrt{\pi}}\displaystyle\int_z^\infty e^{-u^2}du \sim \dfrac{1}{\sqrt{\pi}z}e^{-z^2}$ as $z \to \infty$.]

19E Symmetries and Groups in Physics

(i) State and prove Maschke's theorem for finite-dimensional representations of finite groups.

(ii) S_3 is the group of bijections on $\{1,2,3\}$. Find the irreducible representations of S_3, state their dimensions and give their character table.

Let T_2 be the set of objects $T_2 = \{a_{i_1 i_2} : i_1, i_2 = 1,2,3\}$. The operation of the permutation group S_3 on T_2 is defined by the operation of the elements of S_3 separately on each index i_1 and i_2. For example,

$$P_{12} : a_{13} \to a_{23}, \quad P_{231} : a_{23} \to a_{31}, \quad P_{13} : a_{33} \to a_{11}\,.$$

By considering a representative operator from each conjugacy class of S_3, find the table of group characters for the representation $\mathcal{T}_2$ of S_3 acting on T_2. Hence, deduce the irreducible representations into which $\mathcal{T}_2$ decomposes.

20D Numerical Analysis

(i) Define the Backward Difference Formula (BDF) method for ordinary differential equations and derive its two-step version.

(ii) Prove that the interval $(-\infty, 0)$ belongs to the linear stability domain of the two-step BDF method.