UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2003

List of Courses

1/I/1F Analysis II

Let E be a subset of $\mathbb{R}^n$. Prove that the following conditions on E are equivalent:

(i) E is closed and bounded.

(ii) E has the Bolzano–Weierstrass property (i.e., every sequence in E has a subsequence convergent to a point of E).

(iii) Every continuous real-valued function on E is bounded.

[*The Bolzano–Weierstrass property for bounded closed intervals in* $\mathbb{R}^1$ *may be assumed.*]

1/II/10F Analysis II

Explain briefly what is meant by a *metric space*, and by a *Cauchy sequence* in a metric space.

A function $d : X \times X \to \mathbb{R}$ is called a pseudometric on X if it satisfies all the conditions for a metric except the requirement that $d(x, y) = 0$ implies $x = y$. If d is a pseudometric on X, show that the binary relation R on X defined by $x\,R\,y \Leftrightarrow d(x, y) = 0$ is an equivalence relation, and that the function d induces a metric on the set X/R of equivalence classes.

Now let (X, d) be a metric space. If (x_n) and (y_n) are Cauchy sequences in X, show that the sequence whose nth term is $d(x_n, y_n)$ is a Cauchy sequence of real numbers. Deduce that the function $\overline{d}$ defined by

$$\overline{d}((x_n), (y_n)) = \lim_{n\to\infty} d(x_n, y_n)$$

is a pseudometric on the set C of all Cauchy sequences in X. Show also that there is an isometric embedding (that is, a distance-preserving mapping) $X \to C/R$, where R is the equivalence relation on C induced by the pseudometric $\overline{d}$ as in the previous paragraph. Under what conditions on X is $X \to C/R$ bijective? Justify your answer.

2/I/1F Analysis II

Explain what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ to be *differentiable* at a point (a, b). Show that if the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist in a neighbourhood of (a, b) and are continuous at (a, b) then f is differentiable at (a, b).

Let

$$f(x, y) = \frac{xy}{x^2 + y^2} \qquad ((x, y) \neq (0, 0))$$

and $f(0, 0) = 0$. Do the partial derivatives of f exist at $(0, 0)$? Is f differentiable at $(0, 0)$? Justify your answers.

2/II/10F **Analysis II**

Let V be the space of $n \times n$ real matrices. Show that the function

$$N(A) = \sup\{\|A\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\| = 1\}$$

(where $\| - \|$ denotes the usual Euclidean norm on $\mathbb{R}^n$) defines a norm on V. Show also that this norm satisfies $N(AB) \leqslant N(A)N(B)$ for all A and B, and that if $N(A) < \epsilon$ then all entries of A have absolute value less than ϵ. Deduce that any function $f: V \to \mathbb{R}$ such that $f(A)$ is a polynomial in the entries of A is continuously differentiable.

Now let $d : V \to \mathbb{R}$ be the mapping sending a matrix to its determinant. By considering $d(I + H)$ as a polynomial in the entries of H, show that the derivative $d'(I)$ is the function $H \mapsto \operatorname{tr} H$. Deduce that, for any A, $d'(A)$ is the mapping $H \mapsto \operatorname{tr}((\operatorname{adj} A)H)$, where $\operatorname{adj} A$ is the adjugate of A, i.e. the matrix of its cofactors.

[*Hint: consider first the case when A is invertible. You may assume the results that the set U of invertible matrices is open in V and that its closure is the whole of V, and the identity* $(\operatorname{adj} A)A = \det A.I$.]

3/I/1F **Analysis II**

Let V be the vector space of continuous real-valued functions on $[-1, 1]$. Show that the function

$$\|f\| = \int_{-1}^{1} |f(x)|\, dx$$

defines a norm on V.

Let $f_n(x) = x^n$. Show that (f_n) is a Cauchy sequence in V. Is (f_n) convergent? Justify your answer.

3/II/11F **Analysis II**

State and prove the Contraction Mapping Theorem.

Let (X, d) be a bounded metric space, and let F denote the set of all continuous maps $X \to X$. Let $\rho: F \times F \to \mathbb{R}$ be the function

$$\rho(f, g) = \sup\{d(f(x), g(x)) : x \in X\} .$$

Show that ρ is a metric on F, and that (F, ρ) is complete if (X, d) is complete. [*You may assume that a uniform limit of continuous functions is continuous.*]

Now suppose that (X, d) is complete. Let $C \subseteq F$ be the set of contraction mappings, and let $\theta: C \to X$ be the function which sends a contraction mapping to its unique fixed point. Show that θ is continuous. [*Hint: fix $f \in C$ and consider $d(\theta(g), f(\theta(g)))$, where $g \in C$ is close to f.*]

4/I/1F **Analysis II**

Explain what it means for a sequence of functions (f_n) to converge uniformly to a function f on an interval. If (f_n) is a sequence of continuous functions converging uniformly to f on a finite interval $[a, b]$, show that

$$\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx \qquad \text{as } n \to \infty \ .$$

Let $f_n(x) = x\exp(-x/n)/n^2$, $x \geqslant 0$. Does $f_n \to 0$ uniformly on $[0, \infty)$? Does $\int_0^\infty f_n(x)\,dx \to 0$? Justify your answers.

4/II/10F **Analysis II**

Let $(f_n)_{n\geqslant 1}$ be a sequence of continuous complex-valued functions defined on a set $E \subseteq \mathbb{C}$, and converging uniformly on E to a function f. Prove that f is continuous on E.

State the Weierstrass M-test for uniform convergence of a series $\sum_{n=1}^\infty u_n(z)$ of complex-valued functions on a set E.

Now let $f(z) = \sum_{n=1}^\infty u_n(z)$, where

$$u_n(z) = n^{-2}\sec\left(\pi z/2n\right) \ .$$

Prove carefully that f is continuous on $\mathbb{C} \setminus \mathbb{Z}$.

[*You may assume the inequality* $|\cos z| \geqslant |\cos(\mathrm{Re}\, z)|$.]

1/I/7B Complex Methods

Let $u(x,y)$ and $v(x,y)$ be a pair of conjugate harmonic functions in a domain D. Prove that

$$U(x,y) = e^{-2uv}\cos(u^2 - v^2) \qquad \text{and} \qquad V(x,y) = e^{-2uv}\sin(u^2 - v^2)$$

also form a pair of conjugate harmonic functions in D.

1/II/16B Complex Methods

Sketch the region A which is the intersection of the discs

$$D_0 = \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad D_1 = \{z \in \mathbb{C} : |z - (1+i)| < 1\}\,.$$

Find a conformal mapping that maps A onto the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Also find a conformal mapping that maps A onto D_0.

[*Hint: You may find it useful to consider maps of the form* $w(z) = \frac{az+b}{cz+d}$.]

2/I/7B Complex Methods

(a) Using the residue theorem, evaluate

$$\int_{|z|=1} \left(z - \frac{1}{z}\right)^{2n} \frac{dz}{z}\,.$$

(b) Deduce that

$$\int_0^{2\pi} \sin^{2n} t\, dt = \frac{\pi}{2^{2n-1}} \frac{(2n)!}{(n!)^2}\,.$$

2/II/16B **Complex Methods**

(a) Show that if f satisfies the equation

$$f''(x) - x^2 f(x) = \mu f(x), \quad x \in \mathbb{R}, \qquad (*)$$

where μ is a constant, then its Fourier transform $\widehat{f}$ satisfies the same equation, i.e.

$$\widehat{f}''(\lambda) - \lambda^2 \widehat{f}(\lambda) = \mu \widehat{f}(\lambda)\,.$$

(b) Prove that, for each $n \geq 0$, there is a polynomial $p_n(x)$ of degree n, unique up to multiplication by a constant, such that

$$f_n(x) = p_n(x) e^{-x^2/2}$$

is a solution of $(*)$ for some $\mu = \mu_n$.

(c) Using the fact that $g(x) = e^{-x^2/2}$ satisfies $\widehat{g} = cg$ for some constant c, show that the Fourier transform of f_n has the form

$$\widehat{f_n}(\lambda) = q_n(\lambda) e^{-\lambda^2/2}$$

where q_n is also a polynomial of degree n.

(d) Deduce that the f_n are eigenfunctions of the Fourier transform operator, i.e. $\widehat{f_n}(x) = c_n f_n(x)$ for some constants c_n.

4/I/8B **Complex Methods**

Find the Laurent series centred on 0 for the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in each of the domains

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$.

4/II/17B **Complex Methods**

Let

$$f(z) = \frac{z^m}{1+z^n}, \quad n > m+1, \quad m, n \in \mathbb{N},$$

and let C_R be the boundary of the domain

$$D_R = \{z = re^{i\theta} : \ 0 < r < R, \quad 0 < \theta < \tfrac{2\pi}{n}\}, \quad R > 1.$$

(a) Using the residue theorem, determine

$$\int_{C_R} f(z)\, dz.$$

(b) Show that the integral of $f(z)$ along the circular part γ_R of C_R tends to 0 as $R \to \infty$.

(c) Deduce that

$$\int_0^\infty \frac{x^m}{1+x^n}\, dx = \frac{\pi}{n \sin \frac{\pi(m+1)}{n}}\,.$$

1/I/6C Fluid Dynamics

An unsteady fluid flow has velocity field given in Cartesian coordinates (x, y, z) by $\mathbf{u} = (1, xt, 0)$, where t denotes time. Dye is released into the fluid from the origin continuously. Find the position at time t of the dye particle that was released at time s and hence show that the dye streak lies along the curve

$$y = \tfrac{1}{2}tx^2 - \tfrac{1}{6}x^3.$$

1/II/15C Fluid Dynamics

Starting from the Euler equations for incompressible, inviscid flow

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p, \qquad \nabla \cdot \mathbf{u} = 0,$$

derive the vorticity equation governing the evolution of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

Consider the flow

$$\mathbf{u} = \beta(-x, -y, 2z) + \Omega(t)(-y, x, 0),$$

in Cartesian coordinates (x, y, z), where t is time and β is a constant. Compute the vorticity and show that it evolves in time according to

$$\boldsymbol{\omega} = \omega_0 \mathrm{e}^{2\beta t}\mathbf{k},$$

where ω_0 is the initial magnitude of the vorticity and $\mathbf{k}$ is a unit vector in the z-direction.

Show that the material curve $C(t)$ that takes the form

$$x^2 + y^2 = 1 \quad \text{and} \quad z = 1$$

at $t = 0$ is given later by

$$x^2 + y^2 = a^2(t) \quad \text{and} \quad z = \frac{1}{a^2(t)},$$

where the function $a(t)$ is to be determined.

Calculate the circulation of $\mathbf{u}$ around C and state how this illustrates Kelvin's circulation theorem.

3/I/8C **Fluid Dynamics**

Show that the velocity field

$$\mathbf{u} = \mathbf{U} + \frac{\mathbf{\Gamma} \times \mathbf{r}}{2\pi r^2},$$

where $\mathbf{U} = (U, 0, 0)$, $\mathbf{\Gamma} = (0, 0, \Gamma)$ and $\mathbf{r} = (x, y, 0)$ in Cartesian coordinates (x, y, z), represents the combination of a uniform flow and the flow due to a line vortex. Define and evaluate the circulation of the vortex.

Show that

$$\oint_{C_R} (\mathbf{u} \cdot \mathbf{n})\mathbf{u}\, dl \to \tfrac{1}{2}\mathbf{\Gamma} \times \mathbf{U} \qquad \text{as} \qquad R \to \infty,$$

where C_R is a circle $x^2 + y^2 = R^2$, $z =$ const. Explain how this result is related to the lift force on a two-dimensional aerofoil or other obstacle.

3/II/18C **Fluid Dynamics**

State the form of Bernoulli's theorem appropriate for an unsteady irrotational motion of an inviscid incompressible fluid in the absence of gravity.

Water of density ρ is driven through a tube of length L and internal radius a by the pressure exerted by a spherical, water-filled balloon of radius $R(t)$ attached to one end of the tube. The balloon maintains the pressure of the water entering the tube at $2\gamma/R$ in excess of atmospheric pressure, where γ is a constant. It may be assumed that the water exits the tube at atmospheric pressure. Show that

$$R^3\ddot{R} + 2R^2\dot{R}^2 = -\frac{\gamma a^2}{2\rho L}. \qquad (\dagger)$$

Solve equation ($\dagger$), by multiplying through by $2R\dot{R}$ or otherwise, to obtain

$$t = R_0^2 \left(\frac{2\rho L}{\gamma a^2}\right)^{1/2} \left[\frac{\pi}{4} - \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right],$$

where $\theta = \sin^{-1}(R/R_0)$ and R_0 is the initial radius of the balloon. Hence find the time when $R = 0$.

4/I/7C **Fluid Dynamics**

Inviscid fluid issues vertically downwards at speed u_0 from a circular tube of radius a. The fluid falls onto a horizontal plate a distance H below the end of the tube, where it spreads out axisymmetrically.

Show that while the fluid is falling freely it has speed

$$u = u_0 \left[1 + \frac{2g}{u_0^2}(H - z)\right]^{1/2},$$

and occupies a circular jet of radius

$$R = a \left[1 + \frac{2g}{u_0^2}(H - z)\right]^{-1/4},$$

where z is the height above the plate and g is the acceleration due to gravity.

Show further that along the plate, at radial distances $r \gg a$ (i.e. far from the falling jet), where the fluid is flowing almost horizontally, it does so as a film of height $h(r)$, where

$$\frac{a^4}{4r^2h^2} = 1 + \frac{2g}{u_0^2}(H - h).$$

4/II/16C **Fluid Dynamics**

Define the terms *irrotational flow* and *incompressible flow*. The two-dimensional flow of an incompressible fluid is given in terms of a streamfunction $\psi(x, y)$ as

$$\mathbf{u} = (u, v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)$$

in Cartesian coordinates (x, y). Show that the line integral

$$\int_{\mathbf{x_1}}^{\mathbf{x_2}} \mathbf{u} \cdot \mathbf{n} \, dl = \psi(\mathbf{x_2}) - \psi(\mathbf{x_1})$$

along any path joining the points $\mathbf{x_1}$ and $\mathbf{x_2}$, where $\mathbf{n}$ is the unit normal to the path. Describe how this result is related to the concept of mass conservation.

Inviscid, incompressible fluid is contained in the semi-infinite channel $x > 0$, $0 < y < 1$, which has rigid walls at $x = 0$ and at $y = 0$, 1, apart from a small opening at the origin through which the fluid is withdrawn with volume flux m per unit distance in the third dimension. Show that the streamfunction for irrotational flow in the channel can be chosen (up to an additive constant) to satisfy the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

and boundary conditions

$$\begin{aligned} &\psi = 0 && \text{on } y = 0,\ x > 0, \\ &\psi = -m && \text{on } x = 0,\ 0 < y < 1, \\ &\psi = -m && \text{on } y = 1,\ x > 0, \\ &\psi \to -my && \text{as } x \to \infty, \end{aligned}$$

if it is assumed that the flow at infinity is uniform. Solve the boundary-value problem above using separation of variables to obtain

$$\psi = -my + \frac{2m}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi y \, e^{-n\pi x}.$$

2/I/4E **Further Analysis**

Let τ_1 be the collection of all subsets $A \subset \mathbb{N}$ such that $A = \emptyset$ or $\mathbb{N} \setminus A$ is finite. Let τ_2 be the collection of all subsets of $\mathbb{N}$ of the form $I_n = \{n, n+1, n+2, \ldots\}$, together with the empty set. Prove that τ_1 and τ_2 are both topologies on $\mathbb{N}$.

Show that a function f from the topological space $(\mathbb{N}, \tau_1)$ to the topological space $(\mathbb{N}, \tau_2)$ is continuous if and only if one of the following alternatives holds:

(i) $f(n) \to \infty$ as $n \to \infty$;

(ii) there exists $N \in \mathbb{N}$ such that $f(n) = N$ for all but finitely many n and $f(n) \leqslant N$ for all n.

2/II/13E **Further Analysis**

(a) Let $f : [1, \infty) \to \mathbb{C}$ be defined by $f(t) = t^{-1} e^{2\pi i t}$ and let X be the image of f. Prove that $X \cup \{0\}$ is compact and path-connected. [*Hint: you may find it helpful to set* $s = t^{-1}$.]

(b) Let $g : [1, \infty) \to \mathbb{C}$ be defined by $g(t) = (1 + t^{-1}) e^{2\pi i t}$, let Y be the image of g and let $\overline{D}$ be the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. Prove that $Y \cup \overline{D}$ is connected. Explain briefly why it is not path-connected.

3/I/3E **Further Analysis**

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leqslant 1 + |z|^{1/2}$ for every z. Prove that f is constant.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $\mathrm{Re}\,(f(z)) \geqslant 0$ for every z. Prove that f is constant.

3/II/13E **Further Analysis**

(a) State Taylor's Theorem.

(b) Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$ be defined whenever $|z - z_0| < r$. Suppose that $z_k \to z_0$ as $k \to \infty$, that no z_k equals z_0 and that $f(z_k) = g(z_k)$ for every k. Prove that $a_n = b_n$ for every $n \geqslant 0$.

(c) Let D be a domain, let $z_0 \in D$ and let (z_k) be a sequence of points in D that converges to z_0, but such that no z_k equals z_0. Let $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ be analytic functions such that $f(z_k) = g(z_k)$ for every k. Prove that $f(z) = g(z)$ for every $z \in D$.

(d) Let D be the domain $\mathbb{C} \setminus \{0\}$. Give an example of an analytic function $f : D \to \mathbb{C}$ such that $f(n^{-1}) = 0$ for every positive integer n but f is not identically 0.

(e) Show that any function with the property described in (d) must have an essential singularity at the origin.

4/I/4E **Further Analysis**

(a) State and prove Morera's Theorem.

(b) Let D be a domain and for each $n \in \mathbb{N}$ let $f_n : D \to \mathbb{C}$ be an analytic function. Suppose that $f : D \to \mathbb{C}$ is another function and that $f_n \to f$ uniformly on D. Prove that f is analytic.

4/II/13E **Further Analysis**

(a) State the residue theorem and use it to deduce the principle of the argument, in a form that involves winding numbers.

(b) Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\mathrm{Im}\,(p(z)) = 0$. Calculate $\mathrm{Re}\,(p(z))$ for each such z. [*It will be helpful to set* $z = e^{i\theta}$. *You may use the addition formulae* $\sin\alpha + \sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$ *and* $\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$.]

(c) Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed path $\theta \mapsto e^{i\theta}$. Use your answer to (b) to give a rough sketch of the path $p \circ \gamma$, paying particular attention to where it crosses the real axis.

(d) Hence, or otherwise, determine for every real t the number of z (counted with multiplicity) such that $|z| < 1$ and $p(z) = t$. (You need not give rigorous justifications for your calculations.)

1/I/4F **Geometry**

Describe the geodesics (that is, hyperbolic straight lines) in **either** the disc model **or** the half-plane model of the hyperbolic plane. Explain what is meant by the statements that two hyperbolic lines are parallel, and that they are ultraparallel.

Show that two hyperbolic lines l and l' have a unique common perpendicular if and only if they are ultraparallel.

[*You may assume standard results about the group of isometries of whichever model of the hyperbolic plane you use.*]

1/II/13F **Geometry**

Write down the Riemannian metric in the half-plane model of the hyperbolic plane. Show that Möbius transformations mapping the upper half-plane to itself are isometries of this model.

Calculate the hyperbolic distance from ib to ic, where b and c are positive real numbers. Assuming that the hyperbolic circle with centre ib and radius r is a Euclidean circle, find its Euclidean centre and radius.

Suppose that a and b are positive real numbers for which the points ib and $a + ib$ of the upper half-plane are such that the hyperbolic distance between them coincides with the Euclidean distance. Obtain an expression for b as a function of a. Hence show that, for any b with $0 < b < 1$, there is a unique positive value of a such that the hyperbolic distance between ib and $a + ib$ coincides with the Euclidean distance.

3/I/4F **Geometry**

Show that any isometry of Euclidean 3-space which fixes the origin can be written as a composite of at most three reflections in planes through the origin, and give (with justification) an example of an isometry for which three reflections are necessary.

3/II/14F **Geometry**

State and prove the Gauss–Bonnet formula for the area of a spherical triangle. Deduce a formula for the area of a spherical n-gon with angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. For what range of values of α does there exist a (convex) regular spherical n-gon with angle α?

Let Δ be a spherical triangle with angles π/p, π/q and π/r where p, q, r are integers, and let G be the group of isometries of the sphere generated by reflections in the three sides of Δ. List the possible values of (p, q, r), and in each case calculate the order of the corresponding group G. If $(p, q, r) = (2, 3, 5)$, show how to construct a regular dodecahedron whose group of symmetries is G.

[*You may assume that the images of* Δ *under the elements of* G *form a tessellation of the sphere.*]

1/I/5E **Linear Mathematics**

Let V be the subset of $\mathbb{R}^5$ consisting of all quintuples $(a_1, a_2, a_3, a_4, a_5)$ such that

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

and

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0 \ .$$

Prove that V is a subspace of $\mathbb{R}^5$. Solve the above equations for a_1 and a_2 in terms of a_3, a_4 and a_5. Hence, exhibit a basis for V, explaining carefully why the vectors you give form a basis.

1/II/14E **Linear Mathematics**

(a) Let U, U' be subspaces of a finite-dimensional vector space V. Prove that $\dim(U + U') = \dim U + \dim U' - \dim(U \cap U')$.

(b) Let V and W be finite-dimensional vector spaces and let α and β be linear maps from V to W. Prove that

$$\operatorname{rank}(\alpha + \beta) \leqslant \operatorname{rank} \alpha + \operatorname{rank} \beta \ .$$

(c) Deduce from this result that

$$\operatorname{rank}(\alpha + \beta) \geqslant |\operatorname{rank} \alpha - \operatorname{rank} \beta| \ .$$

(d) Let $V = W = \mathbb{R}^n$ and suppose that $1 \leqslant r \leqslant s \leqslant n$. Exhibit linear maps $\alpha, \beta \colon V \to W$ such that $\operatorname{rank} \alpha = r$, $\operatorname{rank} \beta = s$ and $\operatorname{rank}(\alpha + \beta) = s - r$. Suppose that $r + s \geqslant n$. Exhibit linear maps $\alpha, \beta : V \to W$ such that $\operatorname{rank} \alpha = r$, $\operatorname{rank} \beta = s$ and $\operatorname{rank}(\alpha + \beta) = n$.

2/I/6E **Linear Mathematics**

Let $a_1, a_2, \ldots, a_n$ be distinct real numbers. For each i let $\mathbf{v}_i$ be the vector $(1, a_i, a_i^2, \ldots, a_i^{n-1})$. Let A be the $n \times n$ matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and let $\mathbf{c}$ be a column vector of size n. Prove that $A\mathbf{c} = \mathbf{0}$ if and only if $\mathbf{c} = \mathbf{0}$. Deduce that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span $\mathbb{R}^n$.

[*You may use general facts about matrices if you state them clearly.*]

2/II/15E Linear Mathematics

(a) Let $A = (a_{ij})$ be an $m \times n$ matrix and for each $k \leqslant n$ let A_k be the $m \times k$ matrix formed by the first k columns of A. Suppose that $n > m$. Explain why the nullity of A is non-zero. Prove that if k is minimal such that A_k has non-zero nullity, then the nullity of A_k is 1.

(b) Suppose that no column of A consists entirely of zeros. Deduce from (a) that there exist scalars $b_1, \ldots, b_k$ (where k is defined as in (a)) such that $\sum_{j=1}^{k} a_{ij} b_j = 0$ for every $i \leqslant m$, but whenever $\lambda_1, \ldots, \lambda_k$ are distinct real numbers there is some $i \leqslant m$ such that $\sum_{j=1}^{k} a_{ij} \lambda_j b_j \neq 0$.

(c) Now let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be bases for the same real m-dimensional vector space. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct real numbers such that for every j the vectors $\mathbf{v}_1 + \lambda_j \mathbf{w}_1, \ldots, \mathbf{v}_m + \lambda_j \mathbf{w}_m$ are linearly dependent. For each j, let $a_{1j}, \ldots, a_{mj}$ be scalars, not all zero, such that $\sum_{i=1}^{m} a_{ij}(\mathbf{v}_i + \lambda_j \mathbf{w}_i) = \mathbf{0}$. By applying the result of (b) to the matrix (a_{ij}), deduce that $n \leqslant m$.

(d) It follows that the vectors $\mathbf{v}_1 + \lambda \mathbf{w}_1, \ldots, \mathbf{v}_m + \lambda \mathbf{w}_m$ are linearly dependent for at most m values of λ. Explain briefly how this result can also be proved using determinants.

3/I/7G Linear Mathematics

Let α be an endomorphism of a finite-dimensional real vector space U and let β be another endomorphism of U that commutes with α. If λ is an eigenvalue of α, show that β maps the kernel of $\alpha - \lambda\,\iota$ into itself, where ι is the identity map. Suppose now that α is diagonalizable with n distinct real eigenvalues where $n = \dim U$. Prove that if there exists an endomorphism β of U such that $\alpha = \beta^2$, then $\lambda \geqslant 0$ for all eigenvalues λ of α.

3/II/17G Linear Mathematics

Define the *determinant* $\det(A)$ of an $n \times n$ complex matrix A. Let $A_1, \ldots, A_n$ be the columns of A, let σ be a permutation of $\{1, \ldots, n\}$ and let A^σ be the matrix whose columns are $A_{\sigma(1)}, \ldots, A_{\sigma(n)}$. Prove from your definition of determinant that $\det(A^\sigma) = \epsilon(\sigma)\det(A)$, where $\epsilon(\sigma)$ is the sign of the permutation σ. Prove also that $\det(A) = \det(A^t)$.

Define the *adjugate* matrix $\operatorname{adj}(A)$ and prove from your definitions that $A \operatorname{adj}(A) = \operatorname{adj}(A)\, A = \det(A)\, I$, where I is the identity matrix. Hence or otherwise, prove that if $\det(A) \neq 0$, then A is invertible.

Let C and D be real $n \times n$ matrices such that the complex matrix $C + iD$ is invertible. By considering $\det(C + \lambda\, D)$ as a function of λ or otherwise, prove that there exists a real number λ such that $C + \lambda\, D$ is invertible. [*You may assume that if a matrix A is invertible, then* $\det(A) \neq 0$.]

Deduce that if two real matrices A and B are such that there exists an invertible complex matrix P with $P^{-1} A\, P = B$, then there exists an invertible **real** matrix Q such that $Q^{-1} A\, Q = B$.

4/I/6G **Linear Mathematics**

Let α be an endomorphism of a finite-dimensional real vector space U such that $\alpha^2 = \alpha$. Show that U can be written as the direct sum of the kernel of α and the image of α. Hence or otherwise, find the characteristic polynomial of α in terms of the dimension of U and the rank of α. Is α diagonalizable? Justify your answer.

4/II/15G **Linear Mathematics**

Let $\alpha \in L(U, V)$ be a linear map between finite-dimensional vector spaces. Let

$$M^l(\alpha) = \{\beta \in L(V, U) : \beta\,\alpha = 0\} \qquad \text{and}$$

$$M^r(\alpha) = \{\beta \in L(V, U) : \alpha\,\beta = 0\}\ .$$

(a) Prove that $M^l(\alpha)$ and $M^r(\alpha)$ are subspaces of $L(V, U)$ of dimensions

$$\dim M^l(\alpha) = (\dim V - \operatorname{rank}\alpha)\dim U \qquad \text{and}$$

$$\dim M^r(\alpha) = \dim\ker(\alpha)\dim V\ .$$

[*You may use the result that there exist bases in U and V so that α is represented by*

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix and r is the rank of α.]

(b) Let $\Phi\colon L(U, V) \to L(V^*, U^*)$ be given by $\Phi(\alpha) = \alpha^*$, where α^* is the dual map induced by α. Prove that Φ is an isomorphism. [*You may assume that Φ is linear, and you may use the result that a finite-dimensional vector space and its dual have the same dimension.*]

(c) Prove that

$$\Phi(M^l(\alpha)) = M^r(\alpha^*) \quad \text{and} \quad \Phi(M^r(\alpha)) = M^l(\alpha^*).$$

[*You may use the results that $(\beta\,\alpha)^* = \alpha^*\,\beta^*$ and that β^{**} can be identified with β under the canonical isomorphism between a vector space and its double dual.*]

(d) Conclude that $\operatorname{rank}(\alpha) = \operatorname{rank}(\alpha^*)$.

1/I/2D **Methods**

Fermat's principle of optics states that the path of a light ray connecting two points will be such that the travel time t is a minimum. If the speed of light varies continuously in a medium and is a function $c(y)$ of the distance from the boundary $y = 0$, show that the path of a light ray is given by the solution to

$$c(y)y'' + c'(y)(1 + y'^2) = 0\,,$$

where $y' = \frac{dy}{dx}$, etc. Show that the path of a light ray in a medium where the speed of light c is a constant is a straight line. Also find the path from $(0, 0)$ to $(1, 0)$ if $c(y) = y$, and sketch it.

1/II/11D **Methods**

(a) Determine the Green's function $G(x, \xi)$ for the operator $\frac{d^2}{dx^2} + k^2$ on $[0, \pi]$ with Dirichlet boundary conditions by solving the boundary value problem

$$\frac{d^2G}{dx^2} + k^2G = \delta(x - \xi)\ , \quad G(0) = 0,\ G(\pi) = 0$$

when k is not an integer.

(b) Use the method of Green's functions to solve the boundary value problem

$$\frac{d^2y}{dx^2} + k^2y = f(x)\ , \quad y(0) = a,\ y(\pi) = b$$

when k is not an integer.

2/I/2C **Methods**

Explain briefly why the second-rank tensor

$$\int_S x_i x_j \, dS(\mathbf{x})$$

is isotropic, where S is the surface of the unit sphere centred on the origin.

A second-rank tensor is defined by

$$T_{ij}(\mathbf{y}) = \int_S (y_i - x_i)(y_j - x_j) \, dS(\mathbf{x}) \, ,$$

where S is the surface of the unit sphere centred on the origin. Calculate $T(\mathbf{y})$ in the form

$$T_{ij} = \lambda \delta_{ij} + \mu y_i y_j \, ,$$

where λ and μ are to be determined.

By considering the action of T on $\mathbf{y}$ and on vectors perpendicular to $\mathbf{y}$, determine the eigenvalues and associated eigenvectors of T.

2/II/11C **Methods**

State the transformation law for an nth-rank tensor $T_{ij\cdots k}$.

Show that the fourth-rank tensor

$$c_{ijkl} = \alpha \, \delta_{ij} \, \delta_{kl} + \beta \, \delta_{ik} \, \delta_{jl} + \gamma \, \delta_{il} \, \delta_{jk}$$

is isotropic for arbitrary scalars α, β and γ.

The stress σ_{ij} and strain e_{ij} in a linear elastic medium are related by

$$\sigma_{ij} = c_{ijkl} \, e_{kl}.$$

Given that e_{ij} is symmetric and that the medium is isotropic, show that the stress-strain relationship can be written in the form

$$\sigma_{ij} = \lambda \, e_{kk} \, \delta_{ij} + 2\mu \, e_{ij}.$$

Show that e_{ij} can be written in the form $e_{ij} = p\delta_{ij} + d_{ij}$, where d_{ij} is a traceless tensor and p is a scalar to be determined. Show also that necessary and sufficient conditions for the stored elastic energy density $E = \frac{1}{2}\sigma_{ij} \, e_{ij}$ to be non-negative for any deformation of the solid are that

$$\mu \geq 0 \qquad \text{and} \qquad \lambda \geq -\tfrac{2}{3}\mu.$$

3/I/2D **Methods**

Consider the path between two arbitrary points on a cone of interior angle 2α. Show that the arc-length of the path $r(\theta)$ is given by

$$\int (r^2 + r'^2 \operatorname{cosec}^2 \alpha)^{1/2}\, d\theta \ ,$$

where $r' = \frac{dr}{d\theta}$. By minimizing the total arc-length between the points, determine the equation for the shortest path connecting them.

3/II/12D **Methods**

The transverse displacement $y(x,t)$ of a stretched string clamped at its ends $x = 0, l$ satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - 2k \frac{\partial y}{\partial t} \ , \quad y(x,0) = 0, \ \frac{\partial y}{\partial t}(x,0) = \delta(x-a)\,,$$

where $c > 0$ is the wave velocity, and $k > 0$ is the damping coefficient. The initial conditions correspond to a sharp blow at $x = a$ at time $t = 0$.

(a) Show that the subsequent motion of the string is given by

$$y(x,t) = \frac{1}{\sqrt{\alpha_n^2 - k^2}} \sum_n 2e^{-kt} \sin \frac{\alpha_n a}{c} \ \sin \frac{\alpha_n x}{c} \ \sin /(\sqrt{\alpha_n^2 - k^2} \ \ t)$$

where $\alpha_n = \pi c n / l$.

(b) Describe what happens in the limits of small and large damping. What critical parameter separates the two cases?

4/I/2D **Methods**

Consider the wave equation in a spherically symmetric coordinate system

$$\frac{\partial^2 u(r,t)}{\partial t^2} = c^2 \Delta u(r,t)\,,$$

where $\Delta u = \frac{1}{r} \frac{\partial^2}{\partial r^2}(ru)$ is the spherically symmetric Laplacian operator.

(a) Show that the general solution to the equation above is

$$u(r,t) = \tfrac{1}{r}[f(r+ct) + g(r-ct)]\,,$$

where $f(x), g(x)$ are arbitrary functions.

(b) Using separation of variables, determine the wave field $u(r,t)$ in response to a pulsating source at the origin $u(0,t) = A \sin \omega t$.

4/II/11D **Methods**

The velocity potential $\phi(r,\theta)$ for inviscid flow in two dimensions satisfies the Laplace equation

$$\Delta\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi(r,\theta) = 0\ .$$

(a) Using separation of variables, derive the general solution to the equation above that is single-valued and finite in each of the domains (i) $0 \leqslant r \leqslant a$; (ii) $a \leqslant r < \infty$.

(b) Assuming ϕ is single-valued, solve the Laplace equation subject to the boundary conditions $\frac{\partial\phi}{\partial r} = 0$ at $r = a$, and $\frac{\partial\phi}{\partial r} \to U\cos\theta$ as $r \to \infty$. Sketch the lines of constant potential.

2/I/5B **Numerical Analysis**

Let

$$A = \begin{pmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ a^2 & a^3 & 1 & a \\ a & a^2 & a^3 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} \gamma \\ 0 \\ 0 \\ \gamma a \end{pmatrix}, \quad \gamma = 1 - a^4 \neq 0.$$

Find the LU factorization of the matrix A and use it to solve the system $Ax = b$.

2/II/14B **Numerical Analysis**

Let

$$f''(0) \approx a_0 f(-1) + a_1 f(0) + a_2 f(1) = \mu(f)$$

be an approximation of the second derivative which is exact for $f \in \mathcal{P}_2$, the set of polynomials of degree ≤ 2, and let

$$e(f) = f''(0) - \mu(f)$$

be its error.

(a) Determine the coefficients a_0, a_1, a_2.

(b) Using the Peano kernel theorem prove that, for $f \in C^3[-1,1]$, the set of three-times continuously differentiable functions, the error satisfies the inequality

$$|e(f)| \leq \frac{1}{3} \max_{x \in [-1,1]} |f'''(x)|\,.$$

3/I/6B Numerical Analysis

Given $(n+1)$ distinct points $x_0, x_1, \ldots, x_n$, let

$$\ell_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - x_k}{x_i - x_k}$$

be the fundamental Lagrange polynomials of degree n, let

$$\omega(x) = \prod_{i=0}^{n} (x - x_i),$$

and let p be any polynomial of degree $\leq n$.

(a) Prove that $\sum_{i=0}^{n} p(x_i)\ell_i(x) \equiv p(x)$.

(b) Hence or otherwise derive the formula

$$\frac{p(x)}{\omega(x)} = \sum_{i=0}^{n} \frac{A_i}{x - x_i}, \quad A_i = \frac{p(x_i)}{\omega'(x_i)},$$

which is the decomposition of $p(x)/\omega(x)$ into partial fractions.

3/II/16B Numerical Analysis

The functions $H_0, H_1, \ldots$ are generated by the Rodrigues formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

(a) Show that H_n is a polynomial of degree n, and that the H_n are orthogonal with respect to the scalar product

$$(f, g) = \int_{-\infty}^{\infty} f(x) g(x) e^{-x^2} \, dx.$$

(b) By induction or otherwise, prove that the H_n satisfy the three-term recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

[*Hint: you may need to prove the equality* $H_n'(x) = 2nH_{n-1}(x)$ *as well.*]

3/I/5H **Optimization**

Two players A and B play a zero-sum game with the pay-off matrix

	B_1	B_2	B_3
A_1	4	−2	−5
A_2	−2	4	3
A_3	−3	6	2
A_4	3	−8	−6

Here, the (i, j) entry of the matrix indicates the pay-off to player A if he chooses move A_i and player B chooses move B_j. Show that the game can be reduced to a zero-sum game with 2×2 pay-off matrix.

Determine the value of the game and the optimal strategy for player A.

3/II/15H **Optimization**

Explain what is meant by a transportation problem where the total demand equals the total supply. Write the Lagrangian and describe an algorithm for solving such a problem. Starting from the north-west initial assignment, solve the problem with three sources and three destinations described by the table

5	9	1	36
3	10	6	84
7	2	5	40
14	68	78	

where the figures in the 3×3 box denote the transportation costs (per unit), the right-hand column denotes supplies, and the bottom row demands.

4/I/5H **Optimization**

State and prove the Lagrangian sufficiency theorem for a general optimization problem with constraints.

4/II/14H **Optimization**

Use the two-phase simplex method to solve the problem

$$\begin{array}{llcrcrcl}
\text{minimize} & 5x_1 & - & 12x_2 & + & 13x_3 & & \\
\\
\text{subject to} & 4x_1 & + & 5x_2 & & & \leq & 9, \\
& 6x_1 & + & 4x_2 & + & x_3 & \geq & 12, \\
& 3x_1 & + & 2x_2 & - & x_3 & \leq & 3, \\
& x_i \geq 0, & & i = 1, 2, 3. & & & &
\end{array}$$

1/I/8G **Quadratic Mathematics**

Let U and V be finite-dimensional vector spaces. Suppose that b and c are bilinear forms on $U \times V$ and that b is non-degenerate. Show that there exist linear endomorphisms S of U and T of V such that $c(x, y) = b(S(x), y) = b(x, T(y))$ for all $(x, y) \in U \times V$.

1/II/17G **Quadratic Mathematics**

(a) Suppose p is an odd prime and a an integer coprime to p. Define the *Legendre symbol* $(\frac{a}{p})$ and state Euler's criterion.

(b) Compute $(\frac{-1}{p})$ and prove that

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

whenever a and b are coprime to p.

(c) Let n be any integer such that $1 \leqslant n \leqslant p-2$. Let m be the unique integer such that $1 \leqslant m \leqslant p-2$ and $mn \equiv 1 \pmod{p}$. Prove that

$$\left(\frac{n(n+1)}{p}\right) = \left(\frac{1+m}{p}\right) .$$

(d) Find

$$\sum_{n=1}^{p-2} \left(\frac{n(n+1)}{p}\right) .$$

2/I/8G **Quadratic Mathematics**

Let U be a finite-dimensional real vector space and b a positive definite symmetric bilinear form on $U \times U$. Let $\psi \colon U \to U$ be a linear map such that $b(\psi(x), y) + b(x, \psi(y)) = 0$ for all x and y in U. Prove that if ψ is invertible, then the dimension of U must be even. By considering the restriction of ψ to its image or otherwise, prove that the rank of ψ is always even.

2/II/17G **Quadratic Mathematics**

Let S be the set of all 2×2 complex matrices A which are *hermitian*, that is, $A^* = A$, where $A^* = \overline{A}^t$.

(a) Show that S is a real 4-dimensional vector space. Consider the real symmetric bilinear form b on this space defined by

$$b(A, B) = \tfrac{1}{2}\left(\mathrm{tr}(A\,B) - \mathrm{tr}(A)\,\mathrm{tr}(B)\right) .$$

Prove that $b(A, A) = -\det A$ and $b(A, I) = -\frac{1}{2}\mathrm{tr}(A)$, where I denotes the identity matrix.

(b) Consider the three matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

Prove that the basis I, A_1, A_2, A_3 of S diagonalizes b. Hence or otherwise find the rank and signature of b.

(c) Let Q be the set of all 2×2 complex matrices C which satisfy $C + C^* = \mathrm{tr}(C)\,I$. Show that Q is a real 4-dimensional vector space. Given $C \in Q$, put

$$\Phi(C) = \frac{1-i}{2}\mathrm{tr}(C)\,I + i\,C.$$

Show that Φ takes values in S and is a linear isomorphism between Q and S.

(d) Define a real symmetric bilinear form on Q by setting $c(C, D) = -\frac{1}{2}\mathrm{tr}(C\,D)$, $C, D \in Q$. Show that $b(\Phi(C), \Phi(D)) = c(C, D)$ for all $C, D \in Q$. Find the rank and signature of the symmetric bilinear form c defined on Q.

3/I/9G **Quadratic Mathematics**

Let $f(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Explain what is meant by the *discriminant* d of f. State a necessary and sufficient condition for some form of discriminant d to represent an odd prime number p. Using this result or otherwise, find the primes p which can be represented by the form $x^2 + 3y^2$.

3/II/19G **Quadratic Mathematics**

Let U be a finite-dimensional real vector space endowed with a positive definite inner product. A linear map $\tau : U \to U$ is said to be an *orthogonal projection* if τ is self-adjoint and $\tau^2 = \tau$.

(a) Prove that for every orthogonal projection τ there is an orthogonal decomposition

$$U = \ker(\tau) \oplus \operatorname{im}(\tau).$$

(b) Let $\phi\colon U \to U$ be a linear map. Show that if $\phi^2 = \phi$ and $\phi\,\phi^* = \phi^*\,\phi$, where ϕ^* is the adjoint of ϕ, then ϕ is an orthogonal projection. [*You may find it useful to prove first that if $\phi\,\phi^* = \phi^*\,\phi$, then ϕ and ϕ^* have the same kernel.*]

(c) Show that given a subspace W of U there exists a unique orthogonal projection τ such that $\operatorname{im}(\tau) = W$. If W_1 and W_2 are two subspaces with corresponding orthogonal projections τ_1 and τ_2, show that $\tau_2 \circ \tau_1 = 0$ if and only if W_1 is orthogonal to W_2.

(d) Let $\phi\colon U \to U$ be a linear map satisfying $\phi^2 = \phi$. Prove that one can define a positive definite inner product on U such that ϕ becomes an orthogonal projection.

1/I/9A **Quantum Mechanics**

A particle of mass m is confined inside a one-dimensional box of length a. Determine the possible energy eigenvalues.

1/II/18A **Quantum Mechanics**

What is the significance of the expectation value

$$\langle Q \rangle = \int \psi^*(x)\ Q\ \psi(x) dx$$

of an observable Q in the normalized state $\psi(x)$? Let Q and P be two observables. By considering the norm of $(Q + i\lambda P)\psi$ for real values of λ, show that

$$\langle Q^2 \rangle \langle P^2 \rangle \geqslant \tfrac{1}{4} |\langle [Q, P] \rangle|^2 \,.$$

The uncertainty ΔQ of Q in the state $\psi(x)$ is defined as

$$(\Delta Q)^2 = \langle (Q - \langle Q \rangle)^2 \rangle \,.$$

Deduce the generalized uncertainty relation,

$$\Delta Q \Delta P \geqslant \tfrac{1}{2} |\langle [Q, P] \rangle| \,.$$

A particle of mass m moves in one dimension under the influence of the potential $\frac{1}{2} m\omega^2 x^2$. By considering the commutator $[x, p]$, show that the expectation value of the Hamiltonian satisfies

$$\langle H \rangle \geqslant \tfrac{1}{2} \hbar\omega \,.$$

2/I/9A **Quantum Mechanics**

What is meant by the statement than an operator is *hermitian*?

A particle of mass m moves in the real potential $V(x)$ in one dimension. Show that the Hamiltonian of the system is hermitian.

Show that

$$\frac{d}{dt}\langle x \rangle = \frac{1}{m}\langle p \rangle \,,$$
$$\frac{d}{dt}\langle p \rangle = \langle -V'(x) \rangle \,,$$

where p is the momentum operator and $\langle A \rangle$ denotes the expectation value of the operator A.

2/II/18A Quantum Mechanics

A particle of mass m and energy E moving in one dimension is incident from the left on a potential barrier $V(x)$ given by

$$V(x) = \begin{cases} V_0 & 0 \leqslant x \leqslant a \\ 0 & \text{otherwise} \end{cases}$$

with $V_0 > 0$.

In the limit $V_0 \to \infty, a \to 0$ with $V_0 a = U$ held fixed, show that the transmission probability is

$$T = \left(1 + \frac{mU^2}{2E\hbar^2}\right)^{-1}.$$

3/II/20A Quantum Mechanics

The radial wavefunction for the hydrogen atom satisfies the equation

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right) + \frac{\hbar^2}{2mr^2}\ell(\ell+1)R(r) - \frac{e^2}{4\pi\epsilon_0 r}R(r) = ER(r)\,.$$

Explain the origin of each term in this equation.

The wavefunctions for the ground state and first radially excited state, both with $\ell = 0$, can be written as

$$R_1(r) = N_1 \exp(-\alpha r)$$
$$R_2(r) = N_2(r+b)\exp(-\beta r)$$

respectively, where N_1 and N_2 are normalization constants. Determine α, β, b and the corresponding energy eigenvalues E_1 and E_2.

A hydrogen atom is in the first radially excited state. It makes the transition to the ground state, emitting a photon. What is the frequency of the emitted photon?

3/I/10A **Special Relativity**

What are the momentum and energy of a photon of wavelength λ?

A photon of wavelength λ is incident on an electron. After the collision, the photon has wavelength λ'. Show that

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta),$$

where θ is the scattering angle and m is the electron rest mass.

4/I/9A **Special Relativity**

Prove that the two-dimensional Lorentz transformation can be written in the form

$$\begin{aligned} x' &= x\cosh\phi - ct\sinh\phi \\ ct' &= -x\sinh\phi + ct\cosh\phi, \end{aligned}$$

where $\tanh\phi = v/c$. Hence, show that

$$\begin{aligned} x' + ct' &= e^{-\phi}(x + ct) \\ x' - ct' &= e^{\phi}(x - ct)\,. \end{aligned}$$

Given that frame S' has speed v with respect to S and S'' has speed v' with respect to S', use this formalism to find the speed v'' of S'' with respect to S.

[*Hint: rotation through a hyperbolic angle ϕ, followed by rotation through ϕ', is equivalent to rotation through $\phi + \phi'$.*]

4/II/18A **Special Relativity**

A pion of rest mass M decays at rest into a muon of rest mass $m < M$ and a neutrino of zero rest mass. What is the speed u of the muon?

In the pion rest frame S, the muon moves in the y-direction. A moving observer, in a frame S' with axes parallel to those in the pion rest frame, wishes to take measurements of the decay along the x-axis, and notes that the pion has speed v with respect to the x-axis. Write down the four-dimensional Lorentz transformation relating S' to S and determine the momentum of the muon in S'. Hence show that in S' the direction of motion of the muon makes an angle θ with respect to the y-axis, where

$$\tan\theta = \frac{M^2 + m^2}{M^2 - m^2}\frac{v}{(c^2 - v^2)^{1/2}}\,.$$

1/I/3H Statistics

Derive the least squares estimators $\hat{\alpha}$ and $\hat{\beta}$ for the coefficients of the simple linear regression model

$$Y_i = \alpha + \beta(x_i - \bar{x}) + \varepsilon_i, \qquad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are given constants, $\bar{x} = n^{-1} \sum_{i=1}^n x_i$, and ε_i are independent with $\mathrm{E}\,\varepsilon_i = 0$, $\mathrm{Var}\,\varepsilon_i = \sigma^2$, $i = 1, \ldots, n$.

A manufacturer of optical equipment has the following data on the unit cost (in pounds) of certain custom-made lenses and the number of units made in each order:

No. of units, x_i	1	3	5	10	12
Cost per unit, y_i	58	55	40	37	22

Assuming that the conditions underlying simple linear regression analysis are met, estimate the regression coefficients and use the estimated regression equation to predict the unit cost in an order for 8 of these lenses.

[*Hint: for the data above*, $S_{xy} = \sum_{i=1}^n (x_i - \bar{x}) y_i = -257.4$.]

1/II/12H Statistics

Suppose that six observations $X_1, \ldots, X_6$ are selected at random from a normal distribution for which both the mean μ_X and the variance σ_X^2 are unknown, and it is found that $S_{XX} = \sum_{i=1}^6 (x_i - \bar{x})^2 = 30$, where $\bar{x} = \frac{1}{6} \sum_{i=1}^6 x_i$. Suppose also that 21 observations $Y_1, \ldots, Y_{21}$ are selected at random from another normal distribution for which both the mean μ_Y and the variance σ_Y^2 are unknown, and it is found that $S_{YY} = 40$. Derive carefully the likelihood ratio test of the hypothesis H_0: $\sigma_X^2 = \sigma_Y^2$ against H_1: $\sigma_X^2 > \sigma_Y^2$ and apply it to the data above at the 0.05 level.

[*Hint:*

Distribution	χ_5^2	χ_6^2	χ_{20}^2	χ_{21}^2	$F_{5,20}$	$F_{6,21}$
95% percentile	11.07	12.59	31.41	32.68	2.71	2.57

]

2/I/3H Statistics

Let $X_1, \ldots, X_n$ be a random sample from the $N(\theta, \sigma^2)$ distribution, and suppose that the prior distribution for θ is $N(\mu, \tau^2)$, where σ^2, μ, τ^2 are known. Determine the posterior distribution for θ, given $X_1, \ldots, X_n$, and the best point estimate of θ under both quadratic and absolute error loss.

2/II/12H **Statistics**

An examination was given to 500 high-school students in each of two large cities, and their grades were recorded as low, medium, or high. The results are given in the table below.

	Low	*Medium*	*High*
City A	103	145	252
City B	140	136	224

Derive carefully the test of homogeneity and test the hypothesis that the distributions of scores among students in the two cities are the same.

[*Hint:*

Distribution	χ_1^2	χ_2^2	χ_3^2	χ_5^2	χ_6^2
99% percentile	6.63	9.21	11.34	15.09	16.81
95% percentile	3.84	5.99	7.81	11.07	12.59

]

4/I/3H **Statistics**

The following table contains a distribution obtained in 320 tosses of 6 coins and the corresponding expected frequencies calculated with the formula for the binomial distribution for $p = 0.5$ and $n = 6$.

No. heads	0	1	2	3	4	5	6
Observed frequencies	3	21	85	110	62	32	7
Expected frequencies	5	30	75	100	75	30	5

Conduct a goodness-of-fit test at the 0.05 level for the null hypothesis that the coins are all fair.

[*Hint:*

Distribution	χ_5^2	χ_6^2	χ_7^2
95% percentile	11.07	12.59	14.07

]

4/II/12H **Statistics**

State and prove the Rao–Blackwell theorem.

Suppose that $X_1, \ldots, X_n$ are independent random variables uniformly distributed over $(\theta, 3\theta)$. Find a two-dimensional sufficient statistic $T(X)$ for θ. Show that an unbiased estimator of θ is $\hat{\theta} = X_1/2$.

Find an unbiased estimator of θ which is a function of $T(X)$ and whose mean square error is no more than that of $\hat{\theta}$.