UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday 1 June 2004 9 to 12

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Additional credit will be given to substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B,** ..., **H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Linear Algebra

Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_{r+1}\}$ is a linearly independent set of distinct elements of a vector space V and $\{\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{f}_{r+1}, \ldots, \mathbf{f}_m\}$ spans V. Prove that $\mathbf{f}_{r+1}, \ldots, \mathbf{f}_m$ may be reordered, as necessary, so that $\{\mathbf{e}_1, \ldots \mathbf{e}_{r+1}, \mathbf{f}_{r+2}, \ldots, \mathbf{f}_m\}$ spans V.

Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a linearly independent set of distinct elements of V and that $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ spans V. Show that $n \leqslant m$.

2F Groups, Rings and Modules

Let G be a finite group of order n. Let H be a subgroup of G. Define the normalizer $N(H)$ of H, and prove that the number of distinct conjugates of H is equal to the index of $N(H)$ in G. If p is a prime dividing n, deduce that the number of Sylow p-subgroups of G must divide n.

[*You may assume the existence and conjugacy of Sylow subgroups.*]

Prove that any group of order 72 must have either 1 or 4 Sylow 3-subgroups.

3G Geometry

Using the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \,,$$

define the length of a curve and the area of a region in the upper half-plane $H = \{x + iy : y > 0\}$.

Find the hyperbolic area of the region $\{(x, y) \in H : 0 < x < 1, y > 1\}$.

4G Analysis II

Define what it means for a sequence of functions $F_n : (0, 1) \to \mathbb{R}$, where $n = 1, 2, \ldots$, to converge uniformly to a function F.

For each of the following sequences of functions on $(0, 1)$, find the pointwise limit function. Which of these sequences converge uniformly? Justify your answers.

(i) $F_n(x) = \frac{1}{n} e^x$

(ii) $F_n(x) = e^{-nx^2}$

(iii) $F_n(x) = \sum_{i=0}^{n} x^i$

5A Complex Methods

Determine the poles of the following functions and calculate their residues there.

$$\text{(i)}\quad \frac{1}{z^2+z^4}, \qquad \text{(ii)}\quad \frac{e^{1/z^2}}{z-1}, \qquad \text{(iii)}\quad \frac{1}{\sin(e^z)}.$$

6B Methods

Write down the general isotropic tensors of rank 2 and 3.

According to a theory of magnetostriction, the mechanical stress described by a second-rank symmetric tensor σ_{ij} is induced by the magnetic field vector B_i. The stress is linear in the magnetic field,

$$\sigma_{ij} = A_{ijk} B_k,$$

where A_{ijk} is a third-rank tensor which depends only on the material. Show that σ_{ij} can be non-zero only in anisotropic materials.

7B Electromagnetism

Write down Maxwell's equations and show that they imply the conservation of charge.

In a conducting medium of conductivity σ, where $\mathbf{J} = \sigma \mathbf{E}$, show that any charge density decays in time exponentially at a rate to be determined.

8D Quantum Mechanics

From the time-dependent Schrödinger equation for $\psi(x,t)$, derive the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

for $\rho(x,t) = \psi^*(x,t)\psi(x,t)$ and some suitable $j(x,t)$.

Show that $\psi(x,t) = e^{i(kx-\omega t)}$ is a solution of the time-dependent Schrödinger equation with zero potential for suitable $\omega(k)$ and calculate ρ and j. What is the interpretation of this solution?

9C Fluid Dynamics

From the general mass-conservation equation, show that the velocity field $\mathbf{u}(\mathbf{x})$ of an incompressible fluid is solenoidal, i.e. that $\nabla \cdot \mathbf{u} = 0$.

Verify that the two-dimensional flow

$$\mathbf{u} = \left(\frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right)$$

is solenoidal and find a streamfunction $\psi(x, y)$ such that $\mathbf{u} = (\partial\psi/\partial y, -\partial\psi/\partial x)$.

10H Statistics

Use the generalized likelihood-ratio test to derive Student's t-test for the equality of the means of two populations. You should explain carefully the assumptions underlying the test.

11H Markov Chains

Let $P = (P_{ij})$ be a transition matrix. What does it mean to say that P is (a) irreducible, (b) recurrent?

Suppose that P is irreducible and recurrent and that the state space contains at least two states. Define a new transition matrix $\tilde{P}$ by

$$\tilde{P}_{ij} = \begin{cases} 0 & \text{if} \quad i = j, \\ (1 - P_{ii})^{-1} P_{ij} & \text{if} \quad i \neq j. \end{cases}$$

Prove that $\tilde{P}$ is also irreducible and recurrent.

SECTION II

12H Linear Algebra

Let U and W be subspaces of the finite-dimensional vector space V. Prove that both the sum $U+W$ and the intersection $U\cap W$ are subspaces of V. Prove further that

$$\dim U + \dim W = \dim\,(U+W) + \dim\,(U\cap W)\,.$$

Let U, W be the kernels of the maps $A, B : \mathbb{R}^4 \to \mathbb{R}^2$ given by the matrices A and B respectively, where

$$A = \begin{pmatrix} 1 & 2 & -1 & -3 \\ -1 & 1 & 2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & -4 \end{pmatrix}.$$

Find a basis for the intersection $U\cap W$, and extend this first to a basis of U, and then to a basis of $U+W$.

13F Groups, Rings and Modules

State the structure theorem for finitely generated abelian groups. Prove that a finitely generated abelian group A is finite if and only if there exists a prime p such that $A/pA = 0$.

Show that there exist abelian groups $A \neq 0$ such that $A/pA = 0$ for all primes p. Prove directly that your example of such an A is not finitely generated.

14G Geometry

Show that for every hyperbolic line L in the hyperbolic plane H there is an isometry of H which is the identity on L but not on all of H. Call it the *reflection* R_L.

Show that every isometry of H is a composition of reflections.

15G Analysis II

State the axioms for a norm on a vector space. Show that the usual Euclidean norm on $\mathbb{R}^n$,

$$||x|| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}\,,$$

satisfies these axioms.

Let U be any bounded convex open subset of $\mathbb{R}^n$ that contains 0 and such that if $x \in U$ then $-x \in U$. Show that there is a norm on $\mathbb{R}^n$, satisfying the axioms, for which U is the set of points in $\mathbb{R}^n$ of norm less than 1.

16A Complex Methods

Let p and q be two polynomials such that

$$q(z) = \prod_{l=1}^{m} (z - \alpha_l),$$

where $\alpha_1, \ldots, \alpha_m$ are distinct non-real complex numbers and $\deg p \leqslant m-1$. Using contour integration, determine

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{ix} dx\,,$$

carefully justifying all steps.

17B Methods

The equation governing small amplitude waves on a string can be written as

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}.$$

The end points $x = 0$ and $x = 1$ are fixed at $y = 0$. At $t = 0$, the string is held stationary in the waveform,

$$y(x, 0) = x(1 - x) \qquad \text{in} \quad 0 \leq x \leq 1.$$

The string is then released. Find $y(x, t)$ in the subsequent motion.

Given that the energy

$$\int_0^1 \left[\left(\frac{\partial y}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial x} \right)^2 \right] dx$$

is constant in time, show that

$$\sum_{\substack{n \text{ odd} \\ n \geqslant 1}} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

18B Electromagnetism

Inside a volume D there is an electrostatic charge density $\rho(\mathbf{r})$, which induces an electric field $\mathbf{E}(\mathbf{r})$ with associated electrostatic potential $\phi(\mathbf{r})$. The potential vanishes on the boundary of D. The electrostatic energy is

$$W = \frac{1}{2}\int_D \rho\phi \, d^3\mathbf{r}. \tag{1}$$

Derive the alternative form

$$W = \frac{\epsilon_0}{2}\int_D E^2 \, d^3\mathbf{r}. \tag{2}$$

A capacitor consists of three identical and parallel thin metal circular plates of area A positioned in the planes $z=-H$, $z=a$ and $z=H$, with $-H<a<H$, with centres on the z axis, and at potentials 0, V and 0 respectively. Find the electrostatic energy stored, verifying that expressions (1) and (2) give the same results. Why is the energy minimal when $a=0$?

19D Quantum Mechanics

The angular momentum operators are $\mathbf{L} = (L_1, L_2, L_3)$. Write down their commutation relations and show that $[L_i, \mathbf{L}^2] = 0$. Let

$$L_\pm = L_1 \pm iL_2 \,,$$

and show that

$$\mathbf{L}^2 = L_- L_+ + {L_3}^2 + \hbar L_3 \,.$$

Verify that $\mathbf{L}f(r) = 0$, where $r^2 = x_i x_i$, for any function f. Show that

$$L_3(x_1+ix_2)^n f(r) = n\hbar(x_1+ix_2)^n f(r)\,, \qquad L_+(x_1+ix_2)^n f(r) = 0\,,$$

for any integer n. Show that $(x_1+ix_2)^n f(r)$ is an eigenfunction of $\mathbf{L}^2$ and determine its eigenvalue. Why must $L_-(x_1+ix_2)^n f(r)$ be an eigenfunction of $\mathbf{L}^2$? What is its eigenvalue?

20C Fluid Dynamics

A layer of water of depth h flows along a wide channel with uniform velocity $(U, 0)$, in Cartesian coordinates (x, y), with x measured downstream. The bottom of the channel is at $y = -h$, and the free surface of the water is at $y = 0$. Waves are generated on the free surface so that it has the new position $y = \eta(x, t) = a\, e^{i(\omega t - kx)}$.

Write down the equation and the full nonlinear boundary conditions for the velocity potential ϕ (for the perturbation velocity) and the motion of the free surface.

By linearizing these equations about the state of uniform flow, show that

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial y}, \qquad \frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + g\eta = 0 \qquad \text{on} \quad y = 0,$$

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{on} \quad y = -h,$$

where g is the acceleration due to gravity.

Hence, determine the dispersion relation for small-amplitude surface waves

$$(\omega - kU)^2 = gk \tanh kh.$$

21H Statistics

State and prove the Rao–Blackwell Theorem.

Suppose that $X_1, X_2, \ldots, X_n$ are independent, identically-distributed random variables with distribution

$$P\,(X_1 = r) = p^{r-1}(1 - p), \quad r = 1, 2, \ldots,$$

where p, $0 < p < 1$, is an unknown parameter. Determine a one-dimensional sufficient statistic, T, for p.

By first finding a simple unbiased estimate for p, or otherwise, determine an unbiased estimate for p which is a function of T.

22H Markov Chains

Consider the Markov chain with state space $\{1,2,3,4,5,6\}$ and transition matrix

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Determine the communicating classes of the chain, and for each class indicate whether it is open or closed.

Suppose that the chain starts in state 2; determine the probability that it ever reaches state 6.

Suppose that the chain starts in state 3; determine the probability that it is in state 6 after exactly n transitions, $n \geqslant 1$.