UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Friday 6 June 2003 9 to 12

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than **FOUR** *questions.*

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, ..., J** *according to the letter affixed to each question. (For example,* **1F, 3F** *should be in one bundle and* **7H, 10H** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1F Combinatorics

Write an essay on the Kruskal–Katona theorem. As well as stating the theorem and giving a detailed sketch of a proof, you should describe some further results that may be derived from it.

2F Representation Theory

Assume that the group $SL_2(\mathbb{F}_3)$ of 2×2 matrices of determinant 1 with entries from the field $\mathbb{F}_3$ has presentation

$$\langle X, P, Q : X^3 = P^4 = 1,\ P^2 = Q^2,\ PQP^{-1} = Q^{-1},\ XPX^{-1} = Q,\ XQX^{-1} = PQ \rangle\,.$$

Show that the subgroup generated by P^2 is central and that the quotient group can be identified with the alternating group A_4. Assuming further that $SL_2(\mathbb{F}_3)$ has seven conjugacy classes find the character table.

Is it true that every irreducible character is induced up from the character of a 1-dimensional representation of some subgroup?

[*Hint: You may find it useful to note that $SL_2(\mathbb{F}_3)$ may be regarded as a subgroup of SU_2, providing a faithful 2-dimensional representation; the subgroup generated by P and Q is the quaternion group of order 8, acting irreducibly.*]

3F Galois Theory

Write an essay on finite fields and their Galois theory.

4H Differentiable Manifolds

Define the 'pull-back' homomorphism of differential forms determined by the smooth map $f : M \to N$ and state its main properties.

If $\theta : W \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^m$ with coordinates x_i on V and y_j on W and the m-form ω is equal to $f\, dx_1 \wedge \ldots \wedge dx_m$ on V, state and prove the expression for $\theta^*(\omega)$ as a multiple of $dy_1 \wedge \ldots \wedge dy_m$.

Define the integral of an m-form ω over an oriented m-manifold M and prove that it is well-defined.

Show that the inclusion map $f : N \hookrightarrow M$, of an oriented n-submanifold N (without boundary) into M, determines an element ν of $H_n(M) \cong \mathrm{Hom}(H^n(M), \mathbb{R})$. If $M = N \times P$ and $f(x) = (x, p)$, for $x \in N$ and p fixed in P, what is the relation between ν and $\pi^*([\omega_N])$, where $[\omega_N]$ is the fundamental cohomology class of N and π is the projection onto the first factor?

5G Algebraic Topology

State the Mayer–Vietoris theorem. You should give the definition of all the homomorphisms involved.

Compute the homology groups of the union of the 2-sphere with the line segment from the North pole to the South pole.

6G Number Fields

Write an essay on the Dirichlet unit theorem with particular reference to quadratic fields.

7H Hilbert Spaces

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$.

(a) Show that if $\|I - T\| < 1$ then T is invertible.

(b) Prove that if T is invertible and if $S \in \mathcal{B}(H)$ satisfies $\|S - T\| < \|T^{-1}\|^{-1}$, then S is invertible.

(c) Define what it means for T to be *compact*. Prove that the set of compact operators on H is a closed subset of $\mathcal{B}(H)$.

(d) Prove that T is compact if and only if there is a sequence (F_n) in $\mathcal{B}(H)$, where each operator F_n has finite rank, such that $\|F_n - T\| \to 0$ as $n \to \infty$.

(e) Suppose that $T = A + K$, where A is invertible and K is compact. Prove that then, also, $T = B + F$, where B is invertible and F has finite rank.

8G Riemann Surfaces

(a) Define the degree $\deg f$ of a meromorphic function on the Riemann sphere $\mathbb{P}^1$. State the Riemann–Hurwitz theorem.

Let f and g be two rational functions on the sphere $\mathbb{P}^1$. Show that

$$\deg(f+g) \leqslant \deg f + \deg g\,.$$

Deduce that

$$|\deg f - \deg g| \leqslant \deg(f+g) \leqslant \deg f + \deg g\,.$$

(b) Describe the topological type of the Riemann surface defined by the equation $w^2 + 2w = z^5$ in $\mathbb{C}^2$. [*You should analyse carefully the behaviour as w and z approach ∞.*]

9H Algebraic Curves

Let X be a smooth curve of genus 0 over an algebraically closed field k. Show that $k(X) = k(\mathbb{P}^1)$.

Now let C be a plane projective curve defined by an irreducible homogeneous cubic polynomial.

(a) Show that if C is smooth then C is not isomorphic to $\mathbb{P}^1$. Standard results on the canonical class may be assumed without proof, provided these are clearly stated.

(b) Show that if C has a singularity then there exists a non-constant morphism from $\mathbb{P}^1$ to C.

10H Logic, Computation and Set Theory

Write an essay on propositional logic. You should include all relevant definitions, and should cover the Completeness Theorem, as well as the Compactness Theorem and the Decidability Theorem.

[*You may assume that the set of primitive propositions is countable. You do not need to give proofs of simple examples of syntactic implication, such as the fact that* $p \Rightarrow p$ *is a theorem or that* $p \Rightarrow q$ *and* $q \Rightarrow r$ *syntactically imply* $p \Rightarrow r$.]

11G Probability and Measure

Let $f : [a,b] \to \mathbb{R}$ be integrable with respect to Lebesgue measure μ on $[a,b]$. Prove that, if

$$\int_J f \, d\mu = 0$$

for every sub-interval J of $[a,b]$, then $f = 0$ almost everywhere on $[a,b]$.

Now define

$$F(x) = \int_a^x f \, d\mu \, .$$

Prove that F is continuous on $[a,b]$. Show that, if F is zero on $[a,b]$, then f is zero almost everywhere on $[a,b]$.

Suppose now that f is bounded and Lebesgue integrable on $[a,b]$. By applying the Dominated Convergence Theorem to

$$F_n(x) = \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}},$$

or otherwise, show that, if F is differentiable on $[a,b]$, then $F' = f$ almost everywhere on $[a,b]$.

The functions $f_n : [a,b] \to \mathbb{R}$ have the properties:

(a) f_n converges pointwise to a differentiable function g on $[a,b]$,

(b) each f_n has a continuous derivative f_n' with $|f_n'(x)| \leqslant 1$ on $[a,b]$,

(c) f_n' converges pointwise to some function h on $[a,b]$.

Deduce that

$$h(x) = \lim_{n\to\infty} \left(\frac{df_n(x)}{dx} \right) = \frac{d}{dx} \left(\lim_{n\to\infty} f_n(x) \right) = g'(x)$$

almost everywhere on $[a,b]$.

12I Applied Probability

Explain what is meant by a renewal process and by a renewal-reward process.

State and prove the law of large numbers for renewal-reward processes.

A component used in a manufacturing process has a maximum lifetime of 2 years and is equally likely to fail at any time during that period. If the component fails whilst in use, it is replaced immediately by a similar component, at a cost of £1000. The factory owner may alternatively replace the component before failure, at a time of his choosing, at a cost of £200. What should the factory owner do?

13J Information Theory

State and prove the Fano and generalized Fano inequalities.

14J Optimization and Control

The scalars x_t, y_t, u_t, are related by the equations

$$x_t = x_{t-1} + u_{t-1}\,, \quad y_t = x_{t-1} + \eta_{t-1}\,, \quad t = 1, \ldots, T\,,$$

where $\{\eta_t\}$ is a sequence of uncorrelated random variables with means of 0 and variances of 1. Given that $\hat{x}_0$ is an unbiased estimate of x_0 of variance 1, the control variable u_t is to be chosen at time t on the basis of the information W_t, where $W_0 = (\hat{x}_0)$ and $W_t = (\hat{x}_0, u_0, \ldots, u_{t-1}, y_1, \ldots, y_t)$, $t = 1, 2, \ldots, T-1$. Let $\hat{x}_1, \ldots, \hat{x}_T$ be the Kalman filter estimates of $x_1, \ldots, x_T$ computed from

$$\hat{x}_t = \hat{x}_{t-1} + u_{t-1} + h_t(y_t - \hat{x}_{t-1})$$

by appropriate choices of $h_1, \ldots, h_T$. Show that the variance of $\hat{x}_t$ is $V_t = 1/(1+t)$.

Define $F(W_T) = E\left[x_T^2 \mid W_T\right]$ and

$$F(W_t) = \inf_{u_t, \ldots, u_{T-1}} E\left[\sum_{\tau=t}^{T-1} u_\tau^2 + x_T^2 \,\middle|\, W_t\right], \quad t = 0, \ldots, T-1\,.$$

Show that $F(W_t) = \hat{x}_t^2 P_t + d_t$, where $P_t = 1/(T-t+1)$, $d_T = 1/(1+T)$ and $d_{t-1} = V_{t-1} V_t P_t + d_t$.

How would the expression for $F(W_0)$ differ if $\hat{x}_0$ had a variance different from 1?

15I Principles of Statistics

Write an account, with appropriate examples, of inference in multiparameter exponential families. Your account should include a discussion of natural statistics and their properties and of various conditional tests on natural parameters.

16J Stochastic Financial Models

A single-period market contains d risky assets, $S^1, S^2, \ldots, S^d$, initially worth $(S_0^1, S_0^2, \ldots, S_0^d)$, and at time 1 worth random amounts $(S_1^1, S_1^2, \ldots, S_1^d)$ whose first two moments are given by

$$\mu = ES_1, \quad V = \mathrm{cov}(S_1) \equiv E\big[\,(S_1 - ES_1)(S_1 - ES_1)^T\,\big].$$

An agent with given initial wealth w_0 is considering how to invest in the available assets, and has asked for your advice. Develop the theory of the mean-variance efficient frontier far enough to exhibit explicitly the minimum-variance portfolio achieving a required mean return, assuming that V is non-singular. How does your analysis change if a riskless asset S^0 is added to the market? Under what (sufficient) conditions would an agent maximising expected utility actually choose a portfolio on the mean-variance efficient frontier?

17B Dynamical Systems

Let $f : S^1 \to S^1$ be an orientation-preserving invertible map of the circle onto itself, with a lift $F : \mathbb{R} \to \mathbb{R}$. Define the rotation numbers $\rho_0(F)$ and $\rho(f)$.

Suppose that $\rho_0(F) = p/q$, where p and q are coprime integers. Prove that the map f has periodic points of least period q, and no periodic points with any least period not equal to q.

Now suppose that $\rho_0(F)$ is irrational. Explain the distinction between wandering and non-wandering points under f. Let $\Omega(x)$ be the set of limit points of the sequence $\{x, f(x), f^2(x), \ldots\}$. Prove

(a) that the set $\Omega(x) = \Omega$ is independent of x and is the smallest closed, non-empty, f-invariant subset of S^1;

(b) that Ω is the set of non-wandering points of S^1;

(c) that Ω is either the whole of S^1 or a Cantor set in S^1.

18D Partial Differential Equations

Discuss the basic properties of the Fourier transform and how it is used in the study of partial differential equations.

The essay should include: definition and basic properties, inversion theorem, applications to establishing well-posedness of evolution partial differential equations with constant coefficients.

19D Methods of Mathematical Physics

By setting $w(z) = \int_\gamma f(t)e^{-zt}dt$, where γ and $f(t)$ are to be suitably chosen, explain how to find integral representations of the solutions of the equation

$$zw'' - kw = 0 ,$$

where k is a non-zero real constant and z is complex. Discuss γ in the particular case that z is restricted to be real and positive and distinguish the different cases that arise according to the sign of k.

Show that in this particular case, by choosing γ as a closed contour around the origin, it is possible to express a solution in the form

$$w(z) = A \sum_{n=0}^{\infty} \frac{(zk)^{n+1}}{n!(n+1)!} ,$$

where A is a constant.

Show also that for $k > 0$ there are solutions that satisfy

$$w(z) \sim Bz^{1/4} e^{-2\sqrt{kz}} \quad \text{as } z \to \infty ,$$

where B is a constant.

20E Numerical Analysis

Write an essay on the conjugate gradient method. Your essay should include:

(a) a statement of the method and a sketch of its derivation;

(b) discussion, without detailed proofs, but with precise statements of relevant theorems, of the conjugacy of the search directions;

(c) a description of the standard form of the algorithm;

(d) discussion of the connection of the method with Krylov subspaces.

21C Electrodynamics

Describe the physical meaning of the various components of the stress-energy tensor T^{ab} of the electromagnetic field.

Suppose that one is given an electric field $\mathbf{E}(\mathbf{x})$ and a magnetic field $\mathbf{B}(\mathbf{x})$. Show that the angular momentum about the origin of these fields is

$$\mathbf{J} = \frac{1}{\mu_0} \int \mathbf{x} \times (\mathbf{E} \times \mathbf{B})\, d^3\mathbf{x}$$

where the integral is taken over all space.

A point electric charge Q is at the origin, and has electric field

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{x}}{|\mathbf{x}|^3} \,.$$

A point magnetic monopole of strength P is at $\mathbf{y}$ and has magnetic field

$$\mathbf{B} = \frac{\mu_0 P}{4\pi} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \,.$$

Find the component, along the axis between the electric charge and the magnetic monopole, of the angular momentum of the electromagnetic field about the origin.

[*Hint: You may find it helpful to express both* $\mathbf{E}$ *and* $\mathbf{B}$ *as gradients of scalar potentials.*]

22C Foundations of Quantum Mechanics

Discuss the quantum mechanics of the one-dimensional harmonic oscillator using creation and annihilation operators, showing how the energy levels are calculated.

A quantum mechanical system consists of two interacting harmonic oscillators and has the Hamiltonian

$$H = \frac{1}{2}\hat{p}_1^2 + \frac{1}{2}\hat{x}_1^2 + \frac{1}{2}\hat{p}_2^2 + \frac{1}{2}\hat{x}_2^2 + \lambda \hat{x}_1 \hat{x}_2 \,.$$

For $\lambda = 0$, what are the degeneracies of the three lowest energy levels? For $\lambda \neq 0$ compute, to lowest non-trivial order in perturbation theory, the energies of the ground state and first excited state.

[*Standard results for perturbation theory may be stated without proof.*]

23A Statistical Physics

A gas of non-interacting identical bosons in volume V, with one-particle energy levels ϵ_r, $r = 1, 2, \ldots, \infty$, is in equilibrium at temperature T and chemical potential μ. Let n_r be the number of particles in the rth one-particle state. Write down an expression for the grand partition function $\mathcal{Z}$. Write down an expression for the probability of finding a given set of occupation numbers n_r of the one-particle states. Hence determine the mean occupation numbers $\bar{n}_r$ (the Bose–Einstein distribution). Write down expressions, in terms of the mean occupation numbers, for the total energy E and total number of particles N.

Write down an expression for the grand potential Ω in terms of $\mathcal{Z}$. Given that

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu}$$

show that S can be written in the form

$$S = k \sum_r f(\bar{n}_r)$$

for some function f, which you should determine. Hence show that $dS = 0$ for any change of the gas that leaves the mean occupation numbers unchanged. Consider a (quasi-static) change of V with this property. Using the formula

$$P = -\left(\frac{\partial E}{\partial V}\right)_{N,S}$$

and given that $\epsilon_r \propto V^{-\sigma}$ $(\sigma > 0)$ for each r, show that

$$P = \sigma(E/V).$$

What is the value of σ for photons?

Let $\mu = 0$, so that E is a function only of T and V. Why does the energy density $\varepsilon = E/V$ depend only on T? Using the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

and the first law of thermodynamics for reversible changes, show that

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

and hence that

$$\varepsilon(T) \propto T^{\gamma}$$

for some power γ that you should determine. Show further that

$$S \propto (TV^{\sigma})^{\frac{1}{\sigma}}.$$

Hence verify, given $\mu = 0$, that $\bar{n}_r$ is left unchanged by a change of V at constant S.

24A Applications of Quantum Mechanics

Atoms of mass m in an infinite one-dimensional periodic array, with interatomic spacing a, have perturbed positions $x_n = na + y_n$, for integer n. The potential between neighbouring atoms is

$$\frac{1}{2}\lambda\left(x_{n+1} - x_n - a\right)^2$$

for positive constant λ. Write down the Lagrangian for the variables y_n. Find the frequency $\omega(k)$ of a normal mode of wavenumber k. Define the Brillouin zone and explain why k may be restricted to lie within it.

Assume now that the array is periodically-identified, so that there are effectively only N atoms in the array and the atomic displacements y_n satisfy the periodic boundary conditions $y_{n+N} = y_n$. Determine the allowed values of k within the Brillouin zone. Show, for allowed wavenumbers k and k', that

$$\sum_{n=0}^{N-1} e^{in(k-k')a} = N\delta_{k,k'}.$$

By writing y_n as

$$y_n = \frac{1}{\sqrt{N}}\sum_k q_k e^{inka}$$

where the sum is over allowed values of k, find the Lagrangian for the variables q_k, and hence the Hamiltonian H as a function of q_k and the conjugate momenta p_k. Show that the Hamiltonian operator $\hat{H}$ of the quantum theory can be written in the form

$$\hat{H} = E_0 + \sum_k \hbar\omega(k) a_k^\dagger a_k$$

where E_0 is a constant and $a_k, a_k^\dagger$ are harmonic oscillator annihilation and creation operators. What is the physical interpretation of a_k and $a_k^\dagger$? How does this show that phonons have quantized energies?

25A General Relativity

What are "inertial coordinates" and what is their physical significance? [*A proof of the existence of inertial coordinates is not required.*] Let O be the origin of inertial coordinates and let $R_{abcd}|_O$ be the curvature tensor at O (with all indices lowered). Show that $R_{abcd}|_O$ can be expressed entirely in terms of second partial derivatives of the metric g_{ab}, evaluated at O. Use this expression to deduce that

(a) $R_{abcd} = -R_{bacd}$

(b) $R_{abcd} = R_{cdab}$

(c) $R_{a[bcd]} = 0$.

Starting from the expression for $R^a{}_{bcd}$ in terms of the Christoffel symbols, show (again by using inertial coordinates) that

$R_{ab[cd;e]} = 0$.

Obtain the contracted Bianchi identities and explain why the Einstein equations take the form

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} - \Lambda g_{ab},$$

where T_{ab} is the energy-momentum tensor of the matter and Λ is an arbitrary constant.

26B Fluid Dynamics II

Show that the complex potential in the complex ζ plane,

$$w = (U - iV)\zeta + (U + iV)\frac{c^2}{\zeta} - \frac{i\kappa}{2\pi}\log\zeta \ ,$$

describes irrotational, inviscid flow past the rigid cylinder $|\zeta| = c$, placed in a uniform flow (U, V) with circulation κ.

Show that the transformation

$$z = \zeta + \frac{c^2}{\zeta}$$

maps the circle $|\zeta| = c$ in the ζ plane onto the flat plate airfoil $-2c < x < 2c$, $y = 0$ in the z plane ($z = x + iy$). Hence, write down an expression for the complex potential, w_p, of uniform flow past the flat plate, with circulation κ. Indicate very briefly how the value of κ might be chosen to yield a physical solution.

Calculate the turning moment, M, exerted on the flat plate by the flow.

(You are given that

$$M = -\frac{1}{2}\rho\,\mathrm{Re}\left\{\oint \left[\frac{\left(\frac{dw}{d\zeta}\right)^2}{\frac{dz}{d\zeta}}\right] z(\zeta)\ d\zeta\right\} ,$$

where ρ is the fluid density and the integral is to be completed around a contour enclosing the circle $|\zeta| = c$).

27E Waves in Fluid and Solid Media

Show that the equations governing isotropic linear elasticity have plane-wave solutions, identifying them as P, SV or SH waves.

A semi-infinite elastic medium in $y < 0$ (where y is the vertical coordinate) with density ρ and Lamé moduli λ and μ is overlaid by a layer of thickness h (in $0 < y < h$) of a second elastic medium with density ρ' and Lamé moduli λ' and μ'. The top surface at $y = h$ is free, i.e. the surface tractions vanish there. The speed of S-waves is lower in the layer, i.e. ${c'_S}^2 = \mu'/\rho' < \mu/\rho = {c_S}^2$. For a time-harmonic SH-wave with horizontal wavenumber k and frequency ω, which oscillates in the slow top layer and decays exponentially into the fast semi-infinite medium, derive the dispersion relation for the apparent wave speed $c(k) = \omega/k$,

$$\tan\left(kh\sqrt{\frac{c^2}{{c'_S}^2} - 1}\right) = \frac{\mu\sqrt{1 - \frac{c^2}{c_S^2}}}{\mu'\sqrt{\frac{c^2}{{c'_S}^2} - 1}} .$$

Show graphically that there is always one root, and at least one higher mode if $\sqrt{c_S^2/{c'_S}^2 - 1} > \pi/kh$.