UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 27th May 2004 9 to 12

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I.

In Section I, you may attempt **all four** *questions.*

In Section II, at most **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Additional credit will be awarded for substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **A**, **B**, **C**, **D**, **E** *and* **F** *according to the code letter affixed to each question. Include in the same bundle questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Algebra and Geometry

The linear map $H : \mathbb{R}^3 \to \mathbb{R}^3$ represents reflection in the plane through the origin with normal $\mathbf{n}$, where $|\mathbf{n}| = 1$, and $\mathbf{n} = (n_1, n_2, n_3)$ referred to the standard basis. The map is given by $\mathbf{x} \mapsto \mathbf{x}' = \mathbf{M}\mathbf{x}$, where $\mathbf{M}$ is a (3×3) matrix.

Show that

$$M_{ij} = \delta_{ij} - 2n_i n_j \, .$$

Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors such that $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ is an orthonormal set. Show that

$$\mathbf{Mn} = -\mathbf{n}, \quad \mathbf{Mu} = \mathbf{u}, \quad \mathbf{Mv} = \mathbf{v} \, ,$$

and find the matrix $\mathbf{N}$ which gives the mapping relative to the basis $(\mathbf{u}, \mathbf{v}, \mathbf{n})$.

2C Algebra and Geometry

Show that

$$\sum_{i=1}^{n} a_i b_i \leqslant \Big(\sum_{i=1}^{n} a_i^2\Big)^{1/2} \Big(\sum_{i=1}^{n} b_i^2\Big)^{1/2}$$

for any real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. Using this inequality, show that if $\mathbf{a}$ and $\mathbf{b}$ are vectors of unit length in $\mathbb{R}^n$ then $|\mathbf{a} \cdot \mathbf{b}| \leqslant 1$.

3D Analysis

Define the *supremum* or *least upper bound* of a non-empty set of real numbers.

State the Least Upper Bound Axiom for the real numbers.

Starting from the Least Upper Bound Axiom, show that if (a_n) is a bounded monotonic sequence of real numbers, then it converges.

4E Analysis

Let $f(x) = (1+x)^{1/2}$ for $x \in (-1,1)$. Show by induction or otherwise that for every integer $r \geq 1$,

$$f^{(r)}(x) = (-1)^{r-1} \frac{(2r-2)!}{2^{2r-1}(r-1)!}(1+x)^{\frac{1}{2}-r}.$$

Evaluate the series

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{(2r-2)!}{8^r r!(r-1)!}\,.$$

[You may use Taylor's Theorem in the form

$$f(x) = f(0) + \sum_{r=1}^{n} \frac{f^{(r)}(0)}{r!} x^r + \int_0^x \frac{(x-t)^n f^{(n+1)}(t)}{n!}\, dt$$

without proof.]

SECTION II

5B Algebra and Geometry

The vector $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfies the equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,,$$

where $\mathbf{A}$ is a (3×3) matrix and $\mathbf{b}$ is a (3×1) column vector. State the conditions under which this equation has (a) a unique solution, (b) an infinity of solutions, (c) no solution for $\mathbf{x}$.

Find all possible solutions for the unknowns x, y and z which satisfy the following equations:

$$\begin{aligned} x + y + z &= 1 \\ x + y + \lambda z &= 2 \\ x + 2y + \lambda z &= 4\,, \end{aligned}$$

in the cases (a) $\lambda = 0$, and (b) $\lambda = 1$.

6A Algebra and Geometry

Express the product $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ in suffix notation and thence prove that the result is zero.

Silver Beard the space pirate believed people relied so much on space-age navigation techniques that he could safely write down the location of his treasure using the ancient art of vector algebra. Spikey the space jockey thought he could follow the instructions, by moving by the sequence of vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$ one stage at a time. The vectors (expressed in 1000 parsec units) were defined as follows:

1. Start at the centre of the galaxy, which has coordinates $(0, 0, 0)$.

2. Vector $\mathbf{a}$ has length $\sqrt{3}$, is normal to the plane $x + y + z = 1$ and is directed into the positive quadrant.

3. Vector $\mathbf{b}$ is given by $\mathbf{b} = (\mathbf{a} \cdot \mathbf{m})\mathbf{a} \times \mathbf{m}$, where $\mathbf{m} = (2, 0, 1)$.

4. Vector $\mathbf{c}$ has length $2\sqrt{2}$, is normal to $\mathbf{a}$ and $\mathbf{b}$, and moves you closer to the x axis.

5. Vector $\mathbf{d} = (1, -2, 2)$.

6. Vector $\mathbf{e}$ has length $\mathbf{a} \cdot \mathbf{b}$. Spikey was initially a little confused with this one, but then realised the orientation of the vector did not matter.

7. Vector $\mathbf{f}$ has unknown length but is parallel to $\mathbf{m}$ and takes you to the treasure located somewhere on the plane $2x - y + 4z = 10$.

Determine the location of the way-points Spikey will use and thence the location of the treasure.

7A Algebra and Geometry

Simplify the fraction

$$\zeta = \cfrac{1}{\bar{z} + \cfrac{1}{z + \cfrac{1}{\bar{z}}}},$$

where $\bar{z}$ is the complex conjugate of z. Determine the geometric form that satisfies

$$\mathrm{Re}(\zeta) = \mathrm{Re}\left(\frac{z + \frac{1}{4}}{|z|^2}\right).$$

Find solutions to

$$\mathrm{Im}(\log z) = \frac{\pi}{3}$$

and

$$z^2 = x^2 - y^2 + 2ix,$$

where $z = x + iy$ is a complex variable. Sketch these solutions in the complex plane and describe the region they enclose. Derive complex equations for the circumscribed and inscribed circles for the region. [The circumscribed circle is the circle that passes through the vertices of the region and the inscribed circle is the largest circle that fits within the region.]

8C Algebra and Geometry

(i) The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$ satisfy $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3 \neq 0$. Are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ necessarily linearly independent? Justify your answer by a proof or a counterexample.

(ii) The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$ have the property that every subset comprising $(n-1)$ of the vectors is linearly independent. Are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ necessarily linearly independent? Justify your answer by a proof or a counterexample.

(iii) For each value of t in the range $0 \leqslant t < 1$, give a construction of a linearly independent set of vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$ satisfying

$$\mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij} + t(1 - \delta_{ij}),$$

where δ_{ij} is the Kronecker delta.

9D Analysis

i) State Rolle's theorem.

Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions which are differentiable on (a, b).

ii) Prove that for some $c \in (a, b)$,

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

iii) Suppose that $f(a) = g(a) = 0$, and that $\lim_{x \to a+} \dfrac{f'(x)}{g'(x)}$ exists and is equal to L. Prove that $\lim_{x \to a+} \dfrac{f(x)}{g(x)}$ exists and is also equal to L.

[You may assume there exists a $\delta > 0$ such that, for all $x \in (a, a + \delta)$, $g'(x) \neq 0$ and $g(x) \neq 0$.]

iv) Evaluate $\lim_{x \to 0} \dfrac{\log \cos x}{x^2}$.

10E Analysis

Define, for an integer $n \geq 0$,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx.$$

Show that for every $n \geq 2$, $nI_n = (n-1)I_{n-2}$, and deduce that

$$I_{2n} = \frac{(2n)!}{(2^n n!)^2} \frac{\pi}{2} \quad \text{and} \quad I_{2n+1} = \frac{(2^n n!)^2}{(2n+1)!}.$$

Show that $0 < I_n < I_{n-1}$, and that

$$\frac{2n}{2n+1} < \frac{I_{2n+1}}{I_{2n}} < 1.$$

Hence prove that

$$\lim_{n \to \infty} \frac{2^{4n+1}(n!)^4}{(2n+1)(2n)!^2} = \pi.$$

11F Analysis

Let f be defined on $\mathbb{R}$, and assume that there exists at least one point $x_0 \in \mathbb{R}$ at which f is continuous. Suppose also that, for every $x, y \in \mathbb{R}$, f satisfies the equation

$$f(x+y) = f(x) + f(y).$$

Show that f is continuous on $\mathbb{R}$.

Show that there exists a constant c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Suppose that g is a continuous function defined on $\mathbb{R}$ and that, for every $x, y \in \mathbb{R}$, g satisfies the equation

$$g(x+y) \;=\; g(x)g(y).$$

Show that if g is not identically zero, then g is everywhere positive. Find the general form of g.

12F Analysis

(i) Show that if $a_n > 0$, $b_n > 0$ and

$$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$$

for all $n \geqslant 1$, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Let

$$c_n = \binom{2n}{n} 4^{-n}.$$

By considering $\log c_n$, or otherwise, show that $c_n \to 0$ as $n \to \infty$.

[*Hint*: $\log(1-x) \leqslant -x$ for $x \in (0,1)$.]

(iii) Determine the convergence or otherwise of

$$\sum_{n=1}^{\infty} \binom{2n}{n} x^n$$

for (a) $x = \frac{1}{4}$, (b) $x = -\frac{1}{4}$.