UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 31st May 2004 9 to 12

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I.

In Section I, you may attempt **all four** *questions.*

In Section II, at most **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Additional credit will be awarded for substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **A** *and* **E** *according to the code letter affixed to each question. Include in the same bundle questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

(a) Use Euclid's algorithm to find positive integers m, n such that $79m - 100n = 1$.

(b) Determine all integer solutions of the congruence

$$237x \equiv 21 \pmod{300}.$$

(c) Find the set of all integers x satisfying the simultaneous congruences

$$x \equiv 8 \pmod{79}$$
$$x \equiv 11 \pmod{100}.$$

2E Numbers and Sets

Prove by induction the following statements:

i) For every integer $n \geq 1$,

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}(4n^3 - n).$$

ii) For every integer $n \geq 1$, $n^3 + 5n$ is divisible by 6.

3A Dynamics

A lecturer driving his car of mass m_1 along the flat at speed U_1 accidentally collides with a stationary vehicle of mass m_2. As both vehicles are old and very solidly built, neither suffers damage in the collision: they simply bounce elastically off each other in a straight line. Determine how both vehicles are moving after the collision if neither driver applied their brakes. State any assumptions made and consider all possible values of the mass ratio $R = m_1/m_2$. You may neglect friction and other such losses.

An undergraduate drives into a rigid rock wall at speed V. The undergraduate's car of mass M is modern and has a crumple zone of length L at its front. As this zone crumples upon impact, it exerts a net force $F = (L - y)^{-1/2}$ on the car, where y is the amount the zone has crumpled. Determine the value of y at the point the car stops moving forwards as a function of V, where $V < 2L^{\frac{1}{4}}/M^{\frac{1}{2}}$.

4A Dynamics

A small spherical bubble of radius a containing carbon dioxide rises in water due to a buoyancy force $\rho g V$, where ρ is the density of water, g is gravitational attraction and V is the volume of the bubble. The drag on a bubble moving at speed u is $6\pi\mu a u$, where μ is the dynamic viscosity of water, and an accelerating bubble acts like a particle of mass $\alpha\rho V$, for some constant α. Find the location at time t of a bubble released from rest at $t = 0$ and show the bubble approaches a steady rise speed

$$U = \frac{2}{9}\frac{\rho g}{\mu}a^2. \qquad (*)$$

Under some circumstances the carbon dioxide gradually dissolves in the water, which leads to the bubble radius varying as $a^2 = a_0^2 - \beta t$, where a_0 is the bubble radius at $t = 0$ and β is a constant. Under the assumption that the bubble rises at speed given by $(*)$, determine the height to which it rises before it disappears.

SECTION II

5E Numbers and Sets

Show that the set of all subsets of $\mathbb{N}$ is uncountable, and that the set of all finite subsets of $\mathbb{N}$ is countable.

Let X be the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$, and let $Y \subset X$ be the set

$$Y = \{f \in X \mid \text{for all but finitely many } n \in \mathbb{N},\ f(n) = n\}.$$

Show that X is uncountable, but that Y is countable.

6E Numbers and Sets

Prove Fermat's Theorem: if p is prime and $(x, p) = 1$ then $x^{p-1} \equiv 1 \pmod{p}$.

Let n and x be positive integers with $(x, n) = 1$. Show that if $n = mp$ where p is prime and $(m, p) = 1$, then

$$x^{n-1} \equiv 1 \pmod{p} \quad \text{if and only if} \quad x^{m-1} \equiv 1 \pmod{p}.$$

Now assume that n is a product of distinct primes. Show that $x^{n-1} \equiv 1 \pmod{n}$ if and only if, for every prime divisor p of n,

$$x^{(n/p)-1} \equiv 1 \pmod{p}.$$

Deduce that if every prime divisor p of n satisfies $(p-1)|(n-1)$, then for every x with $(x, n) = 1$, the congruence

$$x^{n-1} \equiv 1 \pmod{n}$$

holds.

7E Numbers and Sets

Polynomials $P_r(X)$ for $r \geq 0$ are defined by

$$P_0(X) = 1$$
$$P_r(X) = \frac{X(X-1)\cdots(X-r+1)}{r!} = \prod_{i=1}^{r} \frac{X-i+1}{i} \qquad \text{for } r \geq 1.$$

Show that $P_r(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$, and that if $r \geq 1$ then $P_r(X) - P_r(X-1) = P_{r-1}(X-1)$.

Prove that if F is any polynomial of degree d with rational coefficients, then there are unique rational numbers $c_r(F)$ $(0 \leq r \leq d)$ for which

$$F(X) = \sum_{r=0}^{d} c_r(F) P_r(X).$$

Let $\Delta F(X) = F(X+1) - F(X)$. Show that

$$\Delta F(X) = \sum_{r=0}^{d-1} c_{r+1}(F) P_r(X).$$

Show also that, if F and G are polynomials such that $\Delta F = \Delta G$, then $F - G$ is a constant.

By induction on the degree of F, or otherwise, show that if $F(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$, then $c_r(F) \in \mathbb{Z}$ for all r.

8E Numbers and Sets

Let X be a finite set, $X_1, \ldots, X_m$ subsets of X and $Y = X \setminus \bigcup X_i$. Let g_i be the characteristic function of X_i, so that

$$g_i(x) = \begin{cases} 1 & \text{if } x \in X_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $f\colon X \to \mathbb{R}$ be any function. By considering the expression

$$\sum_{x \in X} f(x) \prod_{i=1}^{m} (1 - g_i(x)) \,,$$

or otherwise, prove the Inclusion–Exclusion Principle in the form

$$\sum_{x \in Y} f(x) = \sum_{r \geq 0} (-1)^r \sum_{i_1 < \cdots < i_r} \Bigg(\sum_{x \in X_{i_1} \cap \cdots \cap X_{i_r}} f(x) \Bigg).$$

Let $n > 1$ be an integer. For an integer m dividing n let

$$X_m = \{0 \leq x < n \mid x \equiv 0 \pmod{m}\}.$$

By considering the sets X_p for prime divisors p of n, show that

$$\phi(n) = n \prod_{p \mid n} \Big(1 - \frac{1}{p}\Big)$$

(where ϕ is Euler's function) and

$$\sum_{\substack{0 < x < n \\ (x,n)=1}} x = \frac{n^2}{2} \prod_{p \mid n} \Big(1 - \frac{1}{p}\Big).$$

9A Dynamics

A horizontal table oscillates with a displacement $\mathbf{A}\sin\omega t$, where $\mathbf{A} = (A_x, 0, A_z)$ is the amplitude vector and ω the angular frequency in an inertial frame of reference with the z axis vertically upwards, normal to the table. A block sitting on the table has mass m and linear friction that results in a force $\mathbf{F} = -\lambda\mathbf{u}$, where λ is a constant and $\mathbf{u}$ is the velocity difference between the block and the table. Derive the equations of motion for this block in the frame of reference of the table using axes (ξ, η, ζ) on the table parallel to the axes (x, y, z) in the inertial frame.

For the case where $A_z = 0$, show that at late time the block will approach the steady orbit

$$\xi = \xi_0 - A_x \sin\theta\cos(\omega t - \theta),$$

where

$$\sin^2\theta = \frac{m^2\omega^2}{\lambda^2 + m^2\omega^2}$$

and ξ_0 is a constant.

Given that there are no attractive forces between block and table, show that the block will only remain in contact with the table if $\omega^2 A_z < g$.

10A Dynamics

A small probe of mass m is in low orbit about a planet of mass M. If there is no drag on the probe then its orbit is governed by

$$\ddot{\mathbf{r}} = -\frac{GM}{|\mathbf{r}|^3}\mathbf{r},$$

where $\mathbf{r}$ is the location of the probe relative to the centre of the planet and G is the gravitational constant. Show that the basic orbital trajectory is elliptical. Determine the orbital period for the probe if it is in a circular orbit at a distance r_0 from the centre of the planet.

Data returned by the probe shows that the planet has a very extensive but diffuse atmosphere. This atmosphere induces a drag on the probe that may be approximated by the linear law $\mathbf{D} = -A\dot{\mathbf{r}}$, where $\mathbf{D}$ is the drag force and A is a constant. Show that the angular momentum of the probe about the planet decays exponentially.

11A Dynamics

A particle of mass m and charge q moves through a magnetic field $\mathbf{B}$. There is no electric field or external force so that the particle obeys

$$m\ddot{\mathbf{r}} = q\,\dot{\mathbf{r}} \times \mathbf{B},$$

where $\mathbf{r}$ is the location of the particle. Prove that the kinetic energy of the particle is preserved.

Consider an axisymmetric magnetic field described by $\mathbf{B} = (0, 0, B(r))$ in cylindrical polar coordinates $\mathbf{r} = (r, \theta, z)$. Determine the angular velocity of a circular orbit centred on $\mathbf{r} = \mathbf{0}$.

For a general orbit when $B(r) = B_0/r$, show that the angular momentum about the z-axis varies as $L = L_0 - qB_0(r - r_0)$, where L_0 is the angular momentum at radius r_0. Determine and sketch the relationship between $\dot{r}^2$ and r. [Hint: Use conservation of energy.] What is the escape velocity for the particle?

12A Dynamics

A circular cylinder of radius a, length L and mass m is rolling along a surface. Show that its moment of inertia is given by $\frac{1}{2}ma^2$.

At $t = 0$ the cylinder is at the bottom of a slope making an angle α to the horizontal, and is rolling with velocity V and angular velocity V/a. Assuming slippage does not occur, determine the position of the cylinder as a function of time. What is the maximum height that the cylinder reaches?

The frictional force between the cylinder and surface is given by $\mu mg \cos\alpha$, where μ is the friction coefficient. Show that the cylinder begins to slip rather than roll if $\tan\alpha > 3\mu$. Determine as a function of time the location, speed and angular velocity of the cylinder on the slope if this condition is satisfied. Show that slipping continues as the cylinder ascends and descends the slope. Find also the maximum height the cylinder reaches, and its speed and angular velocity when it returns to the bottom of the slope.